

Asymptotic quasinormal frequencies of different spin fields in spherically symmetric black holes

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We consider the asymptotic quasinormal frequencies of various spin fields in Schwarzschild and Reissner-Nordström black holes. In the Schwarzschild case, the real part of the asymptotic frequency is $\ln 3$ for the spin 0 and the spin 2 fields, while for the spin 1/2, the spin 1, and the spin 3/2 fields it is zero. For the nonextreme charged black holes, the spin 3/2 Rarita-Schwinger field has the same asymptotic frequency as that of the integral spin fields. However, the asymptotic frequency of the Dirac field is different, and its real part is zero. For the extremal case, which is relevant to the supersymmetric consideration, all the spin fields have the same asymptotic frequency, the real part of which is zero. For the imaginary parts of the asymptotic frequencies, it is interesting to see that it has a universal spacing of $1/4M$ for all the spin fields in the single-horizon cases of the Schwarzschild and the extreme Reissner-Nordström black holes. The implications of these results to the universality of the asymptotic quasinormal frequencies are discussed.

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I. INTRODUCTION

Hod [1] was the first to conjecture that the highly damped limit of the black hole quasinormal frequency was related to the fundamental area unit in the quantum theory of gravity. At that time, this limit was known only numerically [2,3],

$$\omega_n = 0.043\,712\,3 \left(\frac{1}{M}\right) - \frac{i}{4M} \left(n + \frac{1}{2}\right) + \dots, \quad (1)$$

as $n \rightarrow \infty$, where M is the mass of the black hole. He noticed that 0.043 712 3 is very close to $\ln 3/8\pi$. Using Bohr's correspondence principle, he was able to derive the area spectrum of the quantum Schwarzschild black hole to be

$$A_n = 4(\ln 3)n, \quad n = 1, 2, 3, \dots \quad (2)$$

Comparing Hod's result with the expressions of the area and entropy spectra obtained in the theory of loop quantum gravity, Dreyer [4] determined the value of the Immirzi parameter, an otherwise arbitrary constant in the theory. At the same time, because of the presence of $\ln 3$, he also suggested that the gauge group should be changed from $SU(2)$ to $SO(3)$. Although this connection between the asymptotic quasinormal frequency and the Immirzi parameter has been questioned [5,6], it has nevertheless aroused a lot of research interests in this direction.

The first analytic evaluation of the asymptotic quasinormal frequency was carried out by Motl and Neitzke [7,8] using the monodromy method. Subsequently, with this method, the calculation has been extended to other kinds of black holes [9]. However, all these calculations are done with respect to fields with integral spins. In this paper, we would like to further consider the asymptotic quasinormal frequencies of fields with half-integral spins like the Dirac

and the Rarita-Schwinger fields, which are lacking so far. On the other hand, we hope our consideration will also shed light on the question of universality of the value of $\ln 3$ studied by several authors [10–12]. It turns out that this value is indeed obtained in most of the cases for single-horizon black holes. We would like to see if this universality can be applied to fields with different spins.

In the next section, we first consider the case of the Schwarzschild black hole. To deal with different spin fields in a unified way, we use the WKB formalism of Andersson and Howls [13], in addition to the monodromy method, to evaluate the asymptotic frequencies. Here we also address the discrepancy on the value of the imaginary part of the Dirac asymptotic frequency in [14,15]. In Sec. III, we turn to the case of the charged Reissner-Nordström black hole. We shall consider both the nonextremal and the extremal black holes. Since the horizon structures are different in these two cases, one cannot take the extremal limit directly. A separate calculation is thus carried out carefully in this section. In so doing, we would also hope to resolve the discrepancies in the value of the asymptotic quasinormal frequencies for the extremal black holes in the literature [9,12,13]. Moreover, this calculation is also interesting because the spin 1, the spin 3/2, and the spin 2 fields together in the extremal black hole spacetime represent the simplest supersymmetric situation. We would like to see how their asymptotic frequencies are related in this case [16]. Conclusions and discussions are presented in Sec. IV.

II. SCHWARZSCHILD BLACK HOLE

For the Schwarzschild black hole, the metric can be written as,

$$ds^2 = -\frac{\Delta}{r^2} dt^2 + \frac{r^2}{\Delta} dr^2 + r^2 d\Omega^2, \quad (3)$$

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where $\Delta = r(r - 2M)$ and M is the mass of the black hole. The radial parts of the wave equations for different spin fields can all be simplified to the form of a Schrödinger-like equation,

$$\frac{d^2 Z(r)}{dr_*^2} + (\omega^2 - V)Z(r) = 0, \quad (4)$$

where ω is the frequency, V is the effective potential, and r_* is the so-called tortoise coordinate with

$$\frac{d}{dr_*} = \left(\frac{\Delta}{r^2}\right) \frac{d}{dr} \Rightarrow r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right). \quad (5)$$

For integral spins, $s = 0, 1$, and 2 [17],

$$V = \frac{\Delta}{r^2} \left[\frac{l(l+1)}{r^2} + \frac{(1-s^2)2M}{r^3} \right], \quad (6)$$

where $l = 0, 1, 2, \dots$ is the angular momentum number. For the Dirac field, $s = 1/2$ [18],

$$V = \frac{\Delta^{1/2}}{r^4} [\kappa^2 \Delta^{1/2} - \kappa(r - 3M)], \quad (7)$$

where $\kappa = j + 1/2$ and $j = l \pm 1/2$, $l = 0, 1, 2, \dots$. For the Rarita-Schwinger field, $s = 3/2$ [19,20],

$$V = \frac{\Delta}{r^6} (\lambda r^2 + 2Mr) - \frac{d}{dr_*} \left[\frac{1}{F} \left(\frac{dF}{dr_*} - \lambda \sqrt{\lambda + 1} \right) \right], \quad (8)$$

where

$$F = \frac{1}{\Delta^{1/2}} (\lambda r^2 + 2Mr), \quad (9)$$

and

$$\lambda = \left(j - \frac{1}{2}\right) \left(j + \frac{3}{2}\right), \quad (10)$$

with $j = l + 3/2$, $l = 0, 1, 2, \dots$. We have listed the effective potentials for various spin fields here for completeness. In the following calculations, we need mainly the asymptotic behaviors of these potentials as $r \rightarrow 0$, that is, near the black hole singularity. We assume that as $r \rightarrow 0$, the asymptotic behavior of the effective potential is

$$V \sim \frac{M^2}{r^4} \alpha. \quad (11)$$

The values of α for various spin fields are listed in Table I.

Going back to Eq. (4), the solutions at infinity, $r_* \rightarrow \pm\infty$, are

$$Z(r) \sim e^{\pm i\omega r_*}, \quad (12)$$

because $V \rightarrow 0$ in this limit. The quasinormal modes correspond to the solutions with the boundary conditions of outgoing wave, $e^{i\omega r_*}$, at $r_* = r = \infty$, and ingoing wave, $e^{-i\omega r_*}$, at the horizon, $r_* = -\infty$ or $r = 2M$. The corresponding spectrum of these modes are complex and discrete.

TABLE I. Some parameters characterizing the asymptotic behaviors of various spin fields for the Schwarzschild black hole. α represents the behavior of the effective potential near the black hole singularity as in Eq. (11). γ is the value of the integral in Eq. (21).

spin	α	γ	$-(1 + 2\cos 2\gamma)$
0	-4	0	-3
1/2	0	$-\pi/2$	1
1	0	$-\pi/2$	1
3/2	0	$-\pi/2$	1
2	12	$-\pi$	-3

In order to use the WKB method of Andersson and Howls to evaluate the asymptotic quasinormal frequencies as $|\text{Im } \omega| \rightarrow \infty$, we define a new function [13],

$$\psi(r) = \frac{\Delta^{1/2}}{r} Z(r). \quad (13)$$

From Eq. (4), one can write the wave equation for ψ as

$$\frac{d^2 \psi}{dr^2} + R(r)\psi = 0, \quad (14)$$

with

$$R(r) = \frac{r^2}{(r - 2M)^2} \left[\omega^2 - V + \frac{2M}{r^3} - \frac{3M^2}{r^4} \right]. \quad (15)$$

The WKB solutions to this equation are [13],

$$f_{1,2}^{(t)}(r) = \frac{1}{\sqrt{Q(r)}} e^{\pm i \int_t^r d\xi Q(\xi)}, \quad (16)$$

where t is a reference point and

$$\begin{aligned} Q^2(r) &= R(r) - \frac{1}{4r^2} \\ &= \frac{r^2}{(r - 2M)^2} \left[\omega^2 - V - \frac{1}{4r^2} + \frac{3M}{r^3} - \frac{4M^2}{r^4} \right]. \end{aligned} \quad (17)$$

Here $Q(r)$ is chosen in such a way to match the behavior of the solutions of $\psi(r)$ near $r = 0$.

The zeros and the poles of $Q(r)$ are important to the behaviors of the WKB solutions $f_{1,2}^{(t)}(r)$. In the Schwarzschild case, as shown in Fig. 1, $Q(r)$ has four zeros. From each zero, three Stokes lines and three anti-Stokes lines emanate. Along the anti-Stokes lines $Q(r)dr$ is purely real, so $f_1^{(t)}(r)$ and $f_2^{(t)}(r)$ are oscillatory functions of comparable magnitudes. Between anti-Stokes lines are regions on the complex r -plane in which one of the two WKB solutions dominates. We have also indicated this in Fig. 1. On the Stokes lines $Q(r)dr$ is purely imaginary. There are also two poles at $r = 0$ and at $r = 2M$. The solution to the wave equation in the WKB approximation is represented by an appropriate combination of $f_1^{(t)}(r)$ and

$f_2^{(t)}(r)$. The behavior of this solution changes as one crosses the Stokes lines. This is the so-called Stokes phenomenon. By incorporating these changes, one can derive the asymptotic behavior of the solution on the whole complex plane.

To start the calculation, we consider the boundary condition of the quasinormal mode at spatial infinity. Assuming that $\text{Re } \omega > 0$, one can analytically continue this boundary condition to the anti-Stokes line labeled a in Fig. 1. With the definition of the WKB solutions in Eq. (16), the boundary condition at a becomes

$$\psi_a = f_1^{(t_1)}, \quad (18)$$

where we have indicated explicitly from which zero the anti-Stokes line emanates. Going to b in the clockwise direction, we cross a Stokes line. Since this Stokes line locates in a region where f_1 dominates, the f_1 part of ψ_a will not change but there will be an additional f_2 part with the coefficient of f_1 in ψ_a (which is 1 here) multiplying $-i$ for crossing the line in the clockwise direction. (If we had crossed the Stokes line in the counterclockwise direction, we would have to multiply by i instead.) Hence, at b ,

$$\psi_b = \psi_a - if_2^{(t_1)} = f_1^{(t_1)} - if_2^{(t_1)}. \quad (19)$$

Next, we have to change the reference point from t_1 to t_2 .

$$f_{1,2}^{(t_2)} = e^{\pm i\gamma_{12}} f_{1,2}^{(t_1)}, \quad (20)$$

where

$$\gamma_{12} = \int_{t_1}^{t_2} d\xi Q(\xi) \equiv \gamma. \quad (21)$$

Now near the zeros, r is small because $\text{Im } \omega \rightarrow -\infty$,

$$Q^2(r) \approx \frac{r^2}{4M^2} \left[\omega^2 - \frac{M^2}{r^4} (\alpha + 4) \right], \quad (22)$$

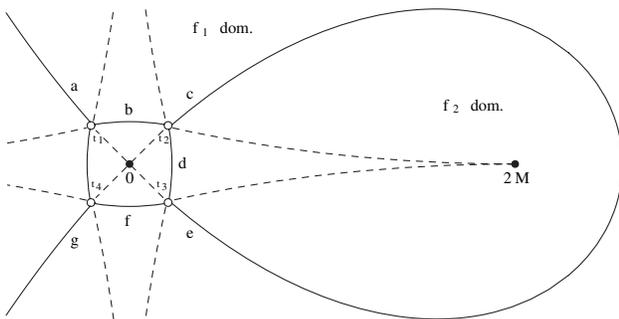


FIG. 1. Stokes structure of the Schwarzschild black hole. Open circles are zeros of $Q(r)$ and filled circles are poles of $Q(r)$. The poles are located at the black hole singularity ($r = 0$) and the event horizon ($r = 2M$). Solid lines are anti-Stokes lines and broken lines are Stokes lines. The regions where f_1 or f_2 dominates are also indicated.

from Eq. (17). Taking $y = \xi^2 \omega / M\sqrt{4 + \alpha}$, we have

$$\begin{aligned} \gamma &= \int_{t_1}^{t_2} d\xi \left(\frac{\omega \xi}{2M} \right) \left[1 - \frac{M^2(\alpha + 4)}{\omega^2 \xi^4} \right]^{1/2} \\ &= \frac{\sqrt{4 + \alpha}}{4} \int_{-1}^1 dy \left(1 - \frac{1}{y^2} \right)^{1/2} = -\frac{\pi}{2} \sqrt{1 + \frac{\alpha}{4}}. \end{aligned} \quad (23)$$

One can also show that $\gamma_{12} = -\gamma_{23} = \gamma_{34} = \gamma$. The value of γ here is crucial in the derivation of the asymptotic quasinormal frequency. They are also listed in Table I.

After changing the reference point to t_2 ,

$$\psi_b = e^{i\gamma} f_1^{(t_2)} - ie^{-i\gamma} f_2^{(t_2)}. \quad (24)$$

Going to c , we cross another anti-Stokes line, so

$$\psi_c = \psi_b - i(e^{i\gamma} f_2^{(t_2)}) = e^{i\gamma} f_1^{(t_2)} - i(e^{i\gamma} + e^{-i\gamma}) f_2^{(t_2)}. \quad (25)$$

Going to d , we cross yet another anti-Stokes line. However, we are in a region where f_2 dominates. Hence,

$$\begin{aligned} \psi_d &= \psi_c - i(-ie^{i\gamma} - ie^{-i\gamma}) f_1^{(t_2)} \\ &= -e^{-i\gamma} f_1^{(t_2)} - i(e^{i\gamma} + e^{-i\gamma}) f_2^{(t_2)}. \end{aligned} \quad (26)$$

Changing the reference point to t_3 , with $\gamma_{23} = -\gamma$,

$$\psi_d = -e^{-2i\gamma} f_1^{(t_3)} - i(1 + e^{2i\gamma}) f_2^{(t_3)}. \quad (27)$$

Going to e ,

$$\begin{aligned} \psi_e &= \psi_d - i(-i - ie^{2i\gamma}) f_1^{(t_3)} \\ &= -(1 + e^{2i\gamma} + e^{-2i\gamma}) f_1^{(t_3)} - i(1 + e^{2i\gamma}) f_2^{(t_3)}. \end{aligned} \quad (28)$$

Going to f and changing the reference point to t_4 , we have

$$\begin{aligned} \psi_f &= \psi_e - i(-1 - e^{2i\gamma} - e^{-2i\gamma}) f_2^{(t_3)} \\ &= -(1 + e^{2i\gamma} + e^{-2i\gamma}) e^{i\gamma} f_1^{(t_4)} + ie^{-3i\gamma} f_2^{(t_4)}. \end{aligned} \quad (29)$$

Going to g ,

$$\begin{aligned} \psi_g &= \psi_f - i(-1 - e^{2i\gamma} - e^{-2i\gamma}) e^{i\gamma} f_2^{(t_4)} \\ &= -e^{i\gamma} (1 + e^{2i\gamma} + e^{-2i\gamma}) f_1^{(t_4)} \\ &\quad + i(e^{3i\gamma} + e^{i\gamma} + e^{-i\gamma} + e^{-3i\gamma}) f_2^{(t_4)}. \end{aligned} \quad (30)$$

Finally we go from g back to a in the counterclockwise direction at infinity completing the trip around the singularity point at $r = 2M$. Since f_1 is dominant in this region, the f_1 part of ψ does not change. However, there will be an additional phase contribution, that is,

$$f_1^{(t_4)} = e^{i\tilde{\gamma}_{41}} f_1^{(t_1)}, \quad (31)$$

with the contour

$$\tilde{\gamma}_{41} + \gamma_{12} + \gamma_{23} + \gamma_{34} = \Gamma \Rightarrow \tilde{\gamma}_{41} = \Gamma - \gamma, \quad (32)$$

where Γ is the closed contour integral around $r = 2M$ (in

the counterclockwise sense),

$$\Gamma = \oint_{r=2M} d\xi Q(\xi) = i4\pi M\omega. \quad (33)$$

With this phase taken into account, the f_1 part of ψ at \bar{a} (back to a after a round trip) is,

$$\begin{aligned} \psi_{\bar{a}} &= -e^{i\gamma}(1 + e^{2i\gamma} + e^{-2i\gamma})e^{i\Gamma}e^{-i\gamma}f_1^{(t_1)} + \dots \\ &= -e^{i\Gamma}(1 + e^{2i\gamma} + e^{-2i\gamma})f_1^{(t_1)} + \dots. \end{aligned} \quad (34)$$

In the monodromy method of Motl and Neitzke [8], the boundary condition at the horizon is translated into the monodromy requirement of the solution around the singular point at $r = 2M$,

$$\psi_{\bar{a}} = e^{-i\Gamma}\psi_a. \quad (35)$$

Considering only the f_1 part of the solution ψ , and using the value of Γ in Eq. (33), we have

$$\begin{aligned} -e^{i\Gamma}(1 + e^{2i\gamma} + e^{-2i\gamma}) &= e^{-i\Gamma} \Rightarrow e^{8\pi M\omega} \\ &= -(1 + 2\cos 2\gamma). \end{aligned} \quad (36)$$

As we can see from Table I, we have $\gamma = 0$ and $-\pi$ for the scalar and the tensor fields, respectively. In both cases, we have

$$e^{8\pi M\omega} = -3 \Rightarrow \omega = \frac{1}{8\pi M} \ln 3 - \frac{i}{4M} \left(n + \frac{1}{2} \right) \quad (37)$$

as $n \rightarrow \infty$. For the Dirac, the Maxwell, and the Rarita-Schwinger fields, we have $\gamma = -\pi/2$,

$$e^{8\pi M\omega} = 1 \Rightarrow \omega = -\frac{i}{4M} n \quad (38)$$

with zero real part as $n \rightarrow \infty$. This result is consistent with [15,21], but in contradiction with that in [14], where the imaginary part of the asymptotic quasinormal frequency is found to be $-in/8M$. In [21], the subleading contribution of $\text{Im } \omega$ is also calculated. Finally, we note that the spacing of the imaginary parts of the asymptotic frequencies is $1/4M$ for all the spin fields.

III. REISSNER-NORDSTRÖM BLACK HOLE

For the charged Reissner-Nordström black hole, the form of the metric is the same as in Eq. (3), but with $\Delta = r^2 - 2Mr + q^2$ where q is the charge of the black hole. Since the pole structures in the complex r -plane of the nonextremal and the extremal cases are different, one cannot take the extremal limit directly from the nonextremal result. We therefore consider the two cases separately in the following subsections.

A. Nonextremal case

The wave equation in this case is still given by Eq. (14), but with

$$R(r) = \frac{r^4}{\Delta^2} \left[\omega^2 - V + \frac{2M}{r^3} - \frac{3(M^2 + q^2)}{r^4} + \frac{6Mq^2}{r^5} - \frac{2q^4}{r^6} \right], \quad (39)$$

and the WKB solutions are as Eq. (16), with

$$\begin{aligned} Q^2(r) &= \frac{r^4}{\Delta^2} \left[\omega^2 - V - \frac{1}{4r^2} + \frac{3M}{r^3} - \frac{(8M^2 + 7q^2)}{2r^4} \right. \\ &\quad \left. + \frac{7Mq^2}{r^5} - \frac{9q^4}{4r^6} \right]. \end{aligned} \quad (40)$$

There are six zeros and three poles for $Q(r)$. The poles are located at $r = 0$ and $r = r_{\pm} = M \pm \sqrt{M^2 - q^2}$, the event horizon and the inner horizon, respectively. The corresponding Stokes structure is indicated in Fig. 2.

We start with the solution at a again.

$$\psi_a = f_1^{(t_1)}. \quad (41)$$

We go from a to b , to c , to d , and to e , crossing four anti-Stokes lines. Using the same procedure as in the last section, we obtain

$$\psi_e = -(1 + e^{2i\gamma} + e^{-2i\gamma})f_1^{(t_3)} - i(1 + e^{2i\gamma})f_2^{(t_3)}, \quad (42)$$

where $\gamma_{12} = -\gamma_{23} = \gamma_{34} = -\gamma_{45} = \gamma_{56} = \gamma$ with

$$\begin{aligned} \gamma &= \int_{t_1}^{t_2} d\xi Q(\xi) = \int_{t_1}^{t_2} d\xi \left(\frac{\xi^2}{q^2} \right) \left[\omega^2 - \frac{(9 + 4\alpha)q^4}{4\xi^6} \right]^{1/2} \\ &= \frac{\pi}{2} \sqrt{1 + \frac{4\alpha}{9}}. \end{aligned} \quad (43)$$

We assume here that the asymptotic behavior of the effec-

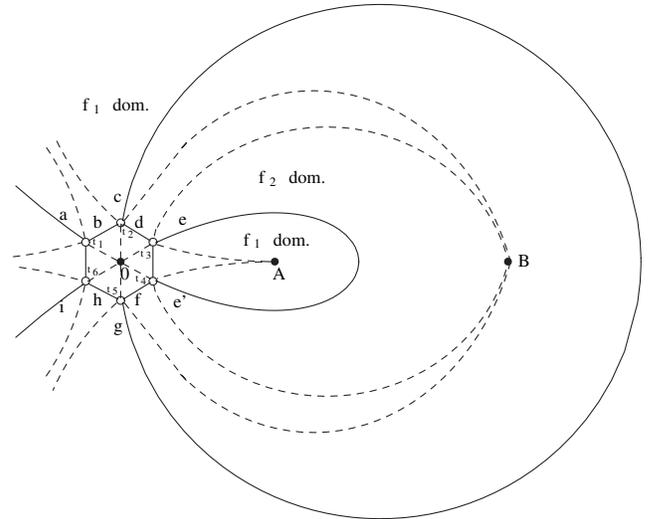


FIG. 2. Stokes structure of the nonextreme Reissner-Nordström black hole. There are six zeros and three poles. One pole is at the black hole singularity ($r = 0$). The other two are at the inner horizon A ($r = r_-$) and at the event horizon B ($r = r_+$).

tive potential is

$$V|_{r \rightarrow 0} \sim \frac{q^4}{r^6} \alpha. \quad (44)$$

The value of α and γ for various spin fields are tabulated in Table II.

To circle the singular point at $r = r_+$, we go along the anti-Stokes line from e to e' . The behavior of ψ does not change, but there will be additional phases. The phase $\tilde{\gamma}_{34}$ satisfies

$$\tilde{\gamma}_{34} + \gamma_{43} = -\Gamma_- \Rightarrow \tilde{\gamma}_{34} = -\Gamma_- + \gamma, \quad (45)$$

where

$$\Gamma_- = \oint_{r=r_-} d\xi Q(\xi) = -i\pi\omega M \frac{(1-\kappa)^2}{\kappa}, \quad (46)$$

with $\kappa = \sqrt{1 - q^2/M^2}$. Since we are considering the non-extremal case, we have $0 < \kappa \leq 1$. Hence, with the contribution of the phase $\tilde{\gamma}_{34}$,

$$\begin{aligned} \psi_{e'} &= -(1 + e^{2i\gamma} + e^{-2i\gamma})e^{i\tilde{\gamma}_{34}}f_1^{(t_4)} - i(1 + e^{2i\gamma})e^{-i\tilde{\gamma}_{34}}f_2^{(t_4)} \\ &= -e^{-i\Gamma_-}e^{i\gamma}(1 + e^{2i\gamma} + e^{-2i\gamma})f_1^{(t_4)} \\ &\quad - ie^{i\Gamma_-}(e^{i\gamma} + e^{-i\gamma})f_2^{(t_4)}. \end{aligned} \quad (47)$$

Going from e' to f , to g , to h , and then to i , we have

$$\begin{aligned} \psi_i &= [-e^{i\Gamma_-}e^{i\gamma}(2 + e^{2i\gamma} + e^{-2i\gamma}) - e^{-i\Gamma_-}e^{i\gamma}(1 + e^{2i\gamma} \\ &\quad + e^{-2i\gamma})]f_1^{(t_6)} + [ie^{i\Gamma_-}(e^{3i\gamma} + 2e^{i\gamma} + 2e^{-i\gamma} + e^{-3i\gamma}) \\ &\quad + ie^{-i\Gamma_-}(e^{i\gamma} + e^{-i\gamma})(1 + e^{2i\gamma} + e^{-2i\gamma})]f_2^{(t_6)}. \end{aligned} \quad (48)$$

To circle the singular point at $r = r_+$, we go back to a at infinity in the counterclockwise direction. The phase contribution satisfies

$$\tilde{\gamma}_{61} + \gamma_{16} = \Gamma_+ + \Gamma_- \Rightarrow \tilde{\gamma}_{61} = \Gamma_+ + \Gamma_- - \gamma, \quad (49)$$

where

$$\Gamma_+ = \oint_{r=r_+} d\xi Q(\xi) = i\pi\omega M \frac{(1+\kappa)^2}{\kappa}. \quad (50)$$

Considering only the f_1 part of the solution,

TABLE II. Some parameters characterizing the asymptotic behaviors of various spin fields for the Reissner-Nordström black hole. α represents the behavior of the effective potential near the black hole singularity as in Eq. (44). γ is the value of the integral in Eq. (43).

spin	α	γ	$-2(1 + \cos 2\gamma)$	$1 + 2\cos 2\gamma + 2\cos 4\gamma$
0	-2	$\pi/6$	-3	1
1/2	0	$\pi/2$	0	1
1	4	$5\pi/6$	-3	1
3/2	-2	$\pi/6$	-3	1
2	4	$5\pi/6$	-3	1

$$\begin{aligned} \psi_{\bar{a}} &= [-e^{i\Gamma_-}e^{i\gamma}(2 + e^{2i\gamma} + e^{-2i\gamma}) - e^{-i\Gamma_-}e^{i\gamma}(1 + e^{2i\gamma} \\ &\quad + e^{-2i\gamma})]e^{i\Gamma_+}e^{i\Gamma_-}e^{-i\gamma}f_1^{(t_1)} + \dots \\ &= -e^{i\Gamma_+}[e^{2i\Gamma_-}(2 + e^{2i\gamma} + e^{-2i\gamma}) \\ &\quad + (1 + e^{2i\gamma} + e^{-2i\gamma})]f_1^{(t_1)} + \dots \end{aligned} \quad (51)$$

The monodromy requirement corresponding to the boundary condition at the event horizon $r = r_+$ is

$$\psi_{\bar{a}} = e^{-i\Gamma_+}\psi_a. \quad (52)$$

Therefore, we have

$$\begin{aligned} -e^{i\Gamma_+}[e^{2i\Gamma_-}(2 + e^{2i\gamma} + e^{-2i\gamma}) + (1 + e^{2i\gamma} + e^{-2i\gamma})] \\ = e^{-i\Gamma_+} \Rightarrow e^{-2i\Gamma_+} = 1 - 2(1 + \cos 2\gamma)(1 + e^{2i\Gamma_-}). \end{aligned} \quad (53)$$

From Table II, we have for the spin 0, the spin 1, and the spin 2 fields, $-2(1 + \cos 2\gamma) = -3$, so

$$\begin{aligned} e^{-2i\Gamma_+} &= -2 - 3e^{2i\Gamma_-} \Rightarrow e^{2\pi\omega M(1+\kappa)^2/\kappa} \\ &= -2 - 3e^{2\pi\omega M(1-\kappa)^2/\kappa}. \end{aligned} \quad (54)$$

Hence, these three fields have the same asymptotic frequency although it cannot be written in a close form as in the Schwarzschild case. Our result agrees with that in Refs. [8,9,13]. For the spin 3/2 Rarita-Schwinger field, the same result is obtained because the corresponding α and γ are identical to that of the integral spin fields. The only field that has a different asymptotic frequency in this case is the Dirac field. For the Dirac field, we have $2(1 + \cos 2\gamma) = 0$ as listed in Table II and

$$e^{-2i\Gamma_+} = 1 \Rightarrow \omega = -in \frac{\kappa}{M(1+\kappa)^2}. \quad (55)$$

The real part of the Dirac asymptotic frequency is zero, and the spacing of the imaginary part varies with q .

B. Extremal case

Suppose we naively take the extremal limit $q \rightarrow M$ or $\kappa \rightarrow 0$ of the result in Eq. (54) in the last subsection, we obtain

$$e^{8\pi\omega M} = -3. \quad (56)$$

for spin 0, 1, 3/2, and 2 fields. This coincides with the result for the scalar and the tensor fields for the Schwarzschild black hole. However, for $s = 1$ and 3/2, it differs from the Schwarzschild result (Eq. (38)). Moreover, for the Dirac case in Eq. (55), the limit $\kappa \rightarrow 0$ is in fact inconsistent. As already pointed out in [13], one cannot take this limit directly to obtain the extremal result. There are only two poles in the extremal case, while in the nonextremal case there are three. The structure of the complex plane is different and a separate analysis has to be carried out. This is what we shall do next.

The functions $R(r)$ and $Q(r)$ in this case can be obtained by taking the extremal limit $q \rightarrow M$ in Eqs. (39) and (40)

$$R(r) = \frac{r^4}{(r-M)^4} \left[\omega^2 - V + \frac{2M}{r^3} - \frac{6M^2}{r^4} + \frac{6M^3}{r^5} - \frac{2M^4}{r^6} \right], \quad (57)$$

and

$$Q^2(r) = \frac{r^4}{(r-M)^4} \left[\omega^2 - V - \frac{1}{4r^2} + \frac{3M}{r^3} - \frac{15M^2}{2r^4} + \frac{7M^3}{r^5} - \frac{9M^4}{4r^6} \right]. \quad (58)$$

There are again six zeros but only two poles for $Q(r)$. At the horizon, $r = M$, we have a double pole, which locates right on one of the Stokes lines. The corresponding Stokes structure is shown in Fig. 3.

Going through the same analysis as before, we start at a ,

$$\psi_a = f_1^{(t_1)}. \quad (59)$$

We then go to b, c, \dots, j , and k in the counterclockwise direction, and we have

$$\begin{aligned} \psi_k = & (e^{5i\gamma} + e^{3i\gamma} + e^{i\gamma} + e^{-i\gamma} + e^{-3i\gamma}) f_1^{(t_6)} \\ & - i(e^{5i\gamma} + e^{3i\gamma} + e^{i\gamma} + e^{-i\gamma} + e^{-3i\gamma} + e^{-5i\gamma}) f_2^{(t_6)}, \end{aligned} \quad (60)$$

where

$$\gamma = \frac{\pi}{2} \sqrt{1 + \frac{4\alpha}{9}}, \quad (61)$$

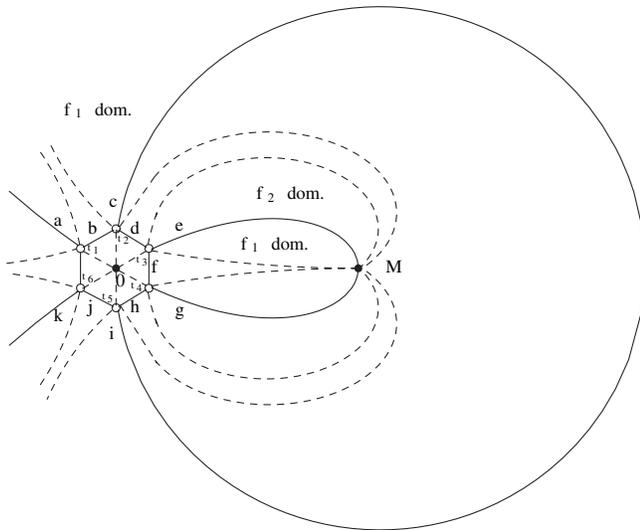


FIG. 3. Stokes structure of the extreme Reissner-Nordström black hole. There are only two poles here. One is at the black hole singularity ($r = 0$) and the other one is at the event horizon ($r = M$).

as given in Eq. (43) in the nonextremal case with the asymptotic behavior of the effective potential

$$V|_{r \rightarrow 0} \sim \frac{M^4}{r^6} \alpha, \quad (62)$$

which is also the same as in the nonextremal case.

Circling the singular point at $r = M$ back to a at infinity as before, we have

$$\tilde{\gamma}_{61} + \gamma_{16} = \Gamma \Rightarrow \tilde{\gamma}_{61} = \Gamma - \gamma, \quad (63)$$

where

$$\Gamma = \oint_{r=M} d\xi Q(\xi) = i4\pi M\omega. \quad (64)$$

Therefore,

$$\begin{aligned} \psi_{\tilde{a}} = & (e^{5i\gamma} + e^{3i\gamma} + e^{i\gamma} + e^{-i\gamma} + e^{-3i\gamma}) e^{i\Gamma} e^{-i\gamma} f_1^{(t_1)} + \dots \\ = & e^{i\Gamma} (e^{4i\gamma} + e^{2i\gamma} + 1 + e^{-2i\gamma} + e^{-4i\gamma}) f_1^{(t_1)} + \dots. \end{aligned} \quad (65)$$

The monodromy condition is again

$$\begin{aligned} e^{i\Gamma} (e^{4i\gamma} + e^{2i\gamma} + 1 + e^{-2i\gamma} + e^{-4i\gamma}) \\ = e^{-i\Gamma} \Rightarrow e^{-2i\Gamma} = 1 + 2\cos 2\gamma + 2\cos 4\gamma. \end{aligned} \quad (66)$$

For all the spin fields, as shown in Table II,

$$1 + 2\cos 2\gamma + 2\cos 4\gamma = 1. \quad (67)$$

Hence, the asymptotic quasinormal frequency in the extremal case for all the spin fields is

$$e^{8\pi M\omega} = 1 \Rightarrow \omega = -in \left(\frac{1}{4M} \right). \quad (68)$$

It is thus curious to see that all the different spin fields have the same asymptotic frequency for the extreme black hole. This result is consistent with that in [9] where the integral spin cases are considered.

IV. CONCLUSIONS AND DISCUSSIONS

We have evaluated the asymptotic quasinormal frequencies for various spin fields in Schwarzschild and Reissner-Nordström black holes, using a combination of the monodromy method of Motl and Neitzke [8] and the WKB formalism of Andersson and Howls [13]. These frequencies are tabulated in Table III. In the Schwarzschild case, the real part of the asymptotic frequency for the spin 0 and the spin 2 fields is $\ln 3$. This value has inspired a lot of interesting in its relation to the black hole area and entropy spectra [1]. However, the real part of the frequency for the spin 1/2, the spin 1, and the spin 3/2 fields is zero. This result casts doubts on the universality of the value of $\ln 3$, even for single-horizon black holes. On the other hand, the imaginary parts of the frequencies all have spacings $1/4M$ or $2\pi T_S$, where T_S is the Hawking temperature of the Schwarzschild black hole. This value is thus universal for all the spin fields [22,23].

Our result for the imaginary part of the Dirac quasinormal frequency agrees with [15,21], but in contradiction with that of [14]. In [14], the imaginary part of the frequency is calculated in two different ways, one analytical and the other numerical. For the analytical calculation, the authors there follow the method of [22,23] in which the imaginary part is derived by identifying the locations of the poles of the scattering amplitude in the Born approximation. This calculation is criticized in [15] where it is shown that the method of [22,23] for the integral spin fields cannot be extended directly to the Dirac case. Hence, the validity of the analytical calculation is in question. As for the numerical analysis, the authors use the continued fraction method of Leaver [24] which converges much slower than the modified method of Nollert [2]. It seems that the method of Nollert cannot be applied to the effective potential of the Dirac field. It is therefore possible that the correct answer has not been reached numerically there. In any case, the conclusion on the spacing of the imaginary part of the frequencies in [14] is not at all reliable.

For the nonextreme Reissner-Nordström black holes, we find that the asymptotic frequency of the spin 3/2 Rarita-Schwinger field is the same as that of the spin 0, the spin 1, and the spin 2 fields which was first evaluated in [7,8]. Since this frequency involves both the mass M and the charge q of the black hole, it cannot be expressed in a close form as that of the Schwarzschild case [13]. Recently, Hod [25] had taken up this problem again. He obtained a value of $\ln 2$ for the real part of the frequency even in the Reissner-Nordström case by considering the quasinormal modes of a charged scalar field. It would be interesting to see if a universal value can be obtained for other spin fields by extending our calculation to charged field cases.

The spin 1/2 Dirac field is special in the Reissner-Nordström case. Its asymptotic quasinormal frequency is different from the other spin fields. The real part is zero, as in the Schwarzschild case. The imaginary part has a spacing of $\kappa/M(1 + \kappa)^2$ or $2\pi T_{\text{RN}}$, where T_{RN} is the Hawking temperature of the Reissner-Nordström black hole. In terms of the Hawking temperature, this spacing has the same form as that for the Schwarzschild black hole.

TABLE III. Quasinormal frequencies of various spin fields for the Schwarzschild and the Reissner-Nordström (RN) black holes (BH). Equation (54) is the equation that determines the frequencies in some Reissner-Nordström cases where the frequencies cannot be written in a close form. $\kappa = \sqrt{1 - q^2/M^2}$.

spin	Schwarzschild BH	RN BH	Extreme RN BH
0	$\frac{1}{8\pi M} \ln 3 - \frac{i}{4M} (n + \frac{1}{2})$	Equation (54)	$-\frac{i}{4M} n$
1/2	$-\frac{i}{4M} n$	$-\frac{i\kappa}{(1+\kappa)^2 M} n$	$-\frac{i}{4M} n$
1	$-\frac{i}{4M} n$	Equation (54)	$-\frac{i}{4M} n$
3/2	$-\frac{i}{4M} n$	Equation (54)	$-\frac{i}{4M} n$
2	$\frac{1}{8\pi M} \ln 3 - \frac{i}{4M} (n + \frac{1}{2})$	Equation (54)	$-\frac{i}{4M} n$

In the extremal case, it is curious to see that all the spin fields have the same asymptotic frequency. Since the extreme Reissner-Nordström black hole is the simplest example of an supersymmetric black hole, one would expect the spin 1, the spin 3/2, and the spin 2 fields to have the same asymptotic frequency [16]. However, it is quite unexpected for the scalar and the Dirac fields to have the same value. The real part of this frequency is zero, while the imaginary part has again a spacing of $1/4M$. This spacing cannot be expressed in terms of the Hawking temperature which is zero in the extremal case, but it is the same as the spacing in the Schwarzschild case. In this respect, it is the spacing of the imaginary part of the asymptotic frequencies that has a universal value of $1/4M$ for all the single-horizon cases of the Schwarzschild and the extreme Reissner-Nordström black holes.

We can see from above that the universality of $\ln 3$ is indeed in question even for the single-horizon black holes. In addition, the relevancy of the asymptotic quasinormal frequency to the microstate description of the black hole entropy is not clear [26] because this frequency depends crucially on the behavior of the effective potential near the black hole singularity as well as that near the event horizon. In spite of this, we still think that the black hole quasinormal spectrum should be important in the understanding of the quantum properties of the black hole [27,28]. The quasinormal modes represent the characteristic oscillations of the black hole. If they are quantized in an appropriate way, which would involve the problem of how to quantize an open system, we should be able to obtain more information on the entropy of the quantum black hole [29].

Finally, it would be desirable to extend our consideration to the case of the Kerr black hole in order to have a further understanding of the universality question of the asymptotic quasinormal frequencies. Up to now, the evaluations of the Kerr asymptotic frequencies are mostly numerical (see, for example, [30–35]). This is because one has to deal with the asymptotic behaviors of the radial equation as well as the angular equation, which involves the spheroidal harmonics. Recently there are a number of studies on the asymptotic behaviors of the spheroidal harmonics [36,37]. Hopefully one would soon be able to carry out a more complete study of the asymptotic quasinormal frequencies in the Kerr black hole case.

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