Charged particles in external fields as physical examples of quasi-exactly-solvable models: A unified treatment

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We present a unified treatment of three cases of quasi-exactly-solvable problems, namely, a charged particle moving in Coulomb and magnetic fields for both the Schrödinger and the Klein-Gordon case, and the relative motion of two charged particles in an external oscillator potential. We show that all these cases are reducible to the same basic equation, which is quasiexactly solvable owing to the existence of a hidden sl_2 algebraic structure. A systematic and unified algebraic solution to the basic equation using the method of factorization is given. Analytical expressions of the energies and the allowed frequencies for the three cases are given in terms of roots of one and the same set of Bethe ansatz equations.

I. INTRODUCTION

It is well known that exact solutions are hard to come by in physics (in fact, in all sciences). Many exactly solvable examples presented in textbooks of physics are only exceptions. More often than not they serve only as paradigms to illustrate the fundamental principles in their respective fields. For real problems, approximation methods are indispensable.

Recently, it was found that for certain quantum-mechanical problems analytical solutions are possible, but only for parts of the energy spectra and for particular values of the fundamental parameters. First it was realized that the problem of two electrons moving in an external oscillator potential belongs to this class of problems [1,2]. Later, it was discovered that the two-dimensional Schrödinger equation of an electron moving in an attractive/repulsive Coulomb field and a homogeneous magnetic field also share similar characteristics [3–5]. More recently, the latter problems were extended to the two-dimensional Klein-Gordon [6] and the Dirac equation [7].

The essential features shared by all these above examples are as follows. The differential (Schrödinger, Klein-Gordon, and Dirac) equations are solved according to the standard procedure. After separating out the asymptotic behaviors of the system, one obtains an equation for the part that can be expanded as a power series of the basic variable. It is at this point that deviation from the standard exactly solvable cases appears: instead of the two-step recursion relations for the coefficients of power series so often encountered in exactly-solvable problems, one gets three-step recursion relations. The complexity of the recursion relations does not allow one to do anything to guarantee normalizability of the eigenfunctions. However, one can impose a sufficient condition for normalizability by terminating the series at a certain order of the power of the variable, i.e., by choosing a polynomial. By doing so one could obtain exact solutions to the original problem, but only for certain energies and for specific values of the parameters of the problem. These parameters are the frequency of the oscillator potential and the external magnetic fields.

It was soon realized [8,7] that the above quantum-mechanical problems are just examples of the so-called quasi-exactly-solvable models, recently discovered by physicists and mathematicians [9–16]. This is a special class of quantum-mechanical problems for which several eigenstates can be found explicitly. The reason for such quasiexact solvability is usually the existence of a hidden Lie-algebraic structure [10–14]. More precisely, a quasi-exactly-solvable Hamiltonian can be reduced to a quadratic combination of the generators of a Lie group with finite-dimensional representations.

In this paper we would like to show that three of the four problems mentioned in the second paragraph, namely, (a) charged particle moving in Coulomb and magnetic fields (the Schrödinger case), (b) charged particle in Coulomb and magnetic fields (the Klein-Gordon case), and (c) relative motion of two charged particles in an external oscillator potential, can be given a unified treatment. We shall show that all these cases are simply variations of the same basic equation [Eq. (10) below], which is quasiexactly solvable owing to the existence of a hidden sl_2 algebraic structure. This algebraic structure was first realized by Turbiner for the case of two electrons in an oscillator potential [8]. We shall give a systematic and unified algebraic solution to the basic equation using the method of factorization presented in [7]. Our method allows one to find the analytic expressions of the energies, and the allowed frequencies once and for all in terms of the roots of a set of Bethe ansatz equations. This is in sharp contrast to the method of solving recursion relations, which must be performed for each and every order of the polynomial part in order to get these expressions.

We will define the three problems in Sec. II. In Sec. III the basic equation is solved by the method of factorization. The Lie-algebraic structure underlying the basic equation is discussed in Sec. IV. Section V then concludes the paper.

II. BRIEF DESCRIPTION OF THE THREE PROBLEMS

In this section we shall give a brief description of the three cases of charged particles moving in external fields, which we will consider in the rest of the paper. Following

the previous works, we adopt the atomic units $\hbar = m = e = 1$ in the cgs system.

A. Electron in Coulomb and magnetic fields: The Schrödinger case

This general case was considered in [3–5,7]. The Hamiltonian of a planar electron in a Coulomb field and a constant magnetic field $\mathbf{B} = B\hat{z}$ ($B > 0$) along the z direction is

$$H = \frac{1}{2}\left(\mathbf{p} + \frac{1}{c}\mathbf{A}\right)^2 - \frac{Z}{r}, \qquad (1)$$

where c is the speed of light, Z (positive or negative) is the charge of the source of the Coulomb field, and the vector potential $\mathbf{A}$ is $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$ in the symmetric gauge.

An ansatz of the eigenfunction in the polar coordinates (r, θ) is

$$\Psi(\mathbf{r}, t) = \frac{u(r)}{\sqrt{r}} \exp(im\theta - iEt), \quad m = 0, \pm 1, \pm 2, \ldots . \tag{2}$$

Here m is the angular momentum quantum number and E is the energy. The radial wave function $u(r)$ satisfies the radial Schrödinger equation

$$\left[\frac{1}{2}\frac{d^2}{dr^2} - \frac{1}{2}\left(m^2 - \frac{1}{4}\right)\frac{1}{r^2} - \frac{1}{2}\omega_L^2 r^2 + \frac{Z}{r} + E - m\omega_L\right]u(r) = 0, \tag{3}$$

where $\omega_L = B/2c$ is the Larmor frequency.

B. Electron in Coulomb and magnetic fields: The Klein-Gordon case

In [6] the above problem is extended to the Klein-Gordon case, assuming the same ansatz of the wave function as in Eq. (2). Now the radial wave function $u(r)$ obeys the following equation:

$$\left[\frac{1}{2}\frac{d^2}{dr^2} - \frac{1}{2}\left(m^2 - \frac{Z^2}{c^2} - \frac{1}{4}\right)\frac{1}{r^2} - \frac{1}{2}\omega_L^2 r^2 + \frac{EZ}{c^2 r} + \frac{E^2}{2c^2} - \frac{c^2}{2}\right.$$
$$\left. - m\omega_L\right]u(r) = 0. \tag{4}$$

But now, as noted in [6], the quantum number m must satisfy the relation

$$m^2 - \frac{Z^2}{c^2} > 0 \tag{5}$$

in order for the solutions to make sense. This relation forbids the existence of the s states ($m = 0$).

C. Relative motion of two electrons in an external oscillator potential

In [2] the author considered the problem of a three-dimensional Schrödinger equation for two electrons (interacting with Coulomb potential) moving in an external harmonic-oscillator potential characterized by frequency ω_{ext}. The Hamiltonian is

$$H = -\frac{1}{2}\nabla_1^2 + \frac{1}{2}\omega_{ext}^2 \mathbf{r}_1^2 - \frac{1}{2}\nabla_2^2 + \frac{1}{2}\omega_{ext}^2 \mathbf{r}_2^2 + \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}. \tag{6}$$

The total wave function is factorizable into three parts that depend, respectively, only on the center of mass, the relative coordinates, and the spins of the electrons. The wave function of the center-of-mass coordinates satisfies the Schrödinger equation of a three-dimensional oscillator, the solution of which is well known. The spin part dictates the parity of the wave function of the relative motion. The Schrödinger equation for the relative motion is

$$\left[-\frac{1}{2}\nabla_r^2 + \frac{1}{2}\omega_r^2 \mathbf{r}^2 + \frac{1}{2r}\right]\phi(\mathbf{r}) = \epsilon' \phi(\mathbf{r}), \tag{7}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $\omega_r = \omega_{ext}/2$, and ϵ' is one-half of the eigenenergy of the relative motion (in the notation of [2]). By assuming an ansatz of the wave function in the spherical coordinates of the form

$$\phi(\mathbf{r}) = \frac{u(r)}{r} Y_{lm}(\hat{\mathbf{r}}), \tag{8}$$

where Y_{lm} are the spherical harmonics, we get from Eq. (7) the following equation

$$\left[\frac{1}{2}\frac{d^2}{dr^2} - \frac{l(l+1)}{2}\frac{1}{r^2} - \frac{1}{2}\omega_r^2 r^2 - \frac{1}{2r} + \epsilon'\right]u(r) = 0. \tag{9}$$

We note here that if we change the sign of the $1/r$ term in the last equation, we get an equation that describes the relative motion in the oscillator potential of an electron and a positron. This case will be included in our discussions.

III. THE BASIC EQUATION AND THE METHOD OF FACTORIZATION

After making some appropriate changes of the parameters, we can recast Eqs. (3), (4), and (9) into the same basic form, namely,

$$\left[\frac{1}{2}\frac{d^2}{dr^2} - \frac{\gamma(\gamma-1)}{2}\frac{1}{r^2} - \frac{1}{2}\omega^2 r^2 + \frac{\beta}{r} + \alpha\right]u(r) = 0. \tag{10}$$

Here β, γ, and ω ($\gamma, \omega > 0$) are real parameters, and α is the eigenvalue of Eq. (10). Explicit expressions of these parameters for the three cases will be given in the next section. That this equation is quasiexactly solvable means that, given

a fixed value of the parameter γ and β (or ω), the equation can be solved exactly only for a particular set of parameter ω (or β) and eigenvalue α.

Now we make the following change of variables: $x \equiv \sqrt{2\omega} r$ and $b \equiv \sqrt{2/\omega}\beta$. Then Eq. (10) becomes

$$\left[\frac{d^2}{dx^2} - \frac{\gamma(\gamma-1)}{x^2} - \frac{x^2}{4} + \frac{b}{x} + \frac{\alpha}{\omega}\right]u(x)=0. \quad (11)$$

The values of α and b in Eq. (11) may be found by means of a method closely resembling the method of factorization in nonrelativistic quantum mechanics [7]. We shall discuss this method briefly below. Let us assume

$$u(x)=x^\gamma \exp(-x^2/4)Q(x), \quad (12)$$

where Q is a polynomial. As mentioned in the Introduction, the assumption that Q be a polynomial is only a sufficient condition for normalizability of the eigenfunction $u(x)$. Substituting Eq. (12) into Eq. (11), we have

$$\left[\frac{d^2}{dx^2} + \left(\frac{2\gamma}{x} - x\right)\frac{d}{dx} + \left(\epsilon + \frac{b}{x}\right)\right]Q(x)=0, \quad (13)$$

where $\epsilon = \alpha/\omega - (\gamma + 1/2)$.

It is seen that the problem of finding the spectrum of Eq. (13) is equivalent to determining the eigenvalues of the operator

$$H = -\frac{d^2}{dx^2} - \left(\frac{2\gamma}{x} - x\right)\frac{d}{dx} - \frac{b}{x}. \quad (14)$$

We want to factorize the operator (14) in the form

$$H = a^\dagger a + \epsilon. \quad (15)$$

The eigenfunctions of the operator H at $\epsilon = 0$ must satisfy the equation

$$a\psi = 0. \quad (16)$$

Suppose polynomial solutions exist for Eq. (13), say $Q = \prod_{k=1}^n (x - x_k)$, where x_k are the zeros of Q and n is the degree of Q (we mention here that the order n in this paper is equal to $(n-1)$ in [2–6] where x^{n-1} is the highest order term in Q). Then the operator a must have the form

$$a = \frac{\partial}{\partial x} - \sum_{k=1}^n \frac{1}{x - x_k}, \quad (17)$$

and the operator $a^\dagger$ has the form

$$a^\dagger = -\frac{\partial}{\partial x} - \frac{2\gamma}{x} + x - \sum_{k=1}^n \frac{1}{x - x_k}. \quad (18)$$

Substituting Eqs. (17) and (18) into Eq. (15) and then comparing the result with Eq. (14), we obtain the following set of equations for the zeros x_k (the so-called Bethe ansatz equations [13]):

$$\frac{2\gamma}{x_k} - x_k - 2\sum_{j \neq k}^n \frac{1}{x_j - x_k} = 0, \quad k=1,\ldots,n \quad (19)$$

as well as the two relations,

$$b = 2\gamma \sum_{k=1}^n x_k^{-1}, \quad \epsilon = n. \quad (20)$$

Summing all the n equations in Eq. (19) enables us to rewrite the first relation in Eq. (20) as

$$b = \sum_{k=1}^n x_k. \quad (21)$$

From the second relation in Eq. (20), one gets

$$\epsilon = n = \alpha/\omega - \left(\gamma + \frac{1}{2}\right). \quad (22)$$

For $n=1,2$ the zeros x_k and the values of the parameter b for which solutions in terms of the polynomial of the corresponding degrees exist can easily be found from Eqs. (19) and (21) in the form

$$n=1, \quad x_1 = \pm\sqrt{2\gamma}, \quad b = \pm\sqrt{2\gamma};$$

$$n=2, \quad x_1 = -x_2 = \sqrt{2\gamma+1}, \quad b=0,$$

$$x_1 = 2\gamma/x_2, \quad x_2 = \pm(1+\sqrt{4\gamma+1})/\sqrt{2},$$

$$b = \pm\sqrt{2(4\gamma+1)}. \quad (23)$$

For general values of n it is difficult to solve for the x_k's from Eq. (19) and one must resort to numerical methods. But some properties of the solutions are known. First, from Eq. (19) we see that if $\{x_k\}$ is a set of solutions to Eq. (19), then so is $\{-x_k\}$. This means, by Eq. (21), that for every possible value of b, there is a corresponding negative value $-b$. Second, as we shall see later in Sec. IV, the number of values of b for a fixed order n is $n+1$.

We now apply the above results to the three cases mentioned previously. The essential step is to solve the Bethe ansatz equations (19) for the roots x_k's for each order n. Then from Eqs. (21) and (22) we obtain the values of the allowed pair of frequency and energy. Here we will give only the general expressions. The reader can easily reproduce the expressions for the two simplest orders (i.e., $n=1$ and 2) given in [2–6] by substituting Eq. (23) into the general expressions.

A. Electron in Coulomb and magnetic fields: The Schrödinger case

In this case, $\gamma = |m| + 1/2$, $\omega = \omega_L$, $\beta = Z = \pm|Z|$, and $\alpha = E - m\omega_L$. The upper (lower) sign in β corresponds to the case of attractive (repulsive) Coulomb interaction. We have

$$\omega_L = \frac{2Z^2}{b^2}, \quad E = \omega_L(n+m+|m|+1). \quad (24)$$

These are the general expressions for the frequency (and hence the magnetic field) and the energy in terms of the values of b. They reproduce the results in [3–5].

B. Electron in Coulomb and magnetic fields: The Klein-Gordon case

For definiteness, we consider positive-energy solutions for the attractive Coulomb potential ($Z > 0$). This is the case considered in [6]. Negative-energy solutions and the case for repulsive Coulomb field can be treated in exactly the same way. In this case, $\gamma = \sqrt{m^2 - Z^2/c^2 + 1/2}$, $\omega = \omega_L$, $\beta = ZE/c^2$, and $\alpha = E^2/2c^2 - c^2/2 - m\omega_L$. In order for the wave function to make sense, γ has to be real. This implies that $m^2 - Z^2/c^2 > 0$, which forbids the existence of the $m = 0$ states (the s states) in the Klein-Gordon case, as noted in [6].

Using $\omega_L = 2\beta^2/b^2$ we get the allowed magnetic field as

$$B = 2c\omega_L = \frac{4Z^2E^2}{b^2c^3} \qquad (25)$$

and from Eq. (22) we obtain the corresponding energy E,

$$E^2 = c^4 \left[1 - \frac{4Z^2}{b^2c^2}(n+1+m+\sqrt{m^2-Z^2/c^2}) \right]^{-1}. \qquad (26)$$

These are the most general expressions for the energy and the frequency.

For negative-energy solutions, the energy is given by the negative roots of Eq. (26). The only difference is that the roots of the Bethe ansatz equations have opposite signs in view of Eq. (21). This only changes the nodal structure of the wave functions. From the expression $\beta = ZE/c^2$, we note the equivalence between the positive- (negative-) energy solutions in the attractive Coulomb case and the negative- (positive-) energy solutions in the repulsive Coulomb case.

Let us mention here that for the cases in Secs. III A and III B, we may consider a dual situation of the original problem: we may consider the magnetic field B (and thus ω_L) as a fixed quantity, and the Bethe ansatz equations instead give the allowed values of the energy and the Coulomb charge Z.

C. Relative motion of two electrons in an external oscillator potential

In this case, $\gamma = l + 1/2$, $\omega = \omega_r$, $\beta = -1/2$, and $\alpha = \epsilon'$. We have the following general solutions:

$$\omega_r = \frac{1}{2b^2}, \quad \epsilon' = \omega_r \left(n + l + \frac{3}{2} \right). \qquad (27)$$

They are also the solutions for the case of an electron and a positron in the oscillator potential ($\beta = +1/2$).

IV. HIDDEN LIE-ALGEBRAIC STRUCTURE OF THE BASIC EQUATION

The basic equation (10), or its equivalent form (13), possesses an underlying Lie-algebraic structure that is responsible for its quasiexact solvability. In fact, Turbiner has identified an sl_2 structure for the case of two charged particles in an oscillator potential [8]. In view of the fact that all the previous cases considered in this paper are related by the same basic equation (10), one expects the same hidden structure to be present in all these cases. This is indeed the case and it is sufficient to show that an sl_2 algebra is in fact the underlying structure possessed by Eqs. (10) or (13). In this section we shall carry out Turbiner's analysis to Eq. (13), with only slight modifications in the parameters to suit the general situation. Only the main ideas are given here, and we refer the reader to [8] for details.

Let us construct three generators in the following manner:

$$J_n^+ = r^2 \frac{d}{dr} - nr,$$

$$J_n^0 = r \frac{d}{dr} - \frac{n}{2},$$

$$J_n^- = \frac{d}{dr}. \qquad (28)$$

These generators realize the sl_2 algebra,

$$[J_n^+, J_n^-] = -2J_n^0, \quad [J_n^0, J_n^\pm] = \pm J_n^\pm \qquad (29)$$

for any value of the parameter n. If n is a non-negative integer, then there exists for the sl_2 algebra a $(n+1)$-dimensional irreducible representation $\mathcal{P}_{n+1}(r) = \langle 1, r, r^2, \dots, r^n \rangle$. From this it is clear that any differential operator formed by taking the polynomial of the generators (28) will have the space $\mathcal{P}_{n+1}$ as the finite-dimensional invariant subspace. This is the main idea underlying the quasi-exactly-solvable operators [8–14].

Now consider the quasi-exactly-solvable operator that is quadratic in the J_n's,

$$T_2 = -J_n^0 J_n^- + 2\omega J_n^+ - \left(\frac{n}{2} + 2\gamma \right) J_n^-. \qquad (30)$$

This operator belongs to the class VIII according to the classification given in [10]. In terms of r, T_2 becomes

$$T_2 = -r \frac{d^2}{dr^2} + 2(\omega r^2 - \gamma) \frac{d}{dr} - 2\omega nr. \qquad (31)$$

Let us now consider the eigenvalue problem

$$T_2 Q(r) = 2\beta(n)Q(r). \qquad (32)$$

This eigenvalue problem possesses $n+1$ eigenvalues $\beta(n)$, and the corresponding eigenfunctions are in the form of a polynomial of the nth power, while other eigenfunctions are nonpolynomial, which in general cannot be found in closed

analytic form [8]. Let us first substitute the form (31) into Eq. (32), then divide the resulting equation by $2\omega r$ and change the variable r to $x \equiv \sqrt{2\omega} r$. This leads to the following equation

$$\left[\frac{d^2}{dx^2} + \left(\frac{2\gamma}{x} - x \right) \frac{d}{dx} + \left\{ n + \sqrt{\frac{2}{\omega}} \beta(n) \frac{1}{x} \right\} \right] Q(x) = 0. \tag{33}$$

This is exactly Eq. (13), provided that $\epsilon = n$ and $b = \sqrt{2/\omega} \beta(n)$. This means that Eq. (13) is quasiexactly solvable if $\epsilon = n$, which is exactly our relation in Eq. (22), and that there are only $n+1$ allowed values of $b = \sqrt{2/\omega} \beta$ in Eq. (13) [cf. Eq. (23)].

Translating back to the original three cases considered in this paper, these results imply the following. In Sec. III C, β is a fixed parameter ($\beta = \pm 1/2$), hence the finite number ($= n+1$) of the values of b implies the same number of the allowed frequency ω_{ext} of the external oscillator potential and the corresponding energy. This is the case found in [2] and presented here again from a new light. For the cases in Secs. III A ($\beta = Z$) and III B ($\beta = ZE/c^2$), the above results mean that, at a fixed order n, there are exactly $n+1$ allowed values of the pair of energy and magnetic field for a fixed Coulomb charge, or of the pair of energy and Coulomb charge for a fixed magnetic field.

Furthermore, it has been shown [8] that there exist $[(n+1)/2]$ positive eigenvalues and the same number of negative eigenvalues of b (here $[a]$ represents the integral part of a). In the general situation considered in this paper, positive (negative) values of b correspond to the attractive (repulsive)

Coulomb field for positive-energy solutions. For negative-energy solutions, the sign of b is reversed for the two kinds of Coulomb field. Hence, our unified treatment together with the Lie-algebraic analysis of these cases give a very simple explanation as to why the number of the positive-energy levels for a fixed order n considered in [2–6] are all equal to $[(n+1)/2]$.

V. CONCLUSIONS

In this paper we have presented a unified treatment of three cases of quasi-exactly-solvable problems, namely, a charged particle moving in Coulomb and magnetic fields for both the Schrödinger and the Klein-Gordon case, and the relative motion of two charged particles in an external oscillator potential. We show that all these cases are reducible to the same basic equation [Eq. (10)], which is quasiexactly solvable owing to the existence of a hidden sl_2 algebraic structure. A systematic and unified algebraic solution to the basic equation using the method of factorization is given. Our method allows one to express the analytic expressions of the energies and the allowed frequencies once and for all in terms of the roots of a set of Bethe ansatz equations. Our treatment also reveals that the eigenenergies and the allowed frequencies in these cases are all given by the roots of the same set of Bethe ansatz equations.

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