Quantum metastability in a class of moving potentials

Chung-Chieh Lee and Choon-Lin Ho
Department of Physics, Tamkang University, Tamsui 25137, Taiwan
(Received 15 August 2001; published 16 January 2002)

In this paper we consider quantum metastability in a class of moving potentials introduced by Berry and Klein. This class of potential has height and width scaled in a specific way so that it can be transformed into a stationary one. While deriving the nondecay probability of the system, we demonstrate that the appropriate technique to use is the less well known method of scattering states. This method is illustrated through two examples, namely, a moving δ-function potential and a moving barrier potential. For expanding potentials, one finds that a small but finite nondecay probability persists at large times. Generalization to scaling potentials of arbitrary shape is briefly outlined.

DOI: 10.1103/PhysRevA.65.022111 PACS number(s): 03.65.Xp, 03.65.Nk

I. INTRODUCTION

An interesting issue in cosmology is the evolution of metastable states in the early universe in the original version and its variants in the inflationary models [1,2]. In these models inflation of the early universe is governed by a Higgs field trapped in a metastable state. Inflation ends when the metastable state decays to the true ground state of the universe. During inflation the universe expands exponentially. It is thus obvious that the metastable state of the Higgs field is trapped in a rapidly varying potential. The problem is therefore a truly time-dependent one. However, owing to the inherent difficulties of the problem, more often than not one considers the decay of the Higgs field in a quasistationary approximation, in which the decay is studied by assuming a static potential [3]. Certainly this approximation is hard to justify, but for the present one has to be content with it. Ultimately one hopes to be able to tackle the nonstationary case. To this end, it is desirable to gain some insights first by studying metastability in time-dependent potentials in simple quantum-mechanical models.

Time-dependent potentials can be broadly divided into three classes. Potentials in the first class are of time-dependent strength. When the strength is small, the Schrödinger equation can be solved by time-dependent perturbation theory. Almost all pedagogical examples belong to this type. When the strength of the potential is not small, other methods of solution must be sought. For example, solutions of time-dependent harmonic oscillator [4] and time-dependent linear [5] potentials can be obtained by the method of invariants. We note that the interesting phenomenon of quantum tunneling induced by an externally driven field has also been examined experimentally and theoretically [6–8]. The second class of potentials involves time-dependent boundaries. Unlike the first class, this class of potentials has attracted much less attention, and almost all previous work in this area concerned only the simplest of all cases, namely, an infinite potential well with a moving wall [9,10]. The last class is a combination of the previous two classes.

We believe that the barrier potential in an inflationary universe is nonstationary; not only the barrier height but also the barrier width should be changing as time elapses. However, it will be extremely difficult to study metastability in such a time-dependent potential in full generality. Thus it would be helpful if the quantum tunneling effect could be studied in any class of moving potential, special though it may be, as a step toward understand the decay of a nonstationary metastable system.

In this paper we consider quantum metastability in a class of scaling potentials that allows one to apply techniques used in the corresponding problem with stationary potentials. This class of potentials was introduced by Berry and Klein [11]. These potentials have heights and widths scaled in a specific way so that one can transform the potential into a stationary one.

The organization of the paper is as follows. In Sec. II we give a general discussion of the solutions of the Schrödinger equation with the scaling form of the time-dependent potential introduced in [11]. It is argued that the most suitable technique for studying quantum metastability in such a potential is the less well known method of scattering states. Two simple examples of such metastable systems, a moving δ-function potential and a moving square barrier, are investigated in Secs. III and IV, respectively. A generalization to an arbitrary barrier is briefly discussed in Sec. V. Section VI concludes the paper.

II. SCHRÖDINGER EQUATION WITH A SCALING POTENTIAL

We consider the problem of quantum metastability of a particle of mass m trapped in a moving potential $V(x,t)$ which has the scaling form proposed by Berry and Klein [11], namely, $V(x,t)=\overline{V}(x/L(t))/L^2(t)$, where $L(t)$ is a time-dependent scaling factor. The Schrödinger equation is

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t}=\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{L^2(t)}\overline{V}\left(\frac{x}{L(t)}\right)\right]\Psi(x,t). \quad (1)$$

So far solution of Eq. (1) is restricted mostly to the special case in which $\overline{V}$ has the functional form of an infinite potential well, i.e., $V(x,t)$ is an infinite well with a moving wall [9,10]. In this case, the scaling factor $L^2(t)$ in front of $\overline{V}$ is immaterial.

We assume $\overline{V}$ to have the generic shape of a potential well that is impenetrable to the left and has a finite barrier to the right, much like that usually employed in the discussion of α decay. The scaling factor $L(t)$ is assumed to be a linear function of time:

$$L(t)=L_0+vt, \quad v=\text{const.} \tag{2}$$

Of course, for $v<0$, the problem is meaningful only for time duration $0<t<L_0/|v|$. Equation (1) cannot be solved by separating the time and spatial coordinates. However, for the scaling form of $V(x,t)$ in Eq. (1) and the linear form of $L(t)$, separation of variables can be achieved through a series of transformations introduced in [12,11] (see also [9,10]). One first transforms the coordinate frame into a rescaled frame with a rescaled coordinate $\overline{x}$ defined by

$$\overline{x}(t)\equiv\frac{x}{L(t)}. \tag{3}$$

In this frame the Schrödinger equation becomes

$$i\hbar\frac{\partial}{\partial t}\Psi(\overline{x},t)=\left[-\frac{\hbar^2}{2mL^2}\frac{\partial^2}{\partial x^2}+i\hbar\frac{v}{L}\overline{x}\frac{\partial}{\partial\overline{x}}+\overline{V}(\overline{x})\right]\Psi(\overline{x},t). \tag{4}$$

Equation (4) can be further simplified by the transformation

$$\Psi(\overline{x},t)=\frac{1}{\sqrt{L(t)}}e^{(im/2\hbar)Lv\overline{x}^2}\Phi(\overline{x},t), \tag{5}$$

and the introduction of a new time variable τ:

$$\tau=\int_0^t\frac{ds}{L^2(s)}=\frac{t}{L_0L(t)}. \tag{6}$$

After substituting Eqs. (5) and (6) into Eq. (4), one obtains the equation

$$i\hbar\frac{\partial}{\partial\tau}\Phi(\overline{x},\tau)=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\overline{x}^2}\Phi(\overline{x},\tau)+\overline{V}(\overline{x})\Phi(\overline{x},\tau), \tag{7}$$

which resembles the Schrödinger equation with a stationary potential. Equation (7) can be solved by separation of variables:

$$\Phi(\overline{x},\tau)=\Phi(\overline{x})e^{-(i/\hbar)\overline{E}\tau}, \tag{8}$$

where $\Phi(\overline{x})$ satisfies the eigenvalue equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\overline{V}(\overline{x})\right]\Phi_k(\overline{x})=\overline{E}_k\Phi_k(\overline{x}). \tag{9}$$

Once Eq. (9) is solved exactly in the rescaled frame, the exact wave function in the original frame is then given by

$$\Psi_k(x,t)=\frac{1}{\sqrt{L(t)}}e^{(im/2\hbar)(v/L)x^2}e^{-(i/\hbar)(1/L_0L)\overline{E}_kt}\Phi_k\left(\frac{x}{L}\right). \tag{10}$$

The set of solutions (10) is complete and orthonormal:

$$\langle\Psi_k(x,t)|\Psi_l(x,t)\rangle=\langle\Phi_k(\overline{x})|\Phi_l(\overline{x})\rangle=\delta_{kl}, \tag{11}$$

so using this set of solutions we can find a solution satisfying any initial condition. Furthermore, if an initial state $\Psi(x,0)$ is expressible in the basis $\{\Psi_k\}$ as

$$\Psi(x,0)=\sum_k c_k\Psi_k(x,0), \quad c_k=\langle\Psi_k(x,0)|\Psi(x,0)\rangle, \tag{12}$$

then at a later time t the state is

$$\Psi(x,t)=\sum_k c_k\Psi_k(x,t). \tag{13}$$

We have now succeeded in transforming the original time-dependent Schrödinger equation into a time-independent one. The problem of calculating the decay probability of a particle confined in $V(x,t)$ at time t is reduced to the corresponding problem with a static potential $\overline{V}(\overline{x})$. Hence techniques used in the time-independent potential for calculating decay rate can be borrowed.

However, there are some subtleties. Naively, one is tempted to employ the best-known method, namely, the complex eigenvalue method, proposed by Gamow in his studies of α decay [13]. In this approach an ''outgoing wave boundary condition'' is imposed on the solutions of the Schrödinger equation for the particle trapped in the well. That means incoming plane wave solutions outside the potential well are discarded right from the beginning. This procedure naturally leads to an eigenvalue problem with complex energy eigenvalues. One then relates the imaginary parts of the energy to the decay rate. While the complex eigenvalue method is straightforward and physically reasonable, it suffers from some conceptual difficulties [14]. For example, how can energy eigenvalues be complex as we are dealing with a Hermitian Hamiltonian? Also, the eigenfunctions are not normalizable, a difficulty directly related to the eigenvalues being complex. Furthermore, the particle trapped in the well cannot be in an eigenstate of the system in the first place, since such states are not completely confined at $t=0$.

Apart from the difficulties mentioned above, the complex eigenvalue method cannot be employed in the present case for other reasons. First, the problem we are interested in is an intrinsically time-dependent one, with a nonconservative Hamiltonian. Hence energy eigenvalues and eigenstates lose their meanings altogether [$\overline{E}$ in Eq. (9) is not an energy eigenvalue]. Second, the ''outgoing wave boundary condition,'' essential to Gamow's method, cannot be imposed in our case. The reason is as follows. As discussed before, in order to fix a moving potential we need to transform our problem to a corresponding static one in a rescaled frame. But in this frame the meaning of an incoming or outgoing

plane wave is rather obscure. In fact, it can be checked that an outgoing plane wave in the original x-t (rescaled $\bar{x}$-τ) frame contains both ''incoming'' and ''outgoing'' components in the rescaled $\bar{x}$-τ (original x-t) frame.

The instanton method is another technique commonly used in the calculation of the decay rate of a metastable state [15]. This semiclassical method amounts essentially to finding the imaginary part of the ground state energy of the system. Since it is based on the idea of eigenvalues of the Hamiltonian, it is therefore not suitable here.

One more method that can be useful in tackling the problem is the complex scaling method [16]. The general idea of this method is to consider a complex rotated Hamiltonian obtained from the usual Hamiltonian by making the position and momentum complex. This resolves the paradox that the eigenenergies are complex, since now the complex rotated Hamiltonian is no longer Hermitian. A pleasant merit of the method is that the eigenfunctions associated with these complex resonant eigenvalues are square integrable, and thus various approximation methods developed for bound states can be applied to the scattering processes. However, this method, while elegant in many respects, is not useful in our present case, since it also requires the idea of eigenenergy.

Since these common methods fail to suit our purpose, we have to look for alternatives. Fortunately, a different method exists, namely, the scattering state method (or virtual level method, as Fermi called it) [17]. This method is much less well known and seldom used in the literature [18]. However, it is conceptually the most satisfying of all the methods. In this method, one first constructs the initial confining state, which is not viewed as an eigenstate, but rather as a linear superposition of scattering states with real energies, and follows its evolution in time. In the course of this evolution, no energy will become complex. Unlike the Gamow states, the scattering states contain both incoming and outgoing components in the region into which the particle escapes. It is this feature of the method that makes it most suitable in the present problem. The method is easily adapted to Eq. (7) by taking the scattering states as the states (8) with real values of $\bar{E}$.

In the next two sections, the scattering state method is applied to two simple examples of the class of scaling potentials. As the scattering state method is not so well known in the literature, we think it appropriate to give some details in order to make this work self-contained. The procedures given in [14] are slightly adapted to our needs.

III. MOVING δ-FUNCTION POTENTIAL

The first example is a uniformly moving δ-function potential

$$V(x,t)=\begin{cases}\infty, & x\leqslant 0,\\ \dfrac{\bar{V}_0}{L(t)}\,\delta(x-a(t))=\dfrac{\bar{V}_0}{L^2(t)}\,\delta\left(\dfrac{x}{L(t)}-\bar{a}\right), & x>0,\end{cases} \tag{14}$$

where $a(t)=\bar{a}L(t)>0$ gives the location of the δ-function potential. This class of potentials was defined in the last section and corresponds in the rescaled frame to $\bar{V}(\bar{x})=\infty$ for $\bar{x}<0$ and $\bar{V}(\bar{x})=\bar{V}_0\delta(\bar{x}-\bar{a})$. One has $\Phi(\bar{x})=0$ in the region $\bar{x}<0$. For $\bar{x}>0$, Eq. (9) is

$$-\frac{\hbar^2}{2m}\frac{d^2\Phi(\bar{x})}{d\bar{x}^2}+\bar{V}_0\delta(\bar{x}-\bar{a})\Phi(\bar{x})=\bar{E}\Phi(\bar{x}). \tag{15}$$

Its general solutions are

$$\Phi(\bar{x})=\begin{cases}\sin(\bar{k}\bar{x}), & 0<\bar{x}<\bar{a},\\ C\cos(\bar{k}\bar{x}+\theta), & \bar{a}<\bar{x},\end{cases} \tag{16}$$

where $\bar{k}=\sqrt{2m\bar{E}}/\hbar$, C is a real constant, and θ is a phase angle. Note that the wave function is real, and includes an incoming wave component in the region $\bar{x}>\bar{a}$. This ensures that $\bar{k}$, and hence $\bar{E}$, is always real. The wave function and its first derivative satisfy the following boundary conditions at $\bar{x}=\bar{a}$:

$$\Phi(\bar{x}=\bar{a}^+)=\Phi(\bar{x}=\bar{a}^-) \tag{17}$$

and

$$\left.\frac{d\Phi(\bar{x})}{d\bar{x}}\right|_{\bar{x}=\bar{a}^+}-\left.\frac{d\Phi(\bar{x})}{d\bar{x}}\right|_{\bar{x}=\bar{a}^-}=\frac{2m}{\hbar^2}\bar{V}_0\Phi(\bar{x}=\bar{a}). \tag{18}$$

From these relations the coefficient C can be determined as a function of $\bar{k}$:

$$C^2(\bar{k})=\sin^2(\bar{k}\bar{a})+\left(\cos(\bar{k}\bar{a})+\frac{2m\bar{V}_0}{\hbar^2\bar{k}\bar{a}}\sin(\bar{k}\bar{a})\right)^2. \tag{19}$$

Physically, the value of $C^2(\bar{k})$ can be interpreted as the ratio of the probability of finding particles in the region $\bar{x}>\bar{a}$ to the probability of finding them within the confined region $0<\bar{x}<\bar{a}$ for a particular $\bar{k}$. The general shape of $C^2(\bar{k})$ is shown in Fig. 1, from which we can assert that the particle can be trapped within the confined region only when C^2 assumes one of its minima, which occur only in the neighborhood of some specific values of $\bar{k}$. In these regions the values of $C^2(\bar{k})$ are extremely small. From Eq. (19) it is obvious that these minima will be centered around $\bar{k}_n=n\pi/\bar{a}$ $(n=1,2,\ldots)$ [i.e., $\sin(\bar{k}_n\bar{a})=0$] as long as $\bar{V}_0$ is large enough so that $2m\bar{V}_0/\hbar^2\gg n\pi$.

For large $\bar{V}_0$ approximate analytic expressions can be obtained and compared with the corresponding results in the time-independent case [19]. According to the scattering state method, one constructs confining states in the potential well by taking a suitable superposition of the scattering states with $\bar{k}$ in the neighborhood of $\bar{k}_n$. To this end, let us first expand C^2 about $\bar{E}_n=\hbar^2\bar{k}_n^2/2m$ (we revert to the variable $\bar{E}$ below):

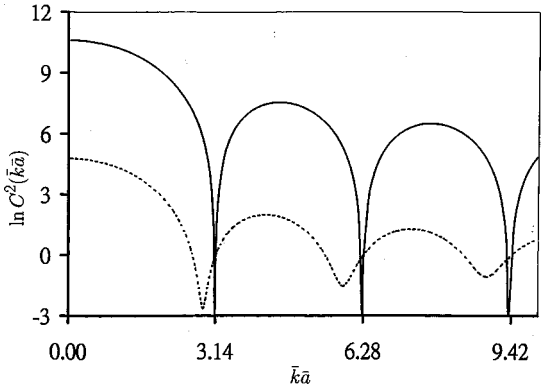

FIG. 1. The shape of $\ln C^2(\bar{k}\bar{a})$ in Eq. (19) as a function of $\bar{k}\bar{a}$ for $2m\bar{V}_0/\hbar^2=10$ (dotted line) and 200 (solid line). The minima of $\ln C^2(\bar{k}\bar{a})$ will be centered around $\bar{k}_n\bar{a}=n\pi$ $(n=1,2,\ldots)$ for large values of $2m\bar{V}_0/\hbar^2$.

$$C^2(\bar{E})\approx\left(\frac{m\bar{a}}{\hbar^2\bar{k}_n}\right)^2\left[1+\left(\frac{2m\bar{V}_0}{\hbar^2\bar{k}_n}\right)^2\right](\bar{E}-\bar{E}_n+\delta)^2 + \left[1+\left(\frac{2m\bar{V}_0}{\hbar^2\bar{k}_n}\right)^2\right]^{-1} \tag{20}$$

$$\equiv G^2(\Delta+\delta)^2+F^2, \tag{21}$$

where $\Delta=\bar{E}-\bar{E}_n$ and the constants

$$\delta=\frac{2\bar{V}_0}{\bar{a}}\left[1+\left(\frac{2m\bar{V}_0}{\hbar^2\bar{k}_n}\right)^2\right]^{-1}, \tag{22}$$

$$G^2=\left(\frac{m\bar{a}}{\hbar^2\bar{k}_n}\right)^2\left[1+\left(\frac{2m\bar{V}_0}{\hbar^2\bar{k}_n}\right)^2\right], \tag{23}$$

$$F^2=\left[1+\left(\frac{2m\bar{V}_0}{\hbar^2\bar{k}_n}\right)^2\right]^{-1}. \tag{24}$$

The scattering states with $\bar{E}$ in the neighborhood of $\bar{E}_n$ can then be written as

$$\psi_\Delta(\bar{x})=\begin{cases}\sqrt{\frac{2}{R}}\frac{1}{\sqrt{G^2(\Delta+\delta)^2+F^2}}\sin(\bar{k}\bar{x}), & 0<\bar{x}<\bar{a},\\ \sqrt{\frac{2}{R}}\cos(\bar{k}\bar{x}+\theta), & \bar{a}<\bar{x}.\end{cases} \tag{25}$$

This system is quantized in the interval $[0,R]$, where $R\gg\bar{a}$, which at the end of the calculation will be set to infinity. With these scattering states, an initial state is constructed that is completely confined within the well by taking a linear combination of the scattering states with different Δ but the same value of $\bar{E}_n$,

$$\Phi(\bar{x},\tau=0)=\sum_\Delta c_\Delta\psi_\Delta(\bar{x})=\begin{cases}\phi_n(\bar{x}), & \bar{x}<\bar{a},\\ 0, & \bar{x}>\bar{a}.\end{cases} \tag{26}$$

The coefficient c_Δ can be calculated from orthogonality of the states $\psi_\Delta(\bar{x})$,

$$c_\Delta=\int_0^R d\bar{x}\psi_\Delta(\bar{x})\Phi(\bar{x},0) = \sqrt{\frac{2}{R}}\frac{1}{\sqrt{G^2(\Delta+\delta)^2+F^2}}\int_0^{\bar{a}}d\bar{x}\sin(\bar{k}\bar{x})\phi_n(\bar{x}). \tag{27}$$

Choosing

$$\phi_n(\bar{x})\approx\sqrt{\frac{2}{\bar{a}}}\sin\left(\frac{n\pi\bar{x}}{\bar{a}}\right), \quad n=1,2,3,\ldots, \tag{28}$$

we get

$$c_\Delta\approx\frac{1}{\sqrt{R\bar{a}}}\frac{2}{\sqrt{G^2(\Delta+\delta)^2+F^2}}\int_0^{\bar{a}}d\bar{x}\sin(\bar{k}\bar{x})\sin\left(\frac{n\pi\bar{x}}{\bar{a}}\right) \tag{29}$$

$$\approx\sqrt{\frac{\bar{a}}{R}}\frac{1}{\sqrt{G^2(\Delta+\delta)^2+F^2}}. \tag{30}$$

The initial state is then given by

$$\Phi(\bar{x},\tau=0)\approx\sqrt{\frac{\bar{a}}{R}}\sum_\Delta\frac{1}{\sqrt{G^2(\Delta+\delta)^2+F^2}}\psi_\Delta(\bar{x}). \tag{31}$$

From Eqs. (10), (12), and (13), the solution at a later time τ is

$$\Phi(\bar{x},\tau)\approx\sqrt{\frac{\bar{a}}{R}}\sum_\Delta\frac{1}{\sqrt{G^2(\Delta+\delta)^2+F^2}}\psi_\Delta(\bar{x})e^{-(i/\hbar)(\bar{E}_n+\Delta)\tau}. \tag{32}$$

As the system is quantized in the interval $[0,R]$, we have $\bar{k}R=n'\pi/2$, where n' is a very large integer $(n'\gg n)$. After replacing the sum by an integral

$$\sum_\Delta\rightarrow\int d\Delta\frac{R}{\pi\hbar}\sqrt{\frac{2m}{\bar{E}_n}}, \tag{33}$$

Eq. (32) becomes

$$\Phi(\bar{x},\tau)\approx\frac{R}{\pi\hbar}\sqrt{\frac{2m}{\bar{E}_n}}\sqrt{\frac{\bar{a}}{R}}\int_{-\infty}^{\infty}d\Delta \times\frac{1}{\sqrt{G^2(\Delta+\delta)^2+F^2}}\psi_\Delta(\bar{x})e^{-(i/\hbar)(\bar{E}_n+\Delta)\tau}. \tag{34}$$

Substituting $\psi_\Delta(\bar{x})$ from Eq. (25), we obtain the approximate wave function of the state confined in the well $(0<\bar{x}<\bar{a})$ as

$$\Phi(\bar{x},\tau)\approx\frac{2}{\pi\hbar}\sqrt{\frac{m\bar{a}}{\bar{E}_n}}\sin(\bar{k}_n\bar{x})e^{-(i/\hbar)\bar{E}_n\tau}\times\int_{-\infty}^{\infty}d\Delta\frac{e^{-(i/\hbar)\Delta\tau}}{G^2(\Delta+\delta)^2+F^2}=\frac{2}{\hbar}\sqrt{\frac{m\bar{a}}{\bar{E}_n}}\frac{\sin(\bar{k}_n\bar{x})}{|FG|}e^{-(i/\hbar)\bar{E}_n\tau}e^{-(1/\hbar)|F/G|\tau}. \tag{35}$$

For metastable systems an important quantity is the nondecay probability $P(t)$ that the particle is still in the well at time t if it is initially confined in the well at $t=0$ $[P(t=0)=1]$. In our case $P(t)$ is defined as

$$P(t)=\frac{\int_0^{a(t)}|\Psi(x,t)|^2dx}{\int_0^{a(0)}|\Psi(x,0)|^2dx}=\frac{\int_0^{\bar{a}}|\Phi(\bar{x},\tau)|^2d\bar{x}}{\int_0^{\bar{a}}|\Phi(\bar{x},0)|^2d\bar{x}}. \tag{36}$$

From Eq. (35) we find

$$P(t)\sim\exp[-\gamma_n(t)], \tag{37}$$

where

$$\gamma_n(t)=2\left|\frac{F}{G}\right|\frac{t}{L_0L(t)} \tag{38}$$

$$=2\left(\frac{\hbar^2\bar{k}_n}{m\bar{a}}\right)\left[1+\left(\frac{2m\bar{V}_0}{\hbar^2\bar{k}_n}\right)^2\right]^{-1}\frac{t}{L_0L(t)}. \tag{39}$$

We have used Eqs. (6), (23), and (24) to obtain the result (39). We note that for an expanding potential $(v>0)$

$$\gamma_n(t)\to2\left|\frac{F}{G}\right|\frac{1}{L_0v} \tag{40}$$

as $t\to\infty$. Unlike the stationary case $(v=0)$, there is a small but finite probability that the particle does not tunnel out of the well. This result is reasonable, since as the barrier moves away from $x=0$ it leaves more room for the particle to stay within the well.

For $\bar{V}_0$ much larger than the characteristic value of $\bar{k}_n$ of the escaping particles, $\gamma(t)$ becomes

$$\gamma_n(t)\to\left[\frac{(\hbar^2\bar{k}_n)^3}{2m^3\bar{a}\bar{V}_0^2}\right]\frac{t}{L_0L(t)}=\left[\frac{\hbar^6(n\pi)^3}{2m^3\bar{a}^4\bar{V}_0^2}\right]\frac{t}{L_0L(t)}, \tag{41}$$

where $\bar{k}_n=n\pi/\bar{a}$ has been substituted. It is proper to compare Eq. (41) with the corresponding result in the stationary case $(v=0)$. In the limit $v\to0$, we have $L(t)\to L_0$, $\bar{a}\to a/L_0$, $\bar{E}_n\to E_nL_0^2$, and $\bar{k}_n\to k_nL_0$, where E_n is the corresponding energy in the static frame, and $k_n=\sqrt{2mE_n}/\hbar$. In this limit $\gamma(v=0)$ is directly proportional to the time t. We can therefore define a decay rate by $\Gamma_n\equiv\gamma_n(v=0)/t$, which in this case is

$$\Gamma_n=\frac{\hbar^6k_n^3}{2m^3a(\bar{V}_0/L_0)^2}=\frac{\hbar^6(n\pi)^3}{2m^3a^4(\bar{V}_0/L_0)^2}. \tag{42}$$

Equation (42) is consistent with the result obtained by the complex eigenvalue method in [19] for a static δ-function potential located at $x=a$ with strength $\bar{V}_0/L_0$.

IV. MOVING SQUARE BARRIER POTENTIAL

For the next example, consider a moving barrier potential

$$V(x,t)=\begin{cases}\infty, & x\leqslant0,\\ \dfrac{1}{L^2(t)}\bar{V}_0, & a(t)<x<b(t),\\ 0, & x>b(t),\end{cases} \tag{43}$$

with $a(t)=\bar{a}L(t)$ and $b(t)=\bar{b}L(t)$ ($\bar{a}$ and $\bar{b}$ are two positive constants). When the problem is transformed to the rescaled frame, it is equivalent to solving Eq. (9) with a stationary potential

$$\bar{V}(\bar{x})=\begin{cases}\infty, & \bar{x}\leqslant0,\\ \bar{V}_0, & \bar{a}<\bar{x}<\bar{b},\\ 0, & \bar{x}>\bar{b}.\end{cases} \tag{44}$$

The general solutions are

$$\Phi(\bar{x})=\begin{cases}\sin(\bar{k}\bar{x}), & 0<\bar{x}<\bar{a},\quad \bar{k}=\sqrt{2m\bar{E}/\hbar^2},\\ Ae^{\bar{k}'\bar{x}}+Be^{-\bar{k}'\bar{x}}, & \bar{a}<\bar{x}<\bar{b},\quad \bar{k}'=\sqrt{2m(\bar{V}_0-\bar{E})/\hbar^2},\\ C\cos(\bar{k}\bar{x}+\theta), & \bar{a}<\bar{x}.\end{cases} \tag{45}$$

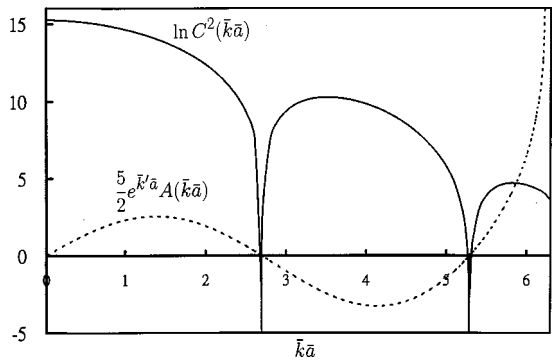

FIG. 2. The shapes of $\ln C^2(\overline{ka})$ (solid line) in Eq. (48) and $5e^{\overline{k'}\overline{a}}A(\overline{ka})/2$ (dotted line) in Eq. (46) as functions of $\overline{ka}$ for $\overline{b}=2\overline{a}$ and $2m\overline{V}_0\overline{a}^2/\hbar^2=40$ showing that the minima of $\ln C^2(\overline{ka})$ occur only in a finite number of neighborhoods of $\overline{k}_n$ ($n=1,2,\ldots$) such that $A(\overline{k}_n\overline{a})=0$.

Here A, B, and C are real constants, and θ is a phase angle. As before, one can set the solutions real in the whole region to ensure that $\overline{k}$ is always real. The solutions and their derivatives need to be continuous at the boundaries $\overline{x}=\overline{a}$ and $\overline{b}$. These boundary conditions determine the values of the coefficients A, B, and C as functions of $\overline{k}$:

$$A(\overline{k})=\frac{1}{2}e^{-\overline{k'}\overline{a}}\left[\sin(\overline{ka})+\frac{\overline{k}}{\overline{k'}}\cos(\overline{ka})\right], \tag{46}$$

$$B(\overline{k})=\frac{1}{2}e^{\overline{k'}\overline{a}}\left[\sin(\overline{ka})-\frac{\overline{k}}{\overline{k'}}\cos(\overline{ka})\right], \tag{47}$$

$$C^2(\overline{k})=\left(1+\frac{\overline{k'}^2}{\overline{k}^2}\right)e^{2\overline{k'}\overline{b}}A^2+2\left(1-\frac{\overline{k'}^2}{\overline{k}^2}\right)AB+\left(1+\frac{\overline{k'}^2}{\overline{k}^2}\right)e^{-2\overline{k'}\overline{b}}B^2. \tag{48}$$

The general shapes of $A(\overline{k})$ and $C^2(\overline{k})$ are shown in Fig. 2. Note that metastable states of the system will occur only in a finite number of neighborhoods of $\overline{k}_n$ ($n=1,2,\ldots$) such that $A(\overline{k}_n)=0$. The roots $\overline{k}_n$ satisfy

$$\sin(\overline{k}_n\overline{a})+\frac{\overline{k}_n}{\overline{k}'_n}\cos(\overline{k}_n\overline{a})=0. \tag{49}$$

Equation (48) implies that $C^2(\overline{k})$ is minimal at $\overline{k}_n$. For a given $\overline{V}_0$, the number of roots $\overline{k}_n$ is restricted by the condition that $\overline{k}'_n$ in Eq. (45),

$$\overline{k}'_n=\sqrt{\frac{2m\overline{V}_0}{\hbar^2}-\overline{k}_n^2}, \tag{50}$$

must be real. Hence the possible values of $\overline{k}_n$ can lie only in the interval $(0,\sqrt{2m\overline{V}_0/\hbar^2})$. For instance, there are only two roots for the parameters assumed in Fig. 2.

We now expand the coefficients $A(\overline{k})$ and $B(\overline{k})$ about $\overline{E}_n$ $(=\hbar^2\overline{k}_n^2/2m)$:

$$A(\overline{E})\approx\left[\frac{dA}{d\overline{E}}\right]_{\overline{E}=\overline{E}_n}(\overline{E}-\overline{E}_n), \tag{51}$$

$$B(\overline{E})\approx B(\overline{E}_n). \tag{52}$$

Inserting Eqs. (51) and (52) into Eq. (48), after some tedious calculations we find that $C^2(\overline{E})$ behaves in the neighborhood of $\overline{E}_n$ as

$$C^2(\overline{E})=G^2(\Delta+\delta)^2+F^2, \tag{53}$$

where $\Delta=\overline{E}-\overline{E}_n$ as in the previous example, and the constants in the present case are

$$G^2=\frac{1}{4}\left(\frac{m\overline{a}}{\hbar^2\overline{k}_n}\right)^2\left(1+\frac{\overline{k'}_n^2}{\overline{k}_n^2}\right)\left[\cos(\overline{k}_n\overline{a})-\frac{\overline{k}_n}{\overline{k}'_n}\sin(\overline{k}_n\overline{a})\right]^2\times e^{2\overline{k}'_n(\overline{b}-\overline{a})}, \tag{54}$$

$$F^2=\left(1+\frac{\overline{k}_n^2}{\overline{k'}_n^2}\right)^{-1}\left[\sin(\overline{k}_n\overline{a})-\frac{\overline{k}_n}{\overline{k}'_n}\cos(\overline{k}_n\overline{a})\right]^2 e^{-2\overline{k}'_n(\overline{b}-\overline{a})}, \tag{55}$$

and

$$\delta=\left(\frac{\hbar^2\overline{k}_n}{m\overline{a}}\right)\left(\frac{\overline{k}_n^2-\overline{k'}_n^2}{\overline{k}_n^2+\overline{k'}_n^2}\right)\times\left(\frac{\overline{k}'_n\sin(\overline{k}_n\overline{a})-\overline{k}_n\cos(\overline{k}_n\overline{a})}{\overline{k}'_n\cos(\overline{k}_n\overline{a})-\overline{k}_n\sin(\overline{k}_n\overline{a})}\right)e^{-2\overline{k}'_n(\overline{b}-\overline{a})}. \tag{56}$$

From the relation (53), the scattering states relevant to this metastable system can be constructed by following exactly the same procedures as in the previous section. The nondecay probability $P(t)$ of finding the particle within the confined region $(0<\overline{x}<\overline{a})$ at time t is again of the form $P(t)\sim\exp(-\gamma_n(t))$, where $\gamma_n(t)$ is now given by

$$\gamma_n(t)=2\left|\frac{F}{G}\right|\frac{t}{L_0L(t)}=\frac{8\hbar^2\overline{k}_n^3}{m\overline{a}}\left(\frac{\overline{k}'_n}{\overline{k}_n^2+\overline{k'}_n^2}\right)^2 e^{-2\overline{k}'_n(\overline{b}-\overline{a})}\frac{t}{L_0L(t)}, \tag{57}$$

where we have used Eqs. (54), (55), and (49). When the barrier height is much larger than the characteristic "energy" of the escaping particles ($\overline{V}_0\gg\overline{E}_n$), which is equivalent to the relation $\overline{k}'_n\gg\overline{k}_n$, $\gamma_n(t)$ becomes

$$\gamma_n(t)\to\frac{8\hbar^2}{m\bar{a}}\frac{\bar{k}_n^3}{\bar{k}'^2_n}e^{-2\bar{k}'_n(\bar{b}-\bar{a})}\frac{t}{L_0L(t)}. \tag{58}$$

As in the previous example, for positive $v>0$ (the expanding case) one finds a small but finite probability that the particle does not tunnel out of the well at large time. In this case not only does the barrier leave more room for the particle to stay within the well as it moves away from $x=0$, but its width also becomes thicker, thus making tunneling difficult.

In order to transform the result (58) to the stationary one, we simply apply the same substitutions as given at the end of the last section, with the addition of $\bar{b}\to b/L_0$ and $\bar{k}'_n\to k'_nL_0$, where $k'_n=\sqrt{2m[(\bar{V}_0/L_0^2)-E_n]}/\hbar$. Once again $\gamma_n(v=0)$ is directly proportional to the time t, in which case a decay rate can be defined: $\Gamma_n\equiv\gamma_n(v=0)/t$. For the present example,

$$\Gamma_n(v=0)=\frac{8\hbar^2}{ma}\frac{k_n^3}{k'^2_n}e^{-2k'_n(b-a)}, \tag{59}$$

which is the same as the result obtained by the complex eigenvalue method for a square barrier with width $(b-a)$ and height $\bar{V}_0/L_0^2$ [20].

V. GENERAL SCALING POTENTIALS

We have calculated the nondecay probabilities of two nonstationary metastable systems explicitly. The potential barriers in the rescaled frame considered in these systems assumed the form of a δ function and a square barrier. These calculations can be immediately generalized to barriers with more general shapes. Without giving further examples, we only discuss briefly the close connection between the nondecay probability $P(t)$ of a particle in a metastable scaling potential $V(x,t)=\bar{V}(x/L(t))/L^2(t)$ and the decay rate Γ of the same particle if it were instead confined in a static potential well $V(x)=\bar{V}(x)$.

From the previous examples, we know that the calculation of $P(t)$ is reduced to the corresponding computation in a static potential $\bar{V}(\bar{x})$ in the rescaled frame. Now this last task would be exactly the same as that carried out in the potential $V(x)=\bar{V}(x)$ in ordinary coordinates. The only difference, as seen from the previous two examples, is that all ordinary parameters, such as E, k, k', t, a, etc., are replaced by the corresponding rescaled ones, i.e., $\bar{E}$, $\bar{k}$, $\bar{k}'$, τ, $\bar{a}$, etc. Application of the scattering state method to the general α decay type of potential $V(x)$ in normal coordinates was given in [14], and can be carried over directly. Following [14] the important step is to determine the discrete values E_n (or equivalently k_n) that minimize the amplitude C of the wave function in the region outside the well. Consider a confining state constructed with E centered around a specific E_n. Minimization of C then gives the two functions $F(E_n)$ and $G(E_n)$ (other parameters in F and G are not indicated). The nondecay probability in $V(x)$ is then given by $\exp(-\Gamma_nt)$, where the decay rate Γ_n is

$$\Gamma_n(E_n)=2\left|\frac{F(E_n)}{G(E_n)}\right|. \tag{60}$$

Suppose all these computations have been done in ordinary coordinates. Then one can immediately write down the expression of the nondecay probability $P(t)\sim\exp[-\gamma_n(t)]$ for the scaling potential $V(x,t)$ as

$$\gamma_n(t)=2\left|\frac{F(\bar{E}_n)}{G(\bar{E}_n)}\right|\frac{t}{L_0L(t)}=\Gamma_n(\bar{E}_n)\frac{t}{L_0L(t)}. \tag{61}$$

Here the functional form of the decay rate Γ_n is taken over directly, but with all the parameters replaced by the corresponding rescaled ones. Equation (61) gives the connection between the nondecay probability in $V(x,t)=\bar{V}(x/L(t))/L^2(t)$ and the decay rate in $V(x)=\bar{V}(x)$. Finally, in the nonmoving limit $v=0$, $V(x,t)$ becomes $V(x)=\bar{V}(x/L_0)/L_0^2$. Setting $v=0$ in Eq. (61) then gives the decay rate in this potential, $\Gamma_n(E_n)/L_0^2$, as seen in the previous cases.

VI. CONCLUSION

The problem of quantum metastability in a class of moving potentials introduced by Berry and Klein is considered. The potential in this class has its height and width scaled in a specific way so that it can be transformed into a stationary one. In deriving the nondecay probability of the system, we employed a method that is less well known but conceptually more satisfactory, namely, the method of scattering states. Nondecay probabilities in a moving δ-function potential and a moving square barrier potential were derived, and a connection between the nondecay probability in a general scaling potential and the decay rate in a related static potential was established.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Council of Republic of China through Grant No. NSC 90-2112-M-032-005. We thank Dr. K. Asokan for a careful reading of the manuscript.

[1] A. H. Guth, Phys. Rev. D **23**, 347 (1981).
[2] D. La and P. J. Steinhardt, Phys. Rev. Lett. **62**, 376 (1989).
[3] E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, New York, 1990).
[4] H. R. Lewis, Jr. and W. B. Riesenfeld, J. Math. Phys. **10**, 1458 (1969).
[5] I. Guedes, Phys. Rev. A **63**, 034 102 (2001).
[6] B. J. Keay *et al.*, Phys. Rev. Lett. **75**, 4098 (1995).
[7] M. Büttiker and R. Landauer, Phys. Rev. Lett. **49**, 1739 (1982).
[8] M. Wagner, Phys. Rev. Lett. **76**, 4010 (1996).

[9] A. J. Makowski and S. T. Dembiński, Phys. Lett. A **154**, 217 (1991).
[10] V. V. Dodonov, A. B. Klimov, and D. E. Nikonov, J. Math. Phys. **34**, 3391 (1993).
[11] M. V. Berry and G. Klein, J. Phys. A **17**, 1805 (1984).
[12] A. Munier, J. R. Burgan, M. Feix, and E. Fijalkow, J. Math. Phys. **22**, 1219 (1981).
[13] G. Gamow, Z. Phys. **51**, 204 (1928).
[14] B. R. Holstein, Am. J. Phys. **64**, 1061 (1996).
[15] S. Coleman, Phys. Rev. D **15**, 2929 (1977); C. Callen and S. Coleman, *ibid.* **16**, 1762 (1977).
[16] B. R. Junker, Adv. At. Mol. Phys. **18**, 207 (1982); W. P. Reinhardt, Annu. Rev. Phys. Chem. **33**, 223 (1982).
[17] E. Fermi, *Nuclear Physics* (University of Chicago Press, Chicago, 1950).
[18] For an excellent review of this method as applied to the theory of α decay, see, e.g., Ref. [14].
[19] H. M. Nussenzveig, *Causality and Dispersion Relations* (Academic Press, New York, 1972), Chap. 4.
[20] A. Bohm, M. Gadella, and G. B. Mainland, Am. J. Phys. **57**, 1103 (1989).