Planar Dirac electron in Coulomb and magnetic fields

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The Dirac equation for an electron in two spatial dimensions in the Coulomb and homogeneous magnetic fields is discussed. This is connected to the problem of the two-dimensional hydrogenlike atom in the presence of an external magnetic field. For weak magnetic fields, the approximate energy values are obtained by a semiclassical method. In the case with strong magnetic fields, we present the exact recursion relations that determine the coefficients of the series expansion of wave functions, the possible energies, and the magnetic fields. It is found that analytic solutions are possible for a denumerably infinite set of magnetic field strengths. This system thus furnishes an example of the so-called quasiexactly solvable models. A distinctive feature in the Dirac case is that, depending on the strength of the Coulomb field, not all total angular momentum quantum numbers allow exact solutions with wave functions in reasonable polynomial forms. Solutions in the nonrelativistic limit with both attractive and repulsive Coulomb fields are briefly discussed by means of the method of factorization.

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I. INTRODUCTION

Planar nonrelativistic electron systems in a uniform magnetic field are fundamental quantum systems which have provided insights into many novel phenomena, such as the quantum Hall effect and the theory of anyons, particles obeying fractional statistics [1,2]. Planar electron systems with energy spectrum described by the Dirac Hamiltonian have also been studied as field-theoretical models for the quantum Hall effect and anyon theory [3]. Related to these field-theoretical models are the recent interesting studies regarding the instability of the naive vacuum and spontaneous magnetization in $(2+1)$-dimensional quantum electrodynamics, which is induced by a bare Chern-Simons term [4]. In view of these developments, it is essential to have a better understanding of the properties of planar Dirac particles in the presence of external electromagnetic fields.

In Ref. [5] we studied exact solutions of planar Dirac equation in the presence of a strong Coulomb field, and the stability of the Dirac vacuum in a regulated Coulomb field. Quite recently, interesting studies on the quantum spectrum of a two-dimensional hydrogen atom in a homogenous magnetic field appeared [6,7]. As is well known, hydrogen atom in a homogeneous magnetic field has attracted great interest in recent years because of its classical chaotic behavior and its rich quantum structures. The main result found in Refs. [6,7] is that, unlike the three-dimensional case, the two-dimensional Schrödinger equation [6] and the Klein-Gordon equation [7] can be solved analytically for a denumerably infinite set of magnetic field strengths. The solutions cannot be expressed in terms of special functions (see also Ref. [8]).

In this paper we discuss the motion of Dirac electron in two spatial dimensions in the Coulomb and homogeneous magnetic fields, and try to obtain exact solutions of a particular form. As in the case of the two-dimensional Schrödinger and the Klein-Gordon equation, by imposing a sufficient condition that guarantees normalizability of the wave functions [see the paragraph after Eq. (35)], we can obtain the exact energy levels for a denumerably infinite set of magnetic fields. In the Dirac case, however, not all values of the total angular momentum j allow exact solutions with the form of wave functions we assumed here. Solutions for the nonrelativistic limit of the Dirac equation in $2+1$ dimensions are briefly discussed by means of the method of factorization.

We emphasize that in this paper, by assuming an ansatz which guarantees normalizability of the wave function, only parts of the energy spectrum of the system are solved exactly. In particular, we do not obtain energy levels with magnitude below the mass value, which include the most interesting ground state solution. This is the same as in the Schrödinger and the Klein-Gordon case. All these three cases can therefore be considered as examples of the newly discovered quasiexactly solvable models [9]. In $3+1$ dimensions, no analytic solutions, even for parts of the spectrum, are possible so far.

II. MOTION OF DIRAC ELECTRON IN THE COULOMB AND MAGNETIC FIELDS

To describe an electron by the Dirac equation in $2+1$ dimensions we need only three anticommuting γ^μ matrices. Hence, the Dirac algebra

$$\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu},\quad g^{\mu\nu}=\mathrm{diag}(1,-1,-1) \tag{1}$$

may be represented in terms of the Pauli matrices as $\gamma^0=\sigma_3$, $\gamma^k=i\sigma_k$, or equivalently, the matrices $(\alpha_1,\alpha_2)=\gamma^0(\gamma^1,\gamma^2)=(-\sigma_2,\sigma_1)$ and $\beta=\gamma^0$ [3]. Then the Dirac equation for an electron minimally coupled to an external electromagnetic field has the form (we set $c=\hbar=1$)

$$(i\partial_t-H_D)\Psi(t,\mathbf{r})=0, \tag{2}$$

where

$$H_D=\alpha\mathbf{P}+\beta m-eA^0\equiv\sigma_1P_2-\sigma_2P_1+\sigma_3m-eA^0 \tag{3}$$

is the Dirac Hamiltonian, $P_k=-i\partial_k+eA_k$ is the operator of generalized momentum of the electron, A_μ the vector potential of the external electromagnetic field, m the rest mass of the electron, and $-e$ ($e>0$) is its electric charge. The Dirac wave function

$$\Psi(t,\mathbf{r})=\begin{pmatrix}\psi_1(t,\mathbf{r})\\ \psi_2(t,\mathbf{r})\end{pmatrix} \tag{4}$$

is a two-component function (i.e., a two spinor). Here $\psi_1(t,\mathbf{r})$ and $\psi_2(t,\mathbf{r})$ are the ''large'' and ''small'' components of the wave functions.

We shall solve for both positive and negative energy solutions of the Dirac equations (2) and (3) in an external Coulomb field and a constant homogeneous magnetic field $B>0$ along the z direction:

$$A^0(r)=Ze/r \quad (e>0),\ \ A_x=-By/2,\ \ A_y=Bx/2. \tag{5}$$

We assume the wave functions to have the form

$$\Psi(t,\mathbf{x})=\frac{1}{\sqrt{2\pi}}\exp(-iEt)\psi_l(r,\varphi), \tag{6}$$

where E is the energy of the electron, and

$$\psi_l(r,\varphi)=\begin{pmatrix}f(r)e^{il\varphi}\\ g(r)e^{i(l+1)\varphi}\end{pmatrix} \tag{7}$$

with integral number l. The function $\psi_l(r,\varphi)$ is an eigenfunction of the conserved total angular momentum $J_z=L_z+S_z=-i\partial/\partial\varphi+\sigma_3/2$ with eigenvalue $j=l+1/2$. One can of course consider wave functions which are eigenfunctions of J_z with eigenvalues $l-1/2$. These functions are of the forms of Eq. (6) with ψ_l given by

$$\psi_l(r,\varphi)=\begin{pmatrix}f(r)e^{i(l-1)\varphi}\\ g(r)e^{il\varphi}\end{pmatrix}. \tag{8}$$

However ansatz (8) is equivalent to ansatz (7) if one makes the change $l\to l-1$. It should be remembered that l is not a good quantum number. This is evident from the fact that the two components of ψ_l depend on the integer l in an asymmetric way. Only the eigenvalues j of the conserved total angular momentum J_z are physically meaningful. For definiteness, in the rest of this paper, all statements and conclusions, whenever angular momentum number l is mentioned, are made with reference to ansatz (6) and (7).

Substituting Eqs. (6) and (7) in Eq. (2), and taking into account the equations

$$P_x\pm iP_y=-ie^{\pm i\varphi}\left[\frac{\partial}{\partial r}\pm\left(\frac{i}{r}\frac{\partial}{\partial\varphi}-\frac{eBr}{2}\right)\right], \tag{9}$$

we obtain

$$\frac{df}{dr}-\left(\frac{l}{r}+\frac{eBr}{2}\right)f+\left(E+m+\frac{Z\alpha}{r}\right)g=0, \tag{10}$$

$$\frac{dg}{dr}+\left(\frac{l+1}{r}+\frac{eBr}{2}\right)g-\left(E-m+\frac{Z\alpha}{r}\right)f=0,$$

where $\alpha\equiv e^2=1/137$ is the fine structure constant. If we let

$$F(r)=\sqrt{r}f(r),\ \ G(r)=\sqrt{r}g(r), \tag{11}$$

Eq. (10) becomes

$$\frac{dF}{dr}-\left(\frac{l+\frac{1}{2}}{r}+\frac{eBr}{2}\right)F+\left(E+m+\frac{Z\alpha}{r}\right)G=0, \tag{12}$$

$$\frac{dG}{dr}+\left(\frac{l+\frac{1}{2}}{r}+\frac{eBr}{2}\right)G-\left(E-m+\frac{Z\alpha}{r}\right)F=0. \tag{13}$$

By eliminating G in Eq. (12) and F in Eq. (13), one can obtain the decoupled second order differential equations for F and G. At large distances, these equations have the asymptotic forms (neglecting r^{-2} terms):

$$\frac{d^2F}{dr^2}+\left[E^2-m^2-eB(l+1)+\frac{2EZ\alpha}{r}-\frac{1}{4}(eBr)^2\right]F=0, \tag{14}$$

$$\frac{d^2G}{dr^2}+\left[E^2-m^2-eBl+\frac{2EZ\alpha}{r}-\frac{1}{4}(eBr)^2\right]G=0. \tag{15}$$

The last term in these two equations, which is proportional to r^2, may be viewed as the ''effective confining potential.''

The exact solutions and the energy eigenvalues with $0<E<m$ corresponding to stationary states of the Dirac equation (10) with $B=0$ were found in Ref. [5]. The electron energy spectrum in the Coulomb field has the form

$$E=m\left[1+\frac{(Z\alpha)^2}{[n_r+\sqrt{(l+1/2)^2-(Z\alpha)^2}]^2}\right]^{-1/2}, \tag{16}$$

where the values of the quantum number n_r are $n_r=0,1,2,\ldots$, if $l\geqslant 0$, and $n_r=1,2,3,\ldots$, if $l<0$. It is seen that

$$E_0=m\sqrt{1-(2Z\alpha)^2} \tag{17}$$

for $l=n_r=0$, and E_0 becomes zero at $Z\alpha=1/2$, whereas in three spatial dimensions E_0 equals zero at $Z\alpha=1$. Thus, in two spatial dimensions the expression for the electron ground state energy in the Coulomb field of a point charge Ze no longer has a physical meaning at $Z\alpha=1/2$. It is worth noting that the corresponding solution of the Dirac equation oscillates near the point $r\to 0$.

For weak magnetic field the wave functions and energy levels with $E<m$ can be found from Eqs. (12) and (13) in the semiclassical approximation. We look for solutions of this system in the standard form

$$F(r)=A(r)\exp[iS(r)],\ \ G(r)=B(r)\exp[iS(r)]. \tag{18}$$

Here $A(r)$ and $B(r)$ are slowly varying functions. Substituting Eq. (18) into Eqs. (12) and (13), we arrive at an ordinary differential equation for $S(r)$ in the form

$$\left(\frac{dS}{dr}\right)^2 \equiv Q = E^2 - m^2 - eB(l+1/2) + \frac{2EZ\alpha}{r} + \frac{(Z\alpha)^2-(l+1/2)^2}{r^2} - \frac{(eBr)^2}{4}. \qquad (19)$$

The energy levels with $E<m$ are defined by the formula

$$\int_{r_{\min}}^{r_{\max}} \sqrt{Q}dr = \pi\left(-\sqrt{(l+1/2)^2-(Z\alpha)^2} + \frac{EZ\alpha}{\sqrt{|m^2+eB(l+1/2)-E^2|}}\right), \qquad (20)$$

where $r_{\max}$ and $r_{\min}$ ($r_{\max}>r_{\min}$) are roots of equation $Q=0$. In obtaining Eq. (20), the term $(eBr)^2$ in Q has been dropped. If we require the energy spectrum to reduce to Eq. (16) when $B=0$, we must equate the right-hand side of Eq. (20) to πn_r. As a result we obtain (for $l\neq 0$)

$$E=\left[m+\frac{eB}{2m}\left(l+\frac{1}{2}\right)\right]\left[1+\frac{(Z\alpha)^2}{[n_r+\sqrt{(l+1/2)^2-(Z\alpha)^2}]^2}\right]^{-1/2}. \qquad (21)$$

In the nonrelativistic approximation the energy spectrum takes the form

$$E_{\text{non}} = -\frac{(Z\alpha)^2 m}{2(n_r+|l+1/2|)^2} + \frac{eB}{2m}\left(l+\frac{1}{2}\right). \qquad (22)$$

Semiclassical motion of electron in the magnetic and Coulomb fields can be characterized by means of the so-called "magnetic length" $l_B=\sqrt{1/eB}$ and the Bohr radius $a_{\text{B}}=1/Z\alpha m$ of a hydrogenlike atom of charge Ze. When the magnetic field is weak so that $l_B \gg a_{\text{B}}$, or equivalently, $B \ll B_{cr} \equiv (Z\alpha)^2 m^2/e$, the energy spectrum is simply the spectrum of a hydrogenlike atom perturbed by a weak magnetic field. We obtain the Zeeman splitting of atomic spectrum depending linearly upon the magnetic field strength and the "magnetic quantum number" $l+1/2$.

In strong magnetic field the asymptotic solutions of $F(r)$ and $G(r)$ have the forms $\exp(-ar^2/2)$ with $a=eB/2$ at large r, and r^γ with

$$\gamma=\sqrt{(l+1/2)^2-(Z\alpha)^2} \qquad (23)$$

for small r. One must have $Z\alpha<1/2$, otherwise the wave function will oscillate as $r\to 0$ when $l=0$ and $l=-1$. In this paper we shall look for solutions of $F(r)$ and $G(r)$ which can be expressed as a product of the asymptotic solutions (for small and large r) and a series in the form

$$F(r)=r^\gamma \exp(-ar^2/2)\sum_{n=0} \alpha_n r^n, \qquad (24)$$

$$G(r)=r^\gamma \exp(-ar^2/2)\sum_{n=0} \beta_n r^n, \qquad (25)$$

with $\alpha_0\neq 0$, $\beta_0\neq 0$. Substituting Eqs. (24) and (25) into Eqs. (12) and (13), we obtain

$$\left[\gamma-\left(l+\frac{1}{2}\right)\right]\alpha_0+Z\alpha\beta_0=0, \qquad (26)$$

$$\left[(\gamma+1)-\left(l+\frac{1}{2}\right)\right]\alpha_1+Z\alpha\beta_1+(E+m)\beta_0=0, \qquad (27)$$

$$\left[(n+\gamma)-\left(l+\frac{1}{2}\right)\right]\alpha_n+Z\alpha\beta_n+(E+m)\beta_{n-1}-2a\alpha_{n-2}=0 \quad (n\geq 2) \qquad (28)$$

from Eq. (12), and

$$\left(\gamma+l+\frac{1}{2}\right)\beta_0-Z\alpha\alpha_0=0, \qquad (29)$$

$$\left(n+\gamma+l+\frac{1}{2}\right)\beta_n-Z\alpha\alpha_n-(E-m)\alpha_{n-1}=0 \quad (n\geq 1) \qquad (30)$$

from Eq. (13).

Equations (26) and (29) allow us to express β_0 in terms of α_0 in two forms:

$$\beta_0=\frac{Z\alpha}{\gamma+l+\frac{1}{2}}\alpha_0 \qquad (31)$$

$$=-\frac{\gamma-l-\frac{1}{2}}{Z\alpha}\alpha_0, \qquad (32)$$

which are equivalent in view of the fact that $\gamma=\sqrt{(l+1/2)^2-(Z\alpha)^2}$. Solving Eqs. (27) and (30) with $n=1$ gives

$$\alpha_1=-\frac{(\gamma+l+\frac{1}{2})(E-m)+(\gamma+l+\frac{3}{2})(E+m)}{(2\gamma+1)(\gamma+l+\frac{1}{2})}Z\alpha\alpha_0, \qquad (33)$$

$$\beta_1=\frac{2(\gamma+l)E-m}{(2\gamma+1)}\alpha_0. \qquad (34)$$

From Eq. (30) one sees that β_n ($n\geq 1$) are obtainable from α_n and α_{n-1}. To determine the recursion relations for the α_n, we simply eliminate β_n and β_{n-1} in Eq. (28) by means of Eq. (30). This leads to (for $n\geq 2$)

$$\left(n+\gamma+l-\frac{1}{2}\right)(n^2+2n\gamma)\alpha_n+Z\alpha\left[\left(n+\gamma+l-\frac{1}{2}\right)(E-m)+\left(n+\gamma+l+\frac{1}{2}\right)(E+m)\right]\alpha_{n-1}+\left(n+\gamma+l+\frac{1}{2}\right)\times\left[E^2-m^2-2a\left(n+\gamma+l-\frac{1}{2}\right)\right]\alpha_{n-2}=0. \quad (35)$$

Following Ref. [6], we impose the sufficient condition that the series parts of $F(r)$ and $G(r)$ should terminate appropriately in order to guarantee normalizability of the eigenfunctions. It follows from Eq. (35) that the solution of $F(r)$ becomes a polynomial of degree $(n-1)$ if the series given by Eq. (35) terminates at a certain n when $\alpha_n=\alpha_{n+1}=0$, and $\alpha_m=0$ $(m\geqslant n+2)$ follow from Eq. (35). Then from Eq. (30) we have $\beta_{n+1}=\beta_{n+2}=\cdots=0$. Thus in general the polynomial part of the function $G(r)$ is of one degree higher than that of F. Now suppose we have calculated α_n in terms of α_0 $(\alpha_0\neq 0)$ from Eqs. (33) and (35) in the form

$$\alpha_n=K(l,n,E,a,Z)\alpha_0. \quad (36)$$

Then two conditions that ensure $\alpha_n=0$ and $\alpha_{n+1}=0$ are

$$K(l,n,E,a,Z)=0 \quad (37)$$

and

$$E^2-m^2=2a\left(n+\gamma+l+\frac{1}{2}\right),\ \ n=1,2,\ldots\ . \quad (38)$$

Since the right hand side of Eq. (38) is always non-negative (for $l\geqslant 0$, this is obvious; for $l\leqslant -1$, one has $-1/2\leqslant\gamma+l+\frac{1}{2}\leqslant 0$, recalling that $Z\alpha<1/2$), we must have $|E|\geqslant m$ for the energy. We note here that, similar to the Schrödinger and the Klein-Gordon case, the adopted ansatz guarantees the normalizability of the wavefunction, but does not provide energy levels with magnitudes below $|E|=m$.

For any integer n, Eqs. (37) and (38) give us a certain number of pairs (E,a) of energy E and the corresponding magnetic field B (or a) which would guarantee normalizability of the wave function. Thus only parts of the whole spectrum of the system are exactly solved. The system can therefore be considered as an example of the quasiexactly solvable models defined in Ref. [9]. In principle the possible values of E and a can be obtained by first expressing the a (or E) in Eq. (37) in terms of E (a) according to Eq. (38). This gives an algebraic equation in E (a) which can be solved for real E (a). The corresponding values of a (E) are then obtained from Eq. (38). In practice the task could be tedious. We shall consider only the simplest cases below, namely, those with $n=1$, 2 and 3. In these cases, the solution of the pair (E,a) is unique for fixed Z and l. In general, for $n>3$, there could exist several pairs of values (E,a) (see Refs. [6,7]). Unlike the nonrelativistic case, here negative energy solutions are possible. As in the case of the $(3+1)$-dimensional Dirac equation [10], the unfilled negative energy solutions are interpreted as positrons with positive energies.

We mention once again that all the exact solutions presented below, including the restrictions for the values of l (or more appropriately, the values of the conserved total quantum number $j=l+1/2$), are obtained according to the ansatz (7), and Eqs. (24) and (25) with polynomial parts. Exact solutions for the other parts of the energy spectrum, if at all possible, would require ansatz of different forms which are not known yet.

(1) $n=1$. In this case we have $\alpha_0\neq 0$ and $\alpha_n=0$ $(n\geqslant 1)$. From Eq. (33) one obtains the energies

$$E=-\frac{m}{2(\gamma+l+1)}. \quad (39)$$

Equation (38) with $n=1$ then gives the corresponding values of magnetic fields a. These results show that, with the ansatz assumed here, solution with positive energy cannot be obtained with $n=1$. Furthermore, the previously mentioned requirement that $E\leqslant -m$ can only be met with $l<0$.

(2) $n=2$. We now consider the next case, in which $\alpha_0,\alpha_1\neq 0$, and $\alpha_n=0$ $(n\geqslant 2)$. This also implies $\beta_n\neq 0$ $(n=0,1,2)$ and $\beta_n=0$ $(n\geqslant 3)$. From Eqs. (38), (35), and (33), we must solve the following set of coupled equations for the possible values of E and a:

$$E^2-m^2=2a\left(2+\gamma+l+\frac{1}{2}\right), \quad (40)$$

$$Z\alpha[(\Gamma+1)(E-m)+(\Gamma+2)(E+m)]\alpha_1+2a(\Gamma+2)\alpha_0=0, \quad (41)$$

$$(2\gamma+1)\Gamma\alpha_1+Z\alpha[\Gamma(E-m)+(\Gamma+1)(E+m)]\alpha_0=0. \quad (42)$$

Here $\Gamma\equiv\gamma+l+1/2$. From these equations one can check that E satisfies the quadratic equation

$$\left[(2\Gamma+1)(2\Gamma+3)-\frac{2\gamma+1}{(Z\alpha)^2}\Gamma\right]E^2+4m(\Gamma+1)E+m^2\times\left[1+\frac{2\gamma+1}{(Z\alpha)^2}\Gamma\right]=0. \quad (43)$$

This can be solved by the standard formula. One must be reminded of the constraint $|E|\geqslant m$. For $l\geqslant 0$, we can obtain analytic solutions with both positive and negative energies. But when $l<0$, analytic solutions can only be obtained for negative energy $E\leqslant -m$. Furthermore, it can be checked that $|E|$ is a monotonic decreasing (increasing) function of $|l|(Z\alpha)$ at fixed $Z\alpha$ (l).

For $Z\alpha\ll 1/2$, i.e. for light hydrogenlike atoms, we can write down approximate expression for energy near the mass value, i.e., $|E|\simeq m$. We can obtain from Eq. (43) the approximate values of E:

$$E_+=m\left[1+\frac{2(Z\alpha)^2}{(2\gamma+1)\Gamma}(\Gamma+1)(\Gamma+2)\right],\quad l\geqslant 0, \tag{44}$$

for positive energies, and

$$E_-=-m\left[1+\frac{2(Z\alpha)^2}{(2\gamma+1)}(\Gamma+1)\right],\quad l\geqslant 0 \text{ and } l<0, \tag{45}$$

for negative energies [in fact, it can be checked from Eq. (43) that for $l<0$, E is always close to $-m$ for any $Z\alpha<1/2$].

When $Z\alpha$ is close to $Z\alpha=1/2$, we have $|E|\gg m$ for $l\geqslant 0$. In this case the energy E can be approximated by

$$E=\pm m\left[1-(Z\alpha)^2\frac{(2\Gamma+1)(2\Gamma+3)}{(2\gamma+1)\Gamma}\right]^{-1/2}. \tag{46}$$

A consequence following from this formula is that, for each $l\geqslant 0$, there is a critical value of Z beyond which polynomial solution with $n=2$ is impossible. The critical value of Z for each l is found by setting the expression in the square root of Eq. (46) to zero. For $l=0$ and $l=1$, the critical values of Z are $Z\alpha=1/2.936$ and 1/2.316, respectively.

In the nonrelativistic limit (see Sec. III), it is the upper, or the large, component $f(r)$ of the Dirac wave function that reduces to the Schrödinger wave function. Hence, in order to compare with the results considered in Ref. [6], it would be appropriate to study the nodal structures of the function $F(r)$ for positive energy solutions in the limit $E\simeq m$. It is easy to see from Eq. (41) or (42) that in this limit, α_0 and α_0 have opposite signs. Thus $F(r)$ has only one node in this limit, which is the same as in the Schrödinger case.

(3) $n=3$. For the case of $n=3$, exact solution of Eqs. (37) and (38) becomes much more tedious. Now the values of E and a are solved by the following coupled equations:

$$E^2-m^2=2a(\Gamma+3), \tag{47}$$

$$Z\alpha[(\Gamma+2)(E-m)+(\Gamma+3)(E+m)]\alpha_2+2a(\Gamma+3)\alpha_1=0, \tag{48}$$

$$4(\gamma+1)(\Gamma+1)\alpha_2+Z\alpha[(\Gamma+1)(E-m)+(\Gamma+2)(E+m)]\alpha_1+4a(\Gamma+2)\alpha_0=0, \tag{49}$$

$$(2\gamma+1)\Gamma\alpha_1+Z\alpha[\Gamma(E-m)+(\Gamma+1)(E+m)]\alpha_0=0. \tag{50}$$

In place of Eq. (43) we now have a cubic equation for the energy E. We shall not attempt to solve it here. It turns out that the equation satisfied by E can be reduced to quadratic ones without linear term in E in the low magnitude ($|E|\approx m$) and the high magnitude ($|E|\gg m$) limit, which correspond to small and large Z, respectively. The results are

$$E_+=m\left[1-\frac{2(Z\alpha)^2(\Gamma+1)(\Gamma+2)(\Gamma+3)}{(2\gamma+1)\Gamma(\Gamma+2)+2(\gamma+1)(\Gamma+1)^2}\right]^{-1/2} \tag{51}$$

and

$$E_-=-m\left[1-\frac{2(Z\alpha)^2(\Gamma+1)(\Gamma+2)(\Gamma+3)}{(2\gamma+1)(\Gamma+2)^2+2(\gamma+1)(\Gamma+1)(\Gamma+3)}\right]^{-1/2} \tag{52}$$

for $|E|\approx m$, and

$$E=\pm m\left[1-\frac{2(Z\alpha)^2(\Gamma+1/2)(\Gamma+3/2)(\Gamma+5/2)(\Gamma+3)}{(2\gamma+1)\Gamma(\Gamma+5/2)(\Gamma+2)+2(\gamma+1)(\Gamma+1/2)(\Gamma+1)(\Gamma+3)}\right]^{-1/2} \tag{53}$$

for $|E|\gg m$. The corresponding values of the magnetic field are obtained by substituting Eqs. (51), (52), or (53) into Eq. (47). For $l=-1$, Eq. (53) is real only for $1/2.65<Z\alpha<1/2$.

As in the $n=2$ case, we shall also investigate the nodal structures of the function $F(r)$ for positive energy solutions in the limit $E\simeq m$. The zeros of the polynomial part of $F(r)$ is given by

$$r_0=\frac{1}{2}\left[-\left(\frac{\alpha_1}{\alpha_2}\right)\pm\sqrt{\left(\frac{\alpha_1}{\alpha_2}\right)^2-4\left(\frac{\alpha_0}{\alpha_2}\right)}\right]. \tag{54}$$

Note that physical solutions of r_0, if exist, must be nonnegative. In the limit $E\simeq m$, Eqs. (48) and (50) give approximately

$$\frac{\alpha_1}{\alpha_2}=-\frac{EZ\alpha}{a}, \tag{55}$$

$$\frac{\alpha_0}{\alpha_2}=-\frac{(2\gamma+1)\Gamma}{2EZ\alpha(\Gamma+1)}\frac{\alpha_1}{\alpha_2}. \tag{56}$$

We see from Eq. (55) that $\alpha_1/\alpha_2<0$ in this limit.

For negative $l<0$, which implies $-1/2\leqslant\Gamma<0$, we also have $\alpha_0/\alpha_2<0$. Equation (54) then implies that there is only one positive zero of $F(r)$. Hence the wave function has only one node for $l<0$.

When $l\geqslant 0$, we have $\Gamma>0$, and hence $\alpha_0/\alpha_2>0$. It can be checked from Eqs. (47), (51), (55), and (56) that

$(\alpha_1/\alpha_2)^2>4(\alpha_0/\alpha_2)$. Thus $F(r)$ has two positive zeros. This is also consistent with the results presented in Ref. [6] for the Schrödinger case (see also the last part of the following section).

III. NONRELATIVISTIC LIMIT AND METHOD OF FACTORIZATION

The electron in $2+1$ dimensions in the nonrelativistic approximation is described by one-component wave function. This can easily be shown in full analogy with the $(3+1)$-dimensional case. Let us represent Ψ in the form

$$\Psi=\exp(-imt)\begin{pmatrix}\psi\\ \chi\end{pmatrix} \tag{57}$$

and substitute Eq. (57) into Eq. (2). This results in, to the first order in $1/c$, the following Schrödinger-type equation (instead of the Schrödinger-Pauli equation in $3+1$ dimensions):

$$i\frac{\partial\psi}{\partial t}=\left(\frac{P_1^2+P_2^2}{2m}+\frac{eB}{2m}-\frac{Ze^2}{r}\right)\psi, \tag{58}$$

where, as before, $P_k=-i\partial_k+eA_k$ denote the generalized momentum operators. The term $eB/2m$ in Eq. (58) indicates that the electron has gyromagnetic factor $g=2$ as in the $(3+1)$-dimensional case [10].

One can now proceed in the same manner as in the Dirac case to solve for the possible energies and magnetic fields. We shall not repeat it here. More simply, we make use of the fact that Eq. (58) differs from the Schrödinger equation discussed in Ref. [6] only by the positive spin correction term $\omega_L=eB/2m$, which is the Larmor frequency. We thus conclude that the denumerably infinite set of magnetic field strengths obtained in [6] are still intact, but the corresponding values of the possible energies are all shifted by an amount ω_L, i.e.,

$$E=\omega_L(n+1+l+|l|). \tag{59}$$

Simply put, the quantum number n in [6] is changed to $n+1$.

Let us note here that the energies and magnetic fields in this case may also be found by means of a method closely resembling the method of factorization in nonrelativistic quantum mechanics. We shall discuss this method briefly below. Both the attractive and repulsive Coulomb interactions will be considered, since planar two electron systems in strong external homogeneous magnetic field (perpendicular to the plane in which the electrons is located) are also of considerable interest for the understanding of the fractional quantum Hall effect. Let us assume

$$\psi(t,\mathbf{x})=\frac{1}{\sqrt{2\pi}}\exp(-iEt+il\varphi)r^{|l|}\exp(-ar^2/2)Q(r), \tag{60}$$

where Q is a polynomial, and $a=eB/2$ as defined before. Substituting Eq. (60) into Eq. (58), we have

$$\left[\frac{d^2}{dx^2}+\left(\frac{2\gamma}{x}-x\right)\frac{d}{dx}+\left(\epsilon\pm\frac{b}{x}\right)\right]Q(x)=0. \tag{61}$$

Here $x=r/l_B$, $l_B=1/\sqrt{eB}$, $\gamma=|l|+1/2$, $b=2m|Z|\alpha l_B=|Z|\alpha\sqrt{2m/\omega_L}$, and $\epsilon=E/\omega_L-(2+l+|l|)$. The upper (lower) sign in Eq. (61) corresponds to the case of attractive (repulsive) Coulomb interaction. This will be assumed throughout the rest of the paper.

It is seen that the problem of finding spectrum for Eq. (61) is equivalent to determining the eigenvalues of the operator

$$H=-\frac{d^2}{dx^2}-\left(\frac{2\gamma}{x}-x\right)\frac{d}{dx}\mp\frac{b}{x}. \tag{62}$$

We want to factorize the operator (62) in the form

$$H=a^+a+p, \tag{63}$$

where the quantum numbers p are related to the eigenvalues of Eq. (61) by $p=\epsilon$. The eigenfunctions of the operator H at $p=0$ must satisfy the equation

$$a\psi=0. \tag{64}$$

Suppose polynomial solutions exist for Eq. (61), say $Q=\Pi_{k=1}^{s}(x-x_k)$, where x_k are the zeros of Q, and s is the degree of Q. Then the operator a must have the form

$$a=\frac{\partial}{\partial x}-\sum_{k=1}^{s}\frac{1}{x-x_k} \tag{65}$$

and the operator a^+ has the form

$$a^+=-\frac{\partial}{\partial x}-\frac{2\gamma}{x}+x-\sum_{k=1}^{s}\frac{1}{x-x_k}. \tag{66}$$

Substituting Eqs. (65) and (66) into (63) and then comparing the result with Eq. (62), we obtain the following set of equations for the zeros x_k (the so-called Bethe ansatz equations [9]):

$$\frac{2\gamma}{x_k}-x_k-2\sum_{j\neq k}^{s}\frac{1}{x_j-x_k}=0,\quad k=1,\ldots,s, \tag{67}$$

as well as the two relations

$$\pm b=2\gamma\sum_{k=1}^{s}x_k^{-1},\quad s=p. \tag{68}$$

Summing all the s equations in Eq. (67) enables us to rewrite the first relation in Eq. (68) as

$$\pm b=\sum_{k=1}^{s}x_k. \tag{69}$$

From these formulas we can find the simplest solutions as well as the values of energy and magnetic field strength. The second relation in (68) gives $E=\omega_L(2+s+l+|l|)$, which is the same as in Eq. (59) noting that $n=s+1$.

For $s=1,2$ the zeros x_k and the values of the parameter b for which solutions in terms of polynomial of the corresponding degrees exist can easily be found from Eqs. (67) and (69) in the form

$$s=1,\ \ x_1=\pm\sqrt{2|l|+1},\ \ b=\sqrt{2|l|+1},$$

$$s=2,\ \ x_1=(2|l|+1)/x_2,\ \ x_2=\pm(1+\sqrt{4|l|+3})/\sqrt{2},$$
$$b=\sqrt{2(4|l|+3)}. \tag{70}$$

From Eq. (70) and the definition of b one has the corresponding values of magnetic field strengths

$$\omega_L=2m\frac{(Z\alpha)^2}{2|l|+1},\ \ s=1,$$

$$\omega_L=m\frac{(Z\alpha)^2}{4|l|+3},\ \ s=2, \tag{71}$$

as well as the energies

$$E_1=\frac{2m(Z\alpha)^2}{2(2|l|+1)}(3+l+|l|),$$

$$E_2=\frac{m(Z\alpha)^2}{(4|l|+3)}(4+l+|l|). \tag{72}$$

The corresponding polynomials are

$$Q_1=x-x_1=x\mp b,$$

$$Q_2=\prod_{k=1}^{2}(x-x_k)=x^2\mp bx+2|l|+1. \tag{73}$$

The wave functions are described by Eq. (60). For $s=1,2$ for the repulsive Coulomb field the wave functions do not have nodes (for $|l|=0,1$), i.e., the states described by them are ground states, while for the attractive Coulomb field the wave function for $s=1$ has one node (first excited state) and the wave function for $s=2$ has two nodes (second excited state).

IV. CONCLUSIONS

In this paper we considered solutions of the Dirac equation in two spatial dimensions in the Coulomb and homogeneous magnetic fields. It was shown by using semiclassical approximation that for weak magnetic fields all discrete energy eigenvalues are negative levels of a hydrogenlike atom perturbed by the magnetic field. For large magnetic fields, analytic solutions of the Dirac equation are possible for a denumerably infinite set of magnetic field strengths, if the two components of the wave function are assumed to have the forms (24) and (25) with terminating polynomial parts. Such forms will guarantee normalizability of the wave functions. We presented the exact recursion relations that determine the coefficients of the series expansion for solutions of the Dirac equation, the possible energies and the magnetic fields. Exact and/or approximate expressions of the energy are explicitly given for the three simplest cases. For low positive energy solutions, we also investigate the nodal structures of the large components of the Dirac wave functions, and find that they are the same as in the Schrödinger case. We emphasize that, by assuming a sufficient condition on the wave function that guarantees normalizability, only parts of the energy spectrum of this system are exactly solved for. In this sense the system can be considered a quasiexactly solvable model as defined in Ref. [9]. As in the Schrödinger and the Klein-Gordon case, energy levels with magnitude below the mass value, which include the most interesting ground state solution, cannot be obtained by our ansatz. For the corresponding case in $3+1$ dimensions, no analytic solutions, even for parts of the spectrum, are possible.

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