On the leadtime effect of two-stages production planning with random point demand

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ABSTRACT

Chen & Chen [4] constructed the production planning model that was based on three assumptions: (1) the demand occurs in the future time and is a random variable, (2) there are two stages in the production process, (3) the unit production cost is a linear function of production in the unit time. Then, the optimal production rates is derived so as to maximize the profit.

The aim of this article is to extend the model under (1) the leadtime between the production process is considered, (2) the object is cost minimization. And then, the phenomenon of optimal solution is discussed.

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1. INTRODUCTION

The classical newsboy problem is a single-period, single product inventory problem which considers the inventory size to be ordered for the sake of meeting random demand so as to maximize expected profit

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while balancing holding and shortage costs. And there are many papers to discuss this problem in recent years. For example, Eppen [6] presented a multi-location newsboy problem with normal distribution of a location's demand, and identical linear holding and penalty cost functions. Chen & Lin [2] extend Eppen's model by considering the concave cost function and with unspecified distribution of demand, then shows that the Eppen's results are still true.

Chen & Chen [4] construct a production planning model of the classical newsboy problem. The main question of Chen & Chen's article is : In two stages production process, how should decision makers control the production rate at each stage to meet the random demand at the end of production period such that the expected profit is optimal? It is also assumed that only the production and holding costs are considered during the production period. However, it neglects the transportation cost between the production process, i.e., leadtime occurs between the production process.

The main result of Chen & Chen's model is that whether the optimal production rates are the same or depend on the relation of parameters of the production and holding costs. It seems that if there are other costs which occur during the production period then the characteristics of optimal solutions would change. Therefore, the aim of this note is to extend the Chen & Chen's model by incorporating the transportation cost during the leadtime and the object of profit maximization is replaced by cost minimization, and then discuss the effects on the optimal production rates and quantity of goods for sale.

2. NOTATIONS AND ASSUMPTIONS

- T : The sailing time.
- L : The leadtime.
- $[0, T-L]$: The available time interval to produce semi-finished goods.
- $[L, T]$: The available time interval to produce finished goods.
- t_1 : The time to begin production of semi-finished goods, where $t_1 \geq 0$.
- t_2: The time to begin production of finished goods, where $t_2 \geq t_1 + L$.
- $[t_{1^*}, T-L]$: The time interval during which the decision maker is actually engaged in production of semi-finished goods.

- $[t_2, T]$: The time interval during which the decision maker is actually engaged in production of finished goods.
- h_1 : The holding cost of unit semi-finished goods in the unit time.
- h_2 : The holding cost of unit finished goods in the unit time.
- h_{11} : The transportation and holding cost of unit semi-finished goods in the unit time. The cost occurs in the time interval of leadtime. And, in this time interval, if there is only holding cost then $h_{11} = h_1$.
- b : The loss or treatment cost of unit surplus goods.
- p : The penalty cost of unit shortage of goods.
- $x_1(t)$: The cumulative production of semi-finished goods at time t.
- $x_1'(t)$: The marginal production of semi-finished goods at time t.
- $x_2(t)$: The cumulative production of finished goods at time t.
- $x_2'(t)$: The marginal production of finished goods at time t.

  When the decision makers make extra production plans in addition to the routine work, the cost will burden them, because of the capital and the human resources. So, the unit production cost will increase as the production increases. Hence, in this article, we assume that the unit production cost is an increasing function of production in the unit time. If they don't have production, they don't have to pay for the cost. Therefore, the assumption we made here is that

- $c_1 x_1'(t)$: The unit production cost of semi-finished goods at time t, where c_1 is constant.
- $c_2 x_2'(t)$: The unit production cost of finished goods at time t, where c_2 is constant.
- S : The quantity of goods in demand at time T. Here S is a random variable, its probability density function is $f(s)$, and its cumulative distribution function is $F(s)$.
- $x_2(T)$: The goods for sale at time T.

where t_1, t_2, $x_1(t)$, $x_2(t)$, $x_1'(t)$, $x_2'(t)$ and $x_2(T)$ are decision variables.

3. MODEL

Using the notation and assumptions in the previous section, we have

- The total production and holding cost of finished goods

$$= \int_{t_2}^{T} [c_2 x_2'^2(t) + h_2 x_2(t)]dt.$$

- The total production and holding of semi-finished goods in the time interval $[t_1, t_2 - L]$

$$= \int_{t_1}^{t_2-L} [c_1 x_1'^2(t) + h_1 x_1(t)]dt.$$

- The total production and holding cost of semi-finished goods in the time interval $[t_2 - L, T - L]$

$$= \int_{t_2-L}^{T-L} [c_1 x_1'^2(t) + h_1(x_1(t) - x_2(t + L))]dt.$$

- The total transportation or holding cost of semi-finished goods in the time interval of leadtime $= h_{11} L x_2(T)$.

  Since the goods for sale $\mathrm{Min}\{x_2(T), S\}$ is a random variable, so

- The expected cost of goods surplus $= \int_0^{x_2(T)} b(x_2(T) - s)f(s)ds.$

- The expected cost of goods lacking $= \int_{x_2(T)}^{\infty} p(s - x_2(T))f(s)ds.$

If the object is cost minimization, then the mathematical model is as follows:

$$\begin{cases} \text{Min} \int_0^{x_2(T)} b(x_2(T)-s)f(s)ds + \int_{x_2(T)}^{\infty} p(s-x_2(T))f(s)ds + h_{11}Lx_2(T) \\ \quad + \int_{t_1}^{T-L} [c_1 x_1'^2(t) + h_1 x_1(t)]dt + \int_{t_2}^{T} [c_2 x_2'^2(t) + h_2 x_2(t)]dt \\ \quad - \int_{t_2-L}^{T-L} h_1 x_2(t+L)]dt \\ \text{s.t. } x_1(t_1) = x_2(t_2) = 0,\ x_1(T-L) = x_2(T),\ x_1'(t) \ge 0\ \ \forall\, t \in [t_1, T-L], \\ \quad x_2'(t) \ge 0\ \ \forall\, t \in [t_2, T],\quad 0 \le t_1 \le t_2 - L, \\ \quad x_1(t-L) \ge x_2(t)\ \ \forall\, t \in [t_2, T] \\ \text{where } t_1, t_2, x_1(T-L) \text{ and } x_2(T) \text{ are free.} \end{cases} \tag{1}$$

Let $(x_1^*(t), x_2^*(t))$ be the optimal solution of (1), and let

$$\begin{cases} x_1(t-L) = y_1(t) \quad \forall\, t \in [t_1 + L, T] \quad \text{and} \quad r_1 = t_1 + L \\ x_2(t) = y_2(t) \quad \forall\, t \in [t_2, T] \quad \text{and} \quad r_2 = t_2 . \end{cases}$$

Then problem (1) becomes

$$\begin{cases} \text{Min} \int_0^{y_2(T)} b(y_2(T)-s)f(s)ds + \int_{y_2(T)}^{\infty} p(s-y_2(T))f(s)ds + h_{11}Ly_2(T) \\ \quad + \int_{r_1}^{T} [c_1 y_1'^2(t) + h_1 y_1(t)]dt + \int_{r_2}^{T} [c_2 y_2'^2(t) + (h_2 - h_1)y_2(t)]dt \\ \text{s.t. } y_1(r_1) = y_2(r_2) = 0,\ y_1(T) = y_2(T),\ y_1'(t) \ge 0\ \ \forall\, t \in [r_1, T] \\ \quad y_2'(t) \ge 0\ \ \forall\, t \in [r_2, T],\quad L \le r_1 \le r_2, \\ \quad y_1(t) \ge y_2(t)\ \ \forall\, t \in [r_2, T] \\ \text{where } r_1, r_2, y_1(T) \text{ and } y_2(T) \text{ are free.} \end{cases} \tag{1'}$$

For the sake of convenience, we first derive the optimal solution (y_1^*, y_2^*) of problem (1′), then transfer it to the optimal solutions $(x_1^*(t), x_2^*(t))$ of problem (1).

4. OPTIMAL SOLUTION

Since problem (1′) is not the standard form of calculus of variation, therefore, we first neglect the constraint $y_1' \geq 0, y_2' \geq 0$, and consider the following problem:

$$\begin{cases} \text{Min} \int_0^{y_2(T)} b(y_2(T)-s)f(s)ds + \int_{y_2(T)}^{\infty} p(s-y_2(T))f(s)ds + h_{11}Ly_2(T) \\ \quad + \int_{r_2}^{T} [c_1 y_1'^2(t) + h_1 y_1(t) + c_2 y_2'^2(t) + (h_2-h_1)y_2(t)]dt \\ \text{s.t. } y_1(r_2) = y_2(r_2) = 0, \quad y_1(T) = y_2(T), \quad h_2 > h_1 \\ \quad y_1(t) \geq y_2(t) \quad \forall\, t \in [r_2, T] \\ \text{where } y_1(T), y_2(T) \text{ and } r_2 \geq L \text{ are free.} \end{cases} \tag{2}$$

Now, suppose $(\bar{y}_1(t), \bar{y}_2(t))$ be the optimal solution of problem (2), and define

$$\begin{cases} z_1(t) = \bar{y}_2(t) \\ z_2(t) = \bar{y}_1(t) \end{cases} \qquad \forall\, t \in [\bar{r}_2, T].$$

Clearly, $(\bar{y}_1(t), z_2(t))$ and $(z_1(t), \bar{y}_2(t))$ are feasible solutions of problem (2). Hence,

the objective value of $(\bar{y}_1, \bar{y}_2) \leq$ the objective value of $(z_1(t), \bar{y}_2)$ .....(3)

and

the objective value of $(\bar{y}_1, \bar{y}_2) \leq$ the objective value of $(\bar{y}_1, z_2)$. .....(4)

If we substitute $(\bar{y}_1(t), \bar{y}_2(t))$ and $(z_1(t), \bar{y}_2(t))$ into (2), then (3) simplifies to

$$\frac{h_1}{c_1} \int_{r_2}^{T} (\bar{y}_1 - \bar{y}_2)dt \leq \int_{r_2}^{T} (\bar{y}_2'^2 - \bar{y}_1'^2)dt \tag{5}$$

Similarly, if we substitute $(\bar{y}_1, \bar{y}_2)$ and $(\bar{y}_1, z_2)$ into (2), then (4) simplifies to

$$\int_{r_2}^{T} (\bar{y}_2'^2 - \bar{y}_1'^2)dt \le \frac{h_2 - h_1}{c_2} \int_{r_2}^{T} (\bar{y}_1 - \bar{y}_2)dt. \tag{6}$$

Combining (5) and (6), we have

$$\left(\frac{h_1}{c_1} - \frac{h_2 - h_1}{c_2} \right) \int_{r_2}^{T} (\bar{y}_1 - \bar{y}_2)dt \le 0. \tag{*}$$

Case 1. If $\frac{h_1}{c_1} > \frac{h_2 - h_1}{c_2}$, then, by property (*), $\bar{y}_1 = \bar{y}_2 \quad \forall \ t \in [\bar{r}_2, T]$. Hence the optimal solution of problem (2) is the same as the following problem:

$$\begin{cases} \text{Min} \int_0^{y_2(T)} b(y_2(T) - s)f(s)ds + \int_{y_2(T)}^{\infty} p(s - y_2(T))f(s)ds \\ \quad + \int_{r_2}^{T} [(c_1 + c_2)y_2'^2(t) + h_2 y_2(t)]dt + h_{11}Ly_2(T) \\ \text{s.t. } y_2(r_2) = 0, \quad y_2(T) \text{ free}, \quad r_2 \ge L \text{ free}. \end{cases} \tag{7}$$

Case (1.1). If $\bar{r}_2 = L$, then the optimal solution $\bar{y}_2(t)$ must satisfy the following necessary conditions [[7], pp. 67-68]:

$$h_2 = 2(c_1 + c_2)\bar{y}_2''(t) \tag{8}$$

$$2(c_1 + c_2)\bar{y}_2'(T) + h_{11}L - p + (p + b)F(\bar{y}_2(T)) = 0. \tag{9}$$

Using the boundary condition $\bar{y}_2(L) = 0$, (8) leads to

$$\bar{y}_2(t) = \frac{h_2}{4(c_1 + c_2)} t^2 + \left[\frac{\bar{y}_2(T)}{T - L} - \frac{h_2}{4(c_1 + c_2)} (T + L) \right] (t - L)$$

$$- \frac{h_2}{4(c_1 + c_2)} L^2. \tag{10}$$

Hence, by (10), (9) becomes

$$h_2 T + 2(c_1 + c_2) \left[\frac{\bar{y}_2(T)}{T - L} - \frac{h_2}{4(c_1 + c_2)} (T + L) \right] = 2(c_1 + c_2)\bar{y}_2'(T)$$

$$= p - h_{11}L - (p+b)F(\bar{y}_2(T)). \tag{11}$$

This means that the value of $\bar{y}_2(T)$ is determined by (11).

Since $\bar{y}_2''(t) = \dfrac{h_2}{2(c_1+c_2)} > 0$, so $\bar{y}_2'(t) \geq 0 \;\; \forall\, t$ if and only if

$$\bar{y}_2'(L) = \frac{\bar{y}_2(T)}{T-L} - \frac{h_2}{4(c_1+c_2)}(T-L) \geq 0. \tag{12}$$

From (11), inequality (12) holds if and only if

$$p \geq h_2(T-L) + h_{11}L + (p+b)F\left(\frac{h_2}{4(c_1+c_2)}(T-L)^2\right). \tag{13}$$

This means that if (13) hold then the optimal solution $\bar{y}_2(t)$ in (10) is also the optimal solution of problem (1′). (14)

Case (1.2). If $\bar{r}_2 > L$, then the optimal solution $\bar{y}_2(t)$ must satisfy the following necessary conditions [[7], pp. 67-68]:

$$h_2 = 2(c_1+c_2)\bar{y}_2''(t) \tag{15}$$

$$(c_1+c_2)\bar{y}_2'^2(\bar{r}_2) + h_2\,\bar{y}_2(\bar{r}_2) - \bar{y}_2'(\bar{r}_2)(2(c_1+c_2)\bar{y}_2'(\bar{r}_2)) = 0 \tag{16}$$

$$2(c_1+c_2)\bar{y}_2'(T) + h_{11}L - p + (p+b)F(\bar{y}_2(T)) = 0. \tag{17}$$

From (16), we know that $\bar{y}_2'(\bar{r}_2) = 0$, then, by $y_2'(\bar{r}_2) = 0$ and $\bar{y}_2(\bar{r}_2) = 0$, (15) becomes

$$\bar{y}_2(t) = \frac{h_2}{4(c_1+c_2)}(t-\bar{r}_2)^2 \qquad \forall\, t \in [\bar{r}_2, T]. \tag{18}$$

Combine (18) and (17), we have

$$p = h_2(T-\bar{r}_2) + h_{11}L + (p+b)F\left(\frac{h_2}{4(c_1+c_2)}(T-\bar{r}_2)^2\right). \tag{19}$$

Hence the value of $\bar{r}_2$ is determined by (19).

Since $\bar{r}_2 > L$ in this case, so (19) yields that

$$p < h_2(T-L) + h_{11}L + (p+b)F\left(\frac{h_2}{4(c_1+c_2)}(T-L)^2\right). \tag{20}$$

Since $\bar{y}_2''(t) = \dfrac{h_2}{2(c_1 + c_2)} > 0$ and $\bar{y}_2'(\bar{r}_2) = 0$, hence $\bar{y}_2'(t) > 0$, $\forall\ t \in [\bar{r}_2, T]$.

Hence, if inequality (20) hold then the optimal solution $\bar{y}_2(t)$ of (18) and (19) is also the optimal solution of problem (1′). (21)

Case 2. If $\dfrac{h_1}{c_1} \le \dfrac{h_2 - h_1}{c_2}$, then consider the following problem:

$$\begin{cases} \text{Min} \displaystyle\int_0^{y_2(T)} b(y_2(T) - s)f(s)ds + \int_{y_2(T)}^{\infty} p(s - y_2(T))f(s)ds + h_{11}Ly_2(T) \\ \quad + \displaystyle\int_{r_1}^{T} [(c_1 y_1'^2(t) + h_1 y_1(t)]dt + \int_{r_2}^{T} [c_2 y_2'^2(t) + (h_2 - h_1)y_2(t)]dt \\ \text{s.t. } y_1(r_1) = y_2(r_2) = 0,\ y_1(T) = y_2(T),\ y_1(t) \ge y_2(t) \quad \forall\ t \in [r_2, T], \\ \quad L \le r_1 \le r_2 \\ \text{where } r_1, r_2, y_1(T) \text{ and } y_2(T) \text{ are free.} \end{cases}$$

Then the above problem can be divided into three subcases:

Case (2.1). If $\bar{r}_1 = \bar{r}_2 = L$, then the optimal solution $(\bar{y}_1, \bar{y}_2)$ must satisfy the following conditions [[7], pp. 105-106]:

$$h_1 = 2c_1 \bar{y}_1''(t) \tag{22}$$

$$h_2 - h_1 = 2c_2 \bar{y}_2''(t) \tag{23}$$

$$2c_1 \bar{y}_1'(T) + 2c_2 \bar{y}_2'(T) + h_{11}L - p + (p + b)F(\bar{y}_2(T)) = 0. \tag{24}$$

From (22) and (23), we have

$$\bar{y}_1(t) = \frac{h_1}{4c_1} t^2 + \left[\frac{\bar{y}_2(T)}{T - L} - \frac{h_1}{4c_1}(T + L) \right](t - L) - \frac{h_1}{4c_1} L^2 \tag{25}$$

$$\bar{y}_2(t) = \frac{h_2 - h_1}{4c_2} t^2 + \left[\frac{\bar{y}_2(T)}{T - L} - \frac{h_2 - h_1}{4c_2}(T + L) \right](t - L) - \frac{h_2 - h_1}{4c_2} L^2. \quad .....(26)$$

Using (25), (26) and the constraint $\bar{y}_1(T) = \bar{y}_2(T)$, then (24) becomes

$$2c_1\left[\frac{\bar{y}_2(T)}{T-L}-\frac{h_1}{4c_1}(T+L)\right]+2c_2\left[\frac{\bar{y}_2(T)}{T-L}-\frac{h_2-h_1}{4c_2}(T+L)\right]$$

$$+h_2T-p+h_{11}L+(p+b)F(\bar{y}_2(T))=0. \tag{27}$$

This means that the value of $\bar{y}_2(T)$ is determined by (27).

It is easy to see that $\bar{y}_1(t)-\bar{y}_2(t)\geq 0, \quad \forall\, t\in[L,T]$.

Since $\bar{y}_1''(t)=\dfrac{h_1}{2c_1}>0,\ \bar{y}_2''(t)=\dfrac{h_2-h_1}{2c_2}>0$ and $\bar{y}_1(T)=\bar{y}_2(T)$, so $\bar{y}_1', \bar{y}_2'\geq 0$ if and only if

$$\bar{y}_2'(L)=\frac{\bar{y}_2(T)}{T-L}-\frac{h_2-h_1}{4c_2}(T-L)\geq 0. \tag{28}$$

Now, let G be the function of the left-hand side of equation (27):

$$G(\bar{y}_2(T))=2c_1\left[\frac{\bar{y}_2(T)}{T-L}-\frac{h_1}{4c_1}(T+L)\right]+2c_2\left[\frac{\bar{y}_2(T)}{T-L}-\frac{h_2-h_1}{4c_2}(T+L)\right]$$

$$+h_2T-p+h_{11}L+(p+b)F(\bar{y}_2(T)).$$

Clearly, $G'(\bar{y}_2(T))>0$. Thus, by (28), $\bar{y}_1'(t),\bar{y}_2'(t)\geq 0\ \ \forall\, t\in[L,T]$ if and only if $G\left(\dfrac{h_2-h_1}{4c_2}(T-L)^2\right)\leq 0$, i.e.,

$$p\geq\frac{T-L}{2}\left(\frac{c_1(h_2-h_1)-c_2h_1}{c_2}\right)+h_2(T-L)+h_{11}L$$

$$+(p+b)F\left(\frac{h_2-h_1}{4c_2}(T-L)^2\right). \tag{29}$$

This shows that if the inequality (29) hold, then the optimal solution $(\bar{y}_1,\bar{y}_2)$ in (25) and (26) is also the optimal solution of problem (1′). (30)

Case (2.2). If $\bar{r}_1=L, \bar{r}_2>L$, then the optimal solution $(\bar{y}_1,\bar{y}_2)$ must satisfy the following necessary conditions [[7], pp. 105-106]:

$$h_1=2c_1\,\bar{y}_1''(t) \tag{31}$$

$$h_2-h_1=2c_2\,\bar{y}_2''(t) \tag{32}$$

$$c_2 \bar{y}_2'^2(\bar{r}_2) + (h_2 - h_1)\bar{y}_2(\bar{r}_2) - \bar{y}_2'(\bar{r}_2)(2c_2 \bar{y}_2'(\bar{r}_2)) = 0 \tag{33}$$

$$2c_1 \bar{y}_1'(T) + 2c_2 \bar{y}_2'(T) + h_{11}L - p + (p + b)F(\bar{y}_2(T)) = 0. \tag{34}$$

From (33), we have $\bar{y}_2'(\bar{r}_2) = 0$, then using the boundary condition $\bar{y}_2(\bar{r}_2) = 0$, (32) yields that

$$\bar{y}_2(t) = \frac{h_2 - h_1}{4c_2}(t - \bar{r}_2)^2. \tag{35}$$

Next, using the constraint $\bar{y}_1(T) = \bar{y}_2(T)$ and the boundary condition $\bar{y}_1(L) = 0$, (31) becomes

$$\bar{y}_1(t) = \frac{h_1}{4c_1}t^2 + \left[\frac{1}{T-L}\frac{h_2 - h_1}{4c_2}(T - \bar{r}_2)^2 - \frac{h_1}{4c_1}(T + L)\right](t - L)$$
$$- \frac{h_1}{4c_1}L^2. \tag{36}$$

Together with (36), (35) and (34), we have

$$2c_1\left[\frac{1}{T-L}\frac{h_2 - h_1}{4c_2}(T - \bar{r}_2)^2 - \frac{h_1}{4c_1}(T + L)\right] + (h_2 - h_1)(T - \bar{r}_2)$$
$$+ h_1 T - p + h_{11}L + (p + b)F\left(\frac{h_2 - h_1}{4c_2}(T - \bar{r}_2)^2\right) = 0. \tag{37}$$

This means that the value of $\bar{r}_2$ is determined by (37).

It is easy to see that $\bar{y}_1(t) - \bar{y}_2(t) \geq 0, \quad \forall\ t \in [\bar{r}_2, T]$.

Since $\bar{y}_1''(t) = \frac{h_1}{2c_1} > 0, \bar{y}_2''(t) = \frac{h_2 - h_1}{2c_2} > 0$ and $\bar{y}_2'(\bar{r}_2) = 0$, so $\bar{y}_2(t)$ must satisfy $\bar{y}_2' \geq 0 \quad \forall\ t$; on the other hand, $\bar{y}_1' \geq 0$ if and only if $\bar{y}_1'(L) = \frac{\bar{y}_2(T)}{T-L} - \frac{h_1}{4c_1}(T - L) \geq 0$, i.e., $\bar{y}_2(T) \geq \frac{h_1}{4c_1}(T - L)^2$. This means

$$\frac{h_2 - h_1}{4c_2}(T - \bar{r}_2)^2 \geq \frac{h_1}{4c_1}(T - L)^2$$

i.e.,

$$L < \bar{r}_2 \leq T - (T - L)\sqrt{\frac{h_1}{c_1}\frac{c_2}{h_2 - h_1}}. \tag{38}$$

Now, let G be the function of $\bar{r}_2$ defined in the left-hand side of equation (37):

$$G(\bar{r}_2) = 2c_1\left[\frac{1}{T-L}\frac{h_2-h_1}{4c_2}(T-\bar{r}_2)^2 - \frac{h_1}{4c_1}(T+L)\right] + (h_2-h_1)(T-\bar{r}_2)$$

$$+ h_1T - p + h_{11}L + (p+b)F\left(\frac{h_2-h_1}{4c_2}(T-\bar{r}_2)^2\right).$$

Differentiate G with $\bar{r}_2$, we have $G'(\bar{r}_2) < 0$.

Hence, by (38), $\bar{y}_1', \bar{y}_2' \geq 0$ if and only if $G\left(T-(T-L)\sqrt{\frac{h_1}{c_1}\frac{c_2}{h_2-h_1}}\right) \leq 0$ and $G(L) > 0$, i.e.,

$$(h_2-h_1)(T-L)\sqrt{\frac{h_1}{c_1}\frac{c_2}{h_2-h_1}} + h_1(T-L) + h_{11}L$$

$$+ (p+b)F\left(\frac{h_1}{4c_1}(T-L)^2\right)$$

$$\leq p$$

$$< \frac{T-L}{2}\left[\frac{c_1(h_2-h_1)-c_2h_1}{c_2}\right] + h_2(T-L) + h_{11}L$$

$$+ (p+b)F\left(\frac{h_2-h_1}{4c_2}(T-L)^2\right). \tag{39}$$

Therefore, if inequality (39) holds, then the optimal $(\bar{y}_1, \bar{y}_2)$ is also optimal solution of problem (1′). (40)

Case (2.3). If $\bar{r}_1 > L, \bar{r}_2 > L$, then the optimal solution $(\bar{y}_1, \bar{y}_2)$ must satisfy the following necessary conditions [[7], pp. 105-106]:

$$h_1 = 2c_1\bar{y}_1''(t) \tag{41}$$

$$h_2 - h_1 = 2c_2\bar{y}_2''(t) \tag{42}$$

$$c_1\bar{y}_1'^2(\bar{r}_1) + h_1\bar{y}_1(\bar{r}_1) - \bar{y}_1'(\bar{r}_1)(2c_1\bar{y}_1'(\bar{r}_1) = 0 \tag{43}$$

$$c_2\bar{y}_2'^2(\bar{r}_2) + (h_2-h_1)\bar{y}_2(\bar{r}_2) - \bar{y}_2'(\bar{r}_2)(2c_2\bar{y}_2'(\bar{r}_2)) = 0 \tag{44}$$

$$2c_1\bar{y}_1'(T)+2c_2\bar{y}_2'(T)+h_{11}L-p+(p+b)F(\bar{y}_2(T))=0. \tag{45}$$

From (43) and (44), we have $\bar{y}_1'(\bar{r}_1)=0$ and $\bar{y}_2'(\bar{r}_2)=0$. And using the boundary condition $\bar{y}_1(\bar{r}_1)=0, \bar{y}_2(\bar{r}_2)=0$, (41) and (42) lead to

$$\begin{cases} \bar{y}_1(t)=\dfrac{h_1}{4c_1}(t-\bar{r}_1)^2 & t\in[\bar{r}_1, T] \\ \bar{y}_2(t)=\dfrac{h_2-h_1}{4c_2}(t-\bar{r}_2)^2 & t\in[\bar{r}_2, T]. \end{cases} \tag{46}$$

Since $\bar{y}_1(T)=\bar{y}_2(T)$, so, by (46), we have

$$\bar{r}_2=T-(T-\bar{r}_1)\sqrt{\frac{h_1}{c_1}\frac{c_2}{h_2-h_1}}\ . \tag{47}$$

Together with (45), (46) and (47), we have

$$(h_2-h_1)\left[(T-\bar{r}_1)\sqrt{\frac{h_1}{c_1}\frac{c_2}{h_2-h_1}}\right]+h_1(T-\bar{r}_1)$$
$$-p+h_{11}L+(p+b)F\left(\frac{h_1}{4c_1}(T-\bar{r}_1)^2\right)=0. \tag{48}$$

This means that the value of $\bar{r}_1$ is determined by (48). And the value of $\bar{r}_2$ is determined by (47).

Furthermore, it is easy to see that $\bar{y}_1(t)-\bar{y}_2(t)\geq 0, \quad \forall\ t\in[\bar{r}_2, T]$.

Since $\bar{y}_1''(t)=\dfrac{h_1}{2c_1}>0, \bar{y}_2''(t)=\dfrac{h_2-h_1}{2c_2}>0$, and $\bar{y}_1'(\bar{r}_1)=0$, $\bar{y}_2'(\bar{r}_2)=0$, so the optimal solution in (46) $(\bar{y}_1, \bar{y}_2)$ satisfy $\bar{y}_1'>0$ and $\bar{y}_2'>0$.

Let G be the function of $\bar{r}_1$ defined in the left-hand side of (48):

$$G(\bar{r}_1)=(h_2-h_1)\left[(T-\bar{r}_1)\sqrt{\frac{h_1}{c_1}\frac{c_2}{h_2-h_1}}\right]+h_1(T-\bar{r}_1)$$
$$-p+h_{11}L+(p+b)F\left(\frac{h_1}{4c_1}(T-\bar{r}_1)^2\right).$$

It is easy to see that $G'(\bar{t}_1) < 0$, so, $\bar{r}_1 > L$ if and only if $G(L) > 0$, i.e.,

$$p < (h_2 - h_1)(T-L)\sqrt{\frac{h_1}{c_1}\frac{c_2}{h_2-h_1}} + h_1(T-L) + h_{11}L$$
$$+ (p+b)F\left(\frac{h_1}{4c_1}(T-L)^2\right). \qquad (49)$$

Hence, if inequality (49) holds then the optimal solution $(\bar{y}_1, \bar{y}_2)$ is also the optimal solutions of problem (1′). (50)

Finally, suppose that (x_1^*, x_2^*), (y_1^*, y_2^*) be the optimal solutions of problem (1) and (1′), respectively. Since $x_1^*(t-L) = y_1^*(t)$, $\forall\ t \in [r_1^*, T]$, $x_2^*(t) = y_2^*(t)$ $\forall\ t \in [r_2^*, T]$ and $r_1^* = t_1^* + L$, $r_2^* = t_2^*$, then by (14), (21) and (30), (40), (50), we have

Case 1. Suppose $\frac{h_1}{c_1} > \frac{h_2 - h_1}{c_2}$.

(1.1) If $p \geq h_2(T-L) + h_{11}L + (p+b)F\left(\frac{h_2}{4(c_1+c_2)}(T-L)^2\right)$, then

$$x_1^*(t) = \frac{h_2}{4(c_1+c_2)}t^2 + \left[\frac{x_2^*(T)}{T-L} - \frac{h_2}{4(c_1+c_2)}(T-L)\right]t, \quad 0 \leq t \leq T-L$$

$$x_2^*(t) = \frac{h_2}{4(c_1+c_2)}t^2 + \left[\frac{x_2^*(T)}{T-L} - \frac{h_2}{4(c_1+c_2)}(T+L)\right](t-L)$$
$$- \frac{h_2}{4(c_1+c_2)}L^2, \qquad L \leq t \leq T$$

where the value of $x_2^*(T)$ is determined by the following equation

$$h_2 T + 2(c_1+c_2)\left[\frac{x_2^*(T)}{T-L} - \frac{h_2}{4(c_1+c_2)}(T+L)\right]$$
$$= p - h_{11}L - (p+b)F(x_2^*(T)). \qquad (51)$$

(1.2) If $p < h_2(T-L) + h_{11}L + (p+b)F\left(\frac{h_2}{4(c_1+c_2)}(T-L)^2\right)$, then

$$x_1^*(t) = \frac{h_2}{4(c_1 + c_2)}(t - t_2^* + L)^2 \qquad \forall\ t \in [t_2^* - L, T - L]$$

$$x_2^*(t) = \frac{h_2}{4(c_1 + c_2)}(t - t_2^*)^2 \qquad \forall\ t \in [t_2^*, T]$$

where the value of t_2^* is determined by the following equation

$$p = h_2(T - t_2^*) + h_{11}L + (p + b)F\left(\frac{h_2}{4(c_1 + c_2)}(T - t_2^*)^2\right). \tag{52}$$

Case 2. Suppose $\frac{h_1}{c_1} \le \frac{h_2 - h_1}{c_2}$.

(2.1) If $p \ge \frac{T - L}{2}\left(\frac{c_1(h_2 - h_1) - c_2 h_1}{c_2}\right) + h_2(T - L) + h_{11}L$

$$+ (p + b)F\left(\frac{h_2 - h_1}{4c_2}(T - L)^2\right),$$

then

$$x_1^*(t) = \frac{h_1}{4c_1}t^2 + \left[\frac{x_2^*(T)}{T - L} - \frac{h_1}{4c_1}(T - L)\right]t, \qquad 0 \le t \le T - L$$

$$x_2^*(t) = \frac{h_2 - h_1}{4c_2}t^2 + \left[\frac{x_2^*(T)}{T - L} - \frac{h_2 - h_1}{4c_2}(T + L)\right](t - L)$$

$$- \frac{h_2 - h_1}{4c_2}L^2, \qquad L \le t \le T$$

where the value of $x_2^*(T)$ is determined by the following equation

$$2c_1\left[\frac{x_2^*(T)}{T - L} - \frac{h_1}{4c_1}(T + L)\right] + 2c_2\left[\frac{x_2^*(T)}{T - L} - \frac{h_2 - h_1}{4c_2}(T + L)\right] + h_2 T$$

$$- p + h_{11}L + (p + b)F(x_2^*(T)) = 0.$$

(2.2) If $(h_2 - h_1)(T - L)\sqrt{\frac{h_1}{c_1}\frac{c_2}{h_2 - h_1}} + (p + b)F\left(\frac{h_1}{4c_1}(T - L)^2\right)$

$$+ h_1(T - L) + h_{11}L$$

$$\le p$$

$$< \frac{T-L}{2}\left(\frac{c_1(h_2-h_1)-c_2h_1}{c_2}\right)+h_2(T-L)+h_{11}L$$

$$+(p+b)F\left(\frac{h_2-h_1}{4c_2}(T-L)^2\right),$$

then

$$x_1^*(t)=\frac{h_1}{4c_1}t^2+\left[\frac{1}{T-L}\frac{h_2-h_1}{4c_2}(T-t_2^*)^2-\frac{h_1}{4c_1}(T-L)\right]t,$$

$$0 \le t \le T-L$$

$$x_2^*(t)=\frac{h_2-h_1}{4c_2}(t-t_2^*)^2, \qquad t_2^* \le t \le T$$

where the value of t_2^* is determined by the following equation

$$2c_1\left[\frac{1}{T-L}\frac{h_2-h_1}{4c_2}(T-t_2^*)^2-\frac{h_1}{4c_1}(T+L)\right]+(h_2-h_1)(T-t_2^*)+h_1T$$

$$-p+h_{11}L+(p+b)F\left(\frac{h_2-h_1}{4c_2}(T-t_2^*)^2\right)=0.$$

(2.3) If $p<(h_2-h_1)(T-L)\sqrt{\frac{h_1}{c_1}\frac{c_2}{h_2-h_1}}+h_1(T-L)+h_{11}L$

$$+(p+b)F\left(\frac{h_1}{4c_1}(T-L)^2\right),$$

then

$$\begin{cases} x_1^*(t)=\frac{h_1}{4c_1}(t-t_1^*)^2 & t\in[t_1^*, T-L] \\ x_2^*(t)=\frac{h_2-h_1}{4c_2}(t-t_2^*)^2 & t\in[t_2^*, T] \end{cases}$$

where the value of t_1^* is determined by the following equation

$$(h_2-h_1)\left[(T-t_1^*-L)\sqrt{\frac{h_1}{c_1}\frac{c_2}{h_2-h_1}}\right]+h_1(T-t_1^*-L)$$

$$-p+h_{11}L+(p+b)F\left(\frac{h_1}{4c_1}(T-t_1^*-L)^2\right)=0$$

and

$$t_2^* = T - (T - t_1^* - L)\sqrt{\frac{h_1}{c_1}\frac{c_2}{h_2 - h_1}}$$

5. CONCLUSION

To analyze the characteristics of the optimal solution, the following equations should be considered first:

$$\frac{h_1}{c_1} = \frac{h_2 - h_1}{c_2} \tag{54}$$

$$[p - h_2(T-L) - h_{11}L]\left(1 - F\left(\frac{h_2}{4(c_1+c_2)}(T-L)^2\right)\right)$$
$$= [h_2(T-L) + h_{11}L + b]F\left(\frac{h_2}{4(c_1+c_2)}(T-L)^2\right). \tag{55}$$

Equation (55) can be interpreted as the unit loss of goods surplus is equal to the unit loss of good lacking. Then, there are some features of the optimal solution which are described as follows:

1. If (54) is satisfied, then the optimal solutions of Case 1 and Case 2 are identical. In other words, to determine whether the production rates of semi-finished and finished goods are the same or not is still dependent on the sign of the value $\frac{h_1}{c_1} - \frac{h_2 - h_1}{c_2}$, but not on L.
2. Equation (55) divides the optimal solution of Case 1 into Cases (1.1) and (1.2). And, in this situation, the optimal quantity of goods for sale $x_2^*(T) = \frac{h_2}{4(c_1+c_2)}(T-L)^2$. It means that if the leadtime, L, increases then the optimal quantity of goods for sale $x_2^*(T)$ will decrease. Furthermore, the optimal quantity of goods for sale $x_2^*(T)$ is proportional to square of, $T - L$, the length of available time interval for production.
3. If the unit loss of goods surplus is larger than the unit loss of goods lacking then the optimal solution is Case (1.1). And the optimal quantity of goods for sale $x_2^*(T) > \frac{h_2}{4(c_1+c_2)}(T-L)^2$.

(1.2). And $x_2^*(T) = \frac{h_2}{4(c_1+c_2)}(T-t_2^*)^2 < \frac{h_2}{4(c_1+c_2)}(T-L)^2$. Then, by (52), (55) is the special case of Case (1.2) as t_2^* approaches L.

4. In Case 1, the optimal production rates of semi-finished and finished goods are the same. Hence the phenomenon of the two-stages production process is equivalent to one-stage production process [3].

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