# Operator Algebra in Chern-Simons Theory on a Torus 

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(Received 28 September 1992)


#### Abstract

We consider Chern-Simons gauge theory on a torus with both nonrelativistic and relativistic matter. It is shown that the Hamiltonian and two total momenta commute among themselves only in the physical Hilbert space. We also discuss relations among degenerate physical states, degenerate vacua, and the existence of multicomponent Schrödinger wave functions.


PACS numbers: 11.15.-q, 05.30.-d, 74.20.Mn

Chern-Simons field theory with matter coupling has attracted intense interest in recent years, owing to its relevance to condensed matter systems such as quantum Hall systems, and possibly high- $T_{c}$ superconductors [1]. Setting aside such physical applicability, Chern-Simons theory is in itself very interesting in view of its rich and beautiful mathematical structures. As such, various aspects of this theory deserve a careful study. Many recent efforts have been directed towards the issue of consistent quantization of the theory [2,3].

While the majority of works in the field is concerned with planar systems, Chern-Simons field theory on compact Riemann surfaces has also captured considerable interests [4-14]. It has even richer structures which, being topological in nature, are absent in planar systems. Among them are the multicomponent structure of manybody wave functions $[9,11,14]$ and the degeneracy of physical states [5]. Moreover, the analysis on a torus is mathematically rigorous, being free from infrared divergences and ambiguity in boundary conditions at space infinity on a plane.

We extend our previous analysis [14], examining algebraic relations among various operators, especially the Hamiltonian $H$ and total momenta $P^{k}$, with an eye on whether translation invariance is maintained or broken in the presence of matter. We shall also discuss a possible link between degeneracy of physical states and the multicomponent structure of wave functions.

It has been argued by Chen et al. [15] that the microscopic translation invariance of the anyon superconductivity model is broken in the mean field approximation, and is restored in the random phase approximation thanks to the presence of the phonon mode. Our consideration is at the microscopic level. We shall show that $H$ and $P^{k}$ do not commute with each other as operators, whereas they do commute in the physical Hilbert space. Therefore, if an effective theory is formulated in terms of physical excitations, the translation invariance must be maintained manifestly in each mode. Similar arguments have been given by Iengo, Lechner, and Li [13] on a torus, and by Banerjee [16] on a plane. Our argument, however, differs from theirs in detail.

We consider two models, Chern-Simons gauge theories on a torus with a nonrelativistic matter field and with a relativistic Dirac field. We shall find algebraic relations universal in both theories.
We first analyze the nonrelativistic case. The Lagrangian is given by

$$
\begin{equation*}
\mathcal{L}=\frac{\kappa}{4 \pi} \varepsilon^{\mu \nu \rho} a_{\mu} \partial_{\nu} a_{\rho}+\mathcal{L}_{\text {matter }} \tag{1}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{L}_{\text {matter }}=\frac{i}{2}\left\{\psi^{\dagger} D_{0} \psi-\left(D_{0} \psi\right)^{\dagger} \psi\right\}-\frac{1}{2 m}\left(D_{k} \psi\right)^{\dagger}\left(D_{k} \psi\right), \tag{2}
\end{equation*}
$$

$D_{0}=\partial_{0}+i a_{0}$, and $D_{k}=\partial_{k}-i a^{k}$. Consistency of the theory in the presence of matter requires that the coefficient of the Chern-Simons term $\kappa$ be a fractional ratio, $\kappa=N /$ $M$, where $N$ and $M$ are coprime integers [7,8]. The fundamental domain of the torus is given by $0 \leq x_{j} \leq L_{j}$, $j=1,2$. Boundary conditions of the fields are then $[4,14]$

$$
\begin{align*}
& a_{\mu}\left[T_{j}(x)\right]=a_{\mu}[x]+\partial_{\mu} \beta_{j}(x), \\
& \psi\left[T_{j}(x)\right]=e^{-i \beta_{j}(x)} \psi(x), \tag{3}
\end{align*}
$$

where $T_{j}: x_{j} \rightarrow x_{j}+L_{j}(j=1,2)$. That is, the fields return to their original values up to gauge transformations after translations along noncontractible loops. The requirement of smoothness of the field operator $\psi(x)$ in the covering space, $\psi\left[T_{1} \cdot T_{2}(x)\right]=\psi\left[T_{2} \cdot T_{1}(x)\right]$, leads to quantization of the Chern-Simons flux, $\Phi=\int d \mathbf{x} f_{12}$ $=2 \pi m$ ( $m$ : integers), and a constraint on the $\beta_{j}$ 's: $\left\{\beta_{1}\left(T_{2} x\right)-\beta_{1}(x)\right\}-\left\{\beta_{2}\left(T_{1} x\right)-\beta_{2}(x)\right\}=-2 \pi m$. Typical $\beta_{j}$ 's which solve this constraint are $\beta_{j}(x)$ $=-\epsilon^{j k} \pi m x_{k} / L_{k}$, which will be taken in the rest of the paper.

Canonical energy-momentum tensors, $T_{c}^{\mu v}$, derived from (1) and (2) are not gauge invariant, and therefore are not well defined on a torus in view of the boundary conditions (3). The gauge-invariant energy-momentum tensors $T_{I}^{\mu \nu}$ are obtained by adding the term $(\kappa / 4 \pi)$ $\times \partial_{\rho}\left(\epsilon^{\rho \mu \sigma} a_{\sigma} a^{\nu}\right)$ to $T_{c}^{\mu \nu}$ and making use of the equations of motion [3]:

$$
\begin{align*}
T_{I}^{\mu \nu}= & {\left[\begin{array}{c}
\frac{i}{2}\left\{\psi^{\dagger} D^{v} \psi-\left(D^{v} \psi\right)^{\dagger} \psi\right\} \\
-\frac{1}{2 m}\left\{\left(D^{k} \psi\right)^{\dagger} D^{\nu} \psi+\left(D^{v} \psi\right)^{\dagger} D^{k} \psi\right\}
\end{array}\right] } \\
& -g^{\mu \nu}\left\{\frac{i}{2}\left[\psi^{\dagger} D^{0} \psi-\left(D^{0} \psi\right)^{\dagger} \psi\right]-\frac{1}{2 m}\left(D^{k} \psi\right)^{\dagger} D^{k} \psi\right\}, \tag{4}
\end{align*}
$$

where the upper (lower) entries in the square brackets give the $\mu=0(\mu=k)$ component of $T_{I}^{\mu \nu}$. From (4) we obtain the Hamiltonian and total momentum operators

$$
\begin{align*}
& H=\frac{1}{2 m} \int d \mathbf{x}\left(D_{k} \psi\right)^{\dagger}\left(D_{k} \psi\right), \\
& P^{k}=-i \int d \mathbf{x} \psi^{\dagger} D_{k} \psi \tag{5}
\end{align*}
$$

To quantize the theory we note that the Chern-Simons field equation $(\kappa / 4 \pi) \varepsilon^{\mu \nu \rho} f_{v \rho}=j^{\mu}$ implies that ChernSimons fields $a_{\mu}(x)$ are determined by the matter field except for nonintegrable phases of Wilson line integrals along noncontractible loops on a torus. Solving the field equations in the radiation gauge diva $=0$, one finds [4]
$a^{j}(x)=\frac{\theta_{j}(t)}{L_{j}}-\frac{\Phi}{2 L_{1} L_{2}} \epsilon^{j k} x_{k}+\hat{a}^{j}(x)$,
$\hat{a}^{j}(x)=\frac{2 \pi}{\kappa} \int d \mathbf{y} \epsilon^{j k} \nabla_{k}^{x} G(\mathbf{x}-\mathbf{y})\left(j^{0}(y)+\frac{\kappa \Phi}{2 \pi L_{1} L_{2}}\right)$,
$a_{0}(x)=-\frac{2 \pi}{\kappa} \int d \mathbf{y} G(\mathbf{x}-\mathbf{y})\left(\partial_{1} j^{2}-\partial_{2} j^{1}\right)(y)$,
where $j^{0}=\psi^{\dagger} \psi, j^{k}=-(i / 2 m)\left\{\psi^{\dagger} D_{k} \psi-\left(D_{k} \psi\right)^{\dagger} \psi\right\}$, and $G(\mathrm{r})$ is the periodic Green's function on a torus, satisfying $\Delta G(\mathrm{r})=\delta(\mathrm{r})-\left(1 / L_{1} L_{2}\right)$.

The constant parts $\theta_{j}$ are nonintegrable phases of the Wilson line integrals. Residual gauge transformations that respect the boundary conditions (3) are given by a gauge function $\Lambda(x)=-\left(\sum_{j} 2 \pi n_{j} x_{j} / L_{j}\right)+\tilde{\Lambda}(x)$ where $n_{j}$ 's are integers and $\tilde{\Lambda}(x)$ is periodic. The degree $\tilde{\Lambda}(x)$ has been made use of to obtain (6), whereas the rest of $\Lambda(x)$ constitutes large gauge transformations inducing $\theta_{j} \rightarrow \theta_{j}+2 \pi n_{j}$. The invariance under large gauge transformations must be imposed on physical quantities.

In quantum theory, physical degrees of freedom are the matter fields $\psi$ (taken to be fermionic), and the nonintegrable phases $\theta_{j}$ 's. $\psi$ 's satisfy $\left\{\psi(\mathbf{x}), \psi^{\dagger}(\mathbf{y})\right\}=\delta(\mathbf{x}-\mathbf{y})$ in the fundamental domain of the torus. $\theta_{1}$ and $\theta_{2}$ are canonical conjugates of each other (Refs. [4,5]), $\left[\theta_{1}, \theta_{2}\right]$ $=2 \pi i / \kappa$, as the Lagrangian contains $(\kappa / 4 \pi)\left(\theta_{2} \dot{\theta}_{1}-\theta_{1} \dot{\theta}_{2}\right)$. In writing the Hamiltonian (5) in terms of these variables with the aid of (6), there arises an ordering ambiguity. We shall take the ordering of $\psi$ and $\psi^{\dagger}$ as it is in (5).

The equation of motion for $\theta_{j}$ is

$$
\begin{equation*}
\dot{\theta}_{j}=\frac{1}{i}\left[\theta_{j}, H\right]=-\epsilon^{j k} \frac{2 \pi}{\kappa L_{k}} J^{k} \tag{7}
\end{equation*}
$$

where $J^{k} \equiv \int d \mathbf{x} j^{k}$. For $\psi(x)$

$$
\begin{equation*}
i \dot{\psi}(x)=\left\{-\frac{1}{2 m} D_{k}^{2}+a_{0}(x)+g(x)\right\} \psi(x) \tag{8}
\end{equation*}
$$

where

$$
g(x)=(1 / 2 m)(2 \pi / \kappa)^{2} \int d \mathbf{y}\left[\nabla_{k} G(\mathbf{x}-\mathbf{y})\right]^{2} \psi^{\dagger} \psi(y)
$$

Equation (8) differs from the classical equation by the $g(x)$ term [14]. (See also Ref. [3] for an analogous result on a plane.)

It is important to recognize that the expression (6) is not completely equivalent to the Chern-Simons field equations $(\kappa / 4 \pi) \varepsilon^{\mu \nu \rho} f_{\nu \rho}=j^{\mu}$. Insertion of (6) into $a_{\mu}$ 's in the equations yields two nontrivial relations, one Eq. (7) and the other

$$
\begin{equation*}
Q+\frac{\kappa}{2 \pi} \Phi \approx 0 \tag{9}
\end{equation*}
$$

where $Q=\int d \mathbf{x} j^{0} . \Phi$ is the flux fixed by the boundary conditions (3) with the given $\beta_{j}(x)$ 's, and $Q$ is conserved as a consequence of Eq. (8). Since the relation (9) does not follow from the Hamiltonian and commutation relations, it has to be imposed as a constraint. We have adopted the notation $\approx$ to signify this.

We now compute commutators of $P^{k}$ and $H$. Note that since $\int d \mathbf{x} \psi^{\dagger} \hat{a}^{k} \psi=0$,

$$
P^{k}=-i \int d \mathbf{x} \psi^{\dagger} D_{k} \psi=-i \int d \mathbf{x} \psi^{\dagger} \bar{D}_{k} \psi
$$

where

$$
\bar{D}_{k} \equiv \partial_{k}-i\left(\theta_{k} / L_{k}\right)+i\left(\epsilon^{k l} x_{l} \Phi / 2 L_{1} L_{2}\right) .
$$

It follows that

$$
\begin{equation*}
\left[P^{k}, \psi(x)\right]=i \bar{D}_{k} \psi(x), \quad\left[P^{k}, \theta_{j}\right]=-\epsilon^{k j} \frac{2 \pi i}{\kappa L_{k}} Q \tag{10}
\end{equation*}
$$

In particular, the change in a gauge-invariant operator generated by $P^{k}$ is a total derivative. For instance, $\left[P^{k}, \psi^{\dagger} \psi(x)\right]=i \partial_{k}\left\{\psi^{\dagger} \psi(x)\right\}$. With the aid of (10) commutators among the operators $P^{k}$ and $H$ are found to be

$$
\begin{align*}
& {\left[P^{j}, P^{k}\right]=i \epsilon^{j k} \frac{2 \pi}{\kappa L_{1} L_{2}} Q\left(Q+\frac{\kappa}{2 \pi} \Phi\right)} \\
& {\left[P^{j}, H\right]=i \epsilon^{j k} \frac{2 \pi}{\kappa L_{1} L_{2}} J^{k}\left(Q+\frac{\kappa}{2 \pi} \Phi\right)} \tag{11}
\end{align*}
$$

Note that $J^{k}=P^{k} / m$ in nonrelativistic theory.
$P^{k}$ s and $H$ commute among themselves only up to the constraint (9). Hence in the physical Hilbert space these operators commute, and translation invariance is maintained. Our conclusion differs from Iengo, Lechner, and Li's claim [13] that $H$ and $P^{k}$ commute among themselves as operators. Jackiw and Nair's construction of relativistic wave equations of anyons [17] and Banerjee's analysis of Poincaré algebra in Chern-Simons theory [16] are in harmony with ours.

There are two other sets of important operators, Wilson line operators $W_{j}=e^{i \theta_{j}}$ and generators of large gauge transformation,

$$
U_{j}=\exp \left\{i \epsilon^{j k} \kappa \theta_{k}-2 \pi i \int d \mathbf{x}\left(x_{j} / L_{j}\right) \psi^{\dagger} \psi(x)\right\}
$$

[ $U_{j}$ is well defined. If the coefficient of the integral were a fraction of $2 \pi i$, there would arise inconsistency in $U_{j} \psi(x) U_{j}^{-1}$ combined with (3).]
$U_{j}$ and $W_{j}$ satisfy dual relations $W_{1} W_{2}=e^{-2 \pi i / \kappa}$ $\times W_{2} W_{1}, \quad U_{1} U_{2}=e^{-2 \pi i \kappa} U_{2} U_{1}$, and $\left[W_{k}, U_{j}\right]=0$. Also $\left[U_{j}, P^{k}\right]=\left[U_{j}, H\right]=0$, as $P^{k}$ and $H$ are gauge invariant. Commutator relations among $W_{j}, P^{k}$, and $H$ are, however, nontrivial:

$$
\begin{align*}
& {\left[W_{j}, P^{k}\right]=\epsilon^{j k} \frac{2 \pi}{\kappa L_{k}} Q W_{j}}  \tag{12}\\
& {\left[W_{j}, H\right]=\epsilon^{j k} \frac{\pi}{\kappa L_{k}}\left(J^{k} W_{j}+W_{j} J^{k}\right) .}
\end{align*}
$$

In the nonrelativistic theory $J^{k}=P^{k} / m$ so that $W_{j}$ 's map an eigenstate into another eigenstate corresponding to different momenta and energy.

Most of the above results can be directly carried over to Chern-Simons gauge theory coupled to a Dirac field. In place of (2) we have

$$
\mathcal{L}_{\text {matter }}=\frac{i}{2}\left\{\bar{\psi} \gamma^{\mu} D_{\mu} \psi-\left(\overline{D_{\mu} \psi}\right) \gamma^{\mu} \psi\right\}-m \bar{\psi} \psi .
$$

The current is given by $j^{\mu}=\bar{\psi} \gamma^{\mu} \psi$. Gauge-invariant energy-momentum tensors are given by $T_{I}^{\mu \nu}=(i / 2)$ $\times\left\{\bar{\psi} \gamma^{\mu} D^{\nu} \psi-\left(\overline{D^{v} \psi}\right) \gamma^{\mu} \psi\right\}$. It follows that

$$
\begin{align*}
& H=\int d \mathbf{x} \bar{\psi}\left(-i \gamma^{k} D_{k}+m\right) \psi \\
& P^{k}=-i \int d \mathbf{x} \psi^{\dagger} D_{k} \psi \tag{13}
\end{align*}
$$

Note that in the Dirac case $J^{k}$ and $P^{k}$ are independent quantities.

Most of the relations obtained for the nonrelativistic case remain valid for the Dirac case with the substitution $j^{\mu}=\bar{\psi} \gamma^{\mu} \psi$ being made. The only change to be made is the equation for $\psi$ : $i \dot{\psi}=\gamma^{0}\left(-i \gamma^{k} D_{k}+m+a_{0} \gamma^{0}\right) \psi(x)$. The relation (6) and the constraint (9) remain intact. Direct computations confirm (7), (10), and particularly the fundamental algebraic relations (11) and (12). $J^{k}$ is not conserved even in the physical Hilbert space, however. Therefore $W_{j}$ no longer maps an eigenstate of $H$ into another.

We stress that the relations (11) and (12) are universal. They are independent of details of theories.

We now return to the nonrelativistic theory and consider the representation of $P^{k}$ and $H$ in the corresponding quantum-mechanical anyon system. The case of an integer $\kappa=N$ has been analyzed in Ref. [14]. There are $N$ degenerate vacua $\left|0_{a}\right\rangle(a=0, \ldots, N-1)$. $q$-body Schrödinger wave functions are given by
$\phi_{a}\left(\mathbf{x}_{1}, \ldots, \mathbf{x}_{q} ; t\right)=(q!)^{-1 / 2}\left\langle 0_{a}\right| \Omega \psi\left(x_{1}\right) \cdots \psi\left(x_{q}\right)\left|\Psi_{q}\right\rangle$, where

$$
\Omega=\exp \left\{-i \sum_{j=1}^{q}\left[\left(x_{1}^{(j)} \theta_{1} / L_{1}\right)+\left(x_{2}^{(j)} \theta_{2} / L_{2}\right)\right]\right\}
$$

The operator $\Omega$ is necessary to insure invariance under large gauge transformations. Wave functions must have $N$ components as a consequence of the vacuum degeneracy. They realize the braid group algebra on a torus.

The representation of $P^{k}$,

$$
\hat{P}^{k} \phi_{a} \equiv(q!)^{-1 / 2}\left\langle 0_{a}\right| \Omega \psi\left(x_{1}\right) \cdots \psi\left(x_{q}\right) P^{k}\left|\Psi_{q}\right\rangle
$$

is found by permuting $P^{k}$ to the left of the $\psi$ 's and $\Omega$. The result is simple:

$$
\begin{equation*}
\hat{P}^{k}=-i \sum_{j=1}^{q} \nabla_{k}^{(j)} \tag{14}
\end{equation*}
$$

The Hamiltonian $\hat{H}$ is [14]

$$
\begin{align*}
& \hat{H}=-\frac{1}{2 m} \sum_{j=1}^{q}\left[\nabla_{k}^{(j)}-i \tilde{a}_{f}^{k}\left(\mathbf{r}_{j}\right)\right]^{2}, \\
& \tilde{a}_{f}^{k}\left(\mathbf{r}_{j}\right)=\frac{2 \pi}{\kappa} \epsilon^{k l} \sum_{p \neq j}\left\{\frac{1}{2 L_{1} L_{2}}\left(x_{l}^{(j)}-x_{l}^{(p)}\right)+\partial_{l}^{(j)} G\left(\mathbf{r}_{j}-\mathbf{r}_{p}\right)\right\} \tag{15}
\end{align*}
$$

It is obvious that all $\hat{P}^{k}$ and $\hat{H}$ commute with each other. The same conclusion has been reached by Iengo, Lechner, and Li [13] in a different formulation in which the nonintegrable phases appear explicitly in the expressions of $\hat{P}^{k}$ and $\hat{H}$. Equations (14) and (15) do not contain the $\theta_{j}$ 's.

For the general case where $\kappa$ is fractional, $\kappa=N / M$ ( $N, M$ coprime), it is known that there are $N M$ degenerate vacua $[7,8]$. At present there are two approaches of interpreting the nature of these vacua.

Following Polychronakos [8], one might consider, of the $N M$ possible vacua, only $N$ distinct physical vacua, each having $M$ gauge copies as generated by $U_{j}$. Hence one considers a fixed combination of these gauge copies to represent a physical vacuum. This can be seen as a sort of "gauge fixing." Adopting the same procedure to our case, we will still have a multicomponent wave function, but now $\left|\Psi_{q}\right\rangle$ and $\left|0_{a}\right\rangle(a=0, \ldots, N-1)$ are linear combinations of $M$ different gauge-equivalent physical states and vacua, respectively.

The situation is different, however, if we do not regard $U_{j}$ as operators of gauge transformation, but instead as physical operators generated by some physical tunneling processes. Such consideration is particularly appropriate when the model defined by (1) and (2) represents an effective theory of, for instance, fractional quantum Hall effect (FQHE) (with external magnetic fields added). This leads to the conclusion that physical states must be degenerate as emphasized by Wen and Niu [5]. What we have seen here is that, not only must physical states be degenerate, but their wave functions must also have multiple components.

To be precise, let us denote the $N M$ degenerate vacua by $\left|0_{a b}\right\rangle(a=0, \ldots, N-1 ; b=0, \ldots, M-1)$. Noting that $U_{1}^{M}$ and $U_{2}^{M}$ commute with each other, we choose them to be eigenstates of $U_{i}^{M}: \quad U_{i}^{M}\left|0_{a b}\right\rangle=e^{i \lambda_{i}}\left|0_{a b}\right\rangle \quad(j$ $=1,2$ ). Then in the $\theta_{1}$ representation one finds [18] that

$$
\begin{aligned}
u_{a b}\left(\theta_{1}\right) & =\left\langle\theta_{1} \mid 0_{a b}\right\rangle \\
& =e^{i b\left(\lambda_{2}+N \theta_{1}\right) / M+i \lambda_{1} \theta_{1} / 2 \pi M} \delta_{2 \pi}\left[\theta_{1}+\left(\lambda_{2}-2 \pi M a\right) / N\right]
\end{aligned}
$$

Actions of the $U_{i}, W_{j}$ on $\left|0_{a b}\right\rangle$ are

$$
\begin{align*}
& U_{1}\left|0_{a b}\right\rangle=e^{i\left(\lambda_{1}+2 \pi N b\right) / M}\left|0_{a b}\right\rangle, \\
& W_{1}\left|0_{a b}\right\rangle=e^{-i\left(\lambda_{2}-2 \pi M a\right) / N}\left|0_{a b}\right\rangle,  \tag{16}\\
& U_{2}\left|0_{a b}\right\rangle=e^{i \lambda_{2} / M}\left|0_{a, b-1}\right\rangle, \\
& W_{2}\left|0_{a b}\right\rangle=e^{i \lambda_{1} / N}\left|0_{a-1, b}\right\rangle .
\end{align*}
$$

We require the physical states $|\Psi\rangle$ to be invariant un$\operatorname{der} U_{i}^{-M}$ as well, $U_{i}^{-M}|\Psi\rangle=e^{i \omega_{i}}|\Psi\rangle$. Since $U_{1}$ and $U_{2}$
commute with $U_{1}^{M}$ and $U_{2}^{M}, U_{i}^{-1}|\Psi\rangle$ is also a physical state with the same eigenvalues. But as $U_{1}$ and $U_{2}$ do not commute, the states generated by them must be degenerate. It follows from $U_{1} U_{2}=e^{-2 \pi i N / M} U_{2} U_{1}$ that the $M$ degenerate states $\left|\Psi^{k}\right\rangle(k=0, \ldots, M-1)$ satisfy [5]

$$
\begin{equation*}
U_{1}^{-1}\left|\Psi^{k}\right\rangle=e^{i \nu k}\left|\Psi^{k}\right\rangle, \quad U_{2}^{-1}\left|\Psi^{k}\right\rangle=e^{i \nu v_{k}^{2}}\left|\Psi^{k+1}\right\rangle . \tag{17}
\end{equation*}
$$

In general, a $q$-particle state $\left|\Psi_{q}^{k}\right\rangle$ can be constructed from the vacua $\left|0_{a b}\right\rangle$ satisfying (16) by

$$
\begin{equation*}
\left|\Psi_{q}^{k}\right\rangle=\frac{1}{\sqrt{q!}} \sum_{a, b} \int d \mathbf{x}_{1} \cdots d \mathbf{x}_{q} \phi_{a b}^{k}\left(\mathbf{x}_{1}, \ldots, \mathbf{x}_{q} ; t\right) \psi^{\dagger}\left(\mathbf{x}_{q}\right) \cdots \psi^{\dagger}\left(x_{1}\right) \Omega^{\dagger}\left|0_{a b}\right\rangle /\left\langle 0_{a b} \mid 0_{a b}\right\rangle \tag{18}
\end{equation*}
$$

where
$\phi_{a b}^{k}\left(\mathbf{x}_{1}, \ldots, \mathbf{x}_{q} ; t\right)=(q!)^{-1 / 2}\left\langle 0_{a b}\right| \Omega \psi\left(x_{1}\right) \cdots \psi\left(x_{q}\right)\left|\Psi_{q}^{k}\right\rangle$.
To satisfy (17), however, $\left|\Psi_{q}^{k}\right\rangle$ can only involve $\left|0_{a k}\right\rangle$, i.e., $b=k$, since $\left|0_{a b}\right\rangle$ pick up different phases for different values of $b$ under the action of $U_{1}^{-1}$. Furthermore, (i) $v_{k}^{\prime}=-\left(\lambda_{1}+2 \pi N k\right) / M, v_{k}^{2}=-\lambda_{2} / M$; (ii) $\phi_{a b}^{k}=0$ for $b \neq k$ and $\phi_{a, k+1}^{k+1}=\phi_{a k}^{k}$; and (iii) $\omega_{j}=-\lambda_{j}$. So states are $M$ fold degenerate, and their wave functions take the form of $(N \times M)$-component matrices $\phi_{a b}^{k}$ with nonvanishing entries only in the $k$ th column.

Particularly, in the $\kappa=1 / M$ case, which is of relevance to FQHE, elementary particles have statistics $\theta_{s}=-M \pi$ and therefore they are either bosons or fermions depending on whether $M$ is odd or even. Many-body states are nevertheless $M$-fold degenerate, and their wave functions have $M$ components. Implications of these degenerate multicomponent wave functions in the braid group structure of quasiparticles have yet to be studied. In any case, independent of the two approaches just discussed, $\hat{P}^{k}$ and $\hat{H}$ are represented by (14) and (15), respectively.

This work is supported in part by the Theoretical Physics Institute and Republic of China Grant No. NSC 81-0208-M032-502 (C.-L.H.) and by the U.S. Department of Energy under Contract No. DE-AC02-83ER-40105 (Y.H.). C.-L.H. would like to thank L. McLerran, M. Voloshin, and the staff of the Theoretical Physics Institute for their hospitality and financial support. We also would like to thank R. Jackiw for useful comments.
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