

Similarity solutions of Fokker-Planck equation with moving boundaries

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Similarity solutions of Fokker-Planck equation with moving boundaries

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In this work we present new exact similarity solutions with moving boundaries of the Fokker-Planck equation having both time-dependent drift and diffusion coefficients.

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I. INTRODUCTION

One of the basic tools which is widely used for studying the effect of fluctuations in macroscopic systems is the Fokker-Planck equation (FPE).¹ This equation has found applications in such diverse areas as physics, chemistry, hydrology, biology, finance, and others. Because of its broad applicability, it is therefore of great interest to obtain solutions of the FPE for various physical situations.

Generally, it is not easy to find analytic solutions of the FPE, except in a few simple cases, such as linear drift and constant diffusion coefficients. In most cases, one can only solve the equation approximately, or numerically. Most of these methods, however, are concerned only with FPEs with time-independent diffusion and drift coefficients (for a review of these methods, see, e.g., Ref. 1).

Solving the FPEs with time-dependent drift and/or diffusion coefficient is in general an even more difficult task. It is therefore not surprising that the number of papers on such kind of FPE is far less than that on the FPE with time-independent coefficients. Some recent works on the FPE with time-dependent diffusion coefficients appear in Refs. 2–4, and works involving time-dependent drift coefficients can be found in Ref. 5–7. References 8–10 consider FPEs with both time-dependent diffusion and drift coefficients. The symmetry properties of the one-dimensional FPE with arbitrary coefficients of drift and diffusion are investigated in Ref. 10. Such properties may in some cases allow one to transform the FPE into one with constant coefficients.

One method of solving a differential equation using symmetry property is the so-called similarity method.¹¹ This method is especially useful when the system under consideration possesses scaling symmetry, i.e., when it has the property of self-similarity either on different time and/or space scales. It is interesting to note that many physical systems do have such property. For instance, systems with fractal structures,¹² systems undergoing 2nd-order phase transitions (as described by the renormalization group theory),¹³ Scaling in turbulence,¹⁴ and the intermediate state of a nuclear explosion (as explained by Taylor),¹⁵ etc. Most relevant to our discussion is the scaling in the well-known diffusion equation:

$$\frac{\partial}{\partial t} W(x, t) = D \frac{\partial^2}{\partial x^2} W(x, t), \quad (1)$$

which is a special case of the FPE. It is clear that this equation is invariant under the scale transformation: $x \rightarrow \epsilon x$, $t \rightarrow \epsilon^2 t$ (ϵ : real parameter). The solution that reflects this symmetry is the well-known solution:

$$W(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{1}{4D} \left(\frac{x}{\sqrt{t}}\right)^2}, \quad (2)$$

which scales as $W(x, t) \rightarrow \epsilon^{-1} W(x, t)$. More recently, scaling behavior is also reported in a generalized diffusion equation that was used to model certain financial market,¹⁶ and in the dynamics of bacterial populations governed by a generalized convection-reaction-diffusion equation.¹⁷

In Ref. 7 we have considered, within the framework of a perturbative approach, the similarity solutions of a class of FPEs which have constant diffusion coefficients and small time-dependent drift coefficients. The solvability of the FPE with both time-dependent drift and diffusion coefficients by means of the similarity method is explored in Ref. 18. One advantage of the similarity method is that it allows one to reduce the FPE to an ordinary differential equation which may be easier to solve, provided that the FPE possesses proper scaling property under certain scaling transformation of the basic variables. Some interesting exactly solvable cases of such FPE on the real line $x \in (-\infty, \infty)$ and the half lines $x \in [0, \infty)$ and $x \in (-\infty, 0]$ were considered in Ref. 18. These domains admit similarity solutions because their boundary points are the fixed points of the scaling transformation considered. This indicates that similarity solutions are not possible for other finite domains.

It is therefore natural to suspect that similarity solutions of FPE on a finite domain may be possible, if its boundary points scale accordingly. We are thus led to consider FPE with moving boundaries. The purpose of this note is to present three new classes of such FPEs. To the best of our knowledge, FPEs with time-dependent coefficients and moving boundaries have not been discussed before.

II. SCALING OF FOKKER-PLANCK EQUATION

We first recapitulate the scaling form of the FPE. The general form of the FPE in $(1 + 1)$ -dimension is

$$\frac{\partial W(x, t)}{\partial t} = \left[-\frac{\partial}{\partial x} D^{(1)}(x, t) + \frac{\partial^2}{\partial x^2} D^{(2)}(x, t) \right] W(x, t), \quad (3)$$

where $W(x, t)$ is the probability distribution function, $D^{(1)}(x, t)$ is the drift coefficient, and $D^{(2)}(x, t)$ is the diffusion coefficient. The drift coefficient represents the external force acting on the particle, while the diffusion coefficient accounts for the effect of fluctuation. $W(x, t)$ as a probability distribution function should be normalized, i.e., $\int_{\text{domain}} W(x, t) dx = 1$ for $t \geq 0$.

We shall be interested in seeking similarity solutions of the FPE, which are possible if the FPE possesses certain scaling symmetry.

Consider the scale transformation:

$$\bar{x} = \varepsilon^a x, \quad \bar{t} = \varepsilon^b t, \quad (4)$$

where ε , a , and b are real parameters. Suppose under this transformation, the probability density function and the two coefficients scale as

$$\bar{W}(\bar{x}, \bar{t}) = \varepsilon^c W(x, t), \quad \bar{D}^{(1)}(\bar{x}, \bar{t}) = \varepsilon^d D^{(1)}(x, t), \quad \bar{D}^{(2)}(\bar{x}, \bar{t}) = \varepsilon^e D^{(2)}(x, t). \quad (5)$$

Here c , d , and e are also some real parameters. It can be checked that the transformed equation in terms of the new variables has the same functional form as Eq.(3) if the scaling indices satisfy $b = a - d = 2a - e$. In this case, the second order FPE can be transformed into an ordinary differential equation which is easier to solve. Such reduction is effected through a new independent variable (called similarity variable), which is certain combinations of the old independent variables such that it is scaling invariant, i.e., no appearance of parameter ε , as a scaling transformation is performed. Here the similarity variable z is defined by

$$z \equiv \frac{x}{t^\alpha}, \quad \text{where } \alpha = \frac{a}{b} \text{ and } a, b \neq 0. \quad (6)$$

For $a, b \neq 0$, one has $\alpha \neq 0, \infty$.

The general scaling form of the probability density function $W(x, t)$ is $W(x, t) = (t^{\delta_1}/x^{\lambda_1})y(z)$, where λ_1 and δ_1 are two real parameters and $y(z)$ is a scale-invariant function under the same transformation (4). From the assumed scaling behavior of $W(x, t)$ in (5), we have $-c/a = \lambda_1 - (\delta_1/\alpha)$. Without loss of generality and for clarity of presentation, we take the parameters $(\lambda_1, \delta_1) = (0, \alpha c/a)$. This gives

$$W(x, t) = t^{\alpha \frac{c}{a}} y(z), \quad (7)$$

where $y(z)$ is a function of z . The normalization of the distribution function is

$$\int_{\text{domain}} W(x, t) dx = \int_{\text{domain}} \left[t^{\alpha(1+\frac{c}{a})} y(z) \right] dz = 1. \quad (8)$$

For the above relation to hold at all $t \geq 0$, the power of t should vanish, and so one must have $c = -a$, and thus

$$W(x, t) = t^{-\alpha} y(z). \quad (9)$$

Similar consideration leads to the following scaling forms of the drift and diffusion coefficients:

$$D^{(1)}(x, t) = t^{\alpha-1} \rho_1(z), \quad D^{(2)}(x, t) = t^{2\alpha-1} \rho_2(z), \quad (10)$$

where $\rho_1(z)$ and $\rho_2(z)$ are scale invariant functions of z .

With Eqs. (6), (7), and (10), the FPE is reduced to

$$\rho_2(z) y''(z) + \left[2\rho_2'(z) - \rho_1(z) + \alpha z \right] y'(z) + \left[\rho_2''(z) - \rho_1'(z) + \alpha \right] y(z) = 0, \quad (11)$$

where the prime denotes the derivative with respect to z . It is really interesting to realize that Eq. (11) is exactly integrable. Integrating it once, we get

$$\rho_2(z) y'(z) + \left[\rho_2'(z) - \rho_1(z) + \alpha z \right] y(z) = C, \quad (12)$$

where C is an integration constant. Solution of Eq. (12) is

$$y(z) = \left(C' + C \int^z dz \frac{e^{-\int^z dz f(z)}}{\rho_2(z)} \right) \exp \left(\int^z dz f(z) \right),$$

$$f(z) \equiv \frac{\rho_1(z) - \rho_2'(z) - \alpha z}{\rho_2(z)}, \quad \rho_2(z) \neq 0, \quad (13)$$

where C' is an integration constant.

To proceed further, let us consider the boundary conditions of the probability density $W(x, t)$ and the associated probability current density $J(x, t)$. From the continuity equation:

$$\frac{\partial}{\partial t} W(x, t) = -\frac{\partial}{\partial x} J(x, t), \quad (14)$$

we have

$$J(x, t) = D^{(1)}(x, t) W(x, t) - \frac{\partial}{\partial x} \left[D^{(2)}(x, t) W(x, t) \right]. \quad (15)$$

Using Eqs. (6), (9), and (10), we get

$$J(x, t) = t^{-1} \left[(\rho_1(z) - \rho_2'(z)) y(z) - \rho_2(z) y'(z) \right]. \quad (16)$$

From Eq. (12), we can reduce the above equation to

$$J(x, t) = \frac{1}{t} [\alpha z y(z) - C] = \frac{1}{t} [\alpha x W(x, t) - C], \quad (17)$$

where $W(x, t) = t^{-\alpha} y(z)$ and $z = xt^\alpha$ have been used in obtaining the second expression.

In this paper we consider boundaries which are impenetrable. Hence we must have

$$W(x, t)|_{\text{boundary}} = J(x, t)|_{\text{boundary}} = 0. \quad (18)$$

These conditions imply that $C = 0$, and that $J(x, t)$ is proportional to $W(x, t)$ and x .

With $C = 0$, the probability density function $W(x, t)$ is given by

$$W(x, t) = A t^{-\alpha} \exp \left(\int_{z=\frac{x}{t^\alpha}}^z dz f(z) \right), \quad (19)$$

where A is the normalization constant. It is interesting to see that the similarity solution of the FPE can be given in such an analytic closed form. Exact similarity solutions of the FPE can be obtained as long as $\rho_1(z)$ and $\rho_2(z)$ are such that the function $f(z)$ in Eq. (19) is an integrable function and

the resulted $W(x, t)$ is normalizable. Equivalently, for any integrable function $f(z)$ such that $W(x, t)$ is normalizable, if one can find a function $\rho_2(z)$ ($\rho_1(z)$ is then determined by $f(z)$ and $\rho_2(z)$), then one obtains an exactly solvable FPE with similarity solution given by (19). Some interesting cases of such FPE on the real line $x \in (-\infty, \infty)$ and the half lines $x \in [0, \infty)$ and $x \in (-\infty, 0]$ were discussed in Ref. 18.

That the half and whole lines can accommodate similarity solutions which allow x to scale is obvious: the points $x = 0$ and $x = \pm\infty$ are the fixed points of the scale transformation (4). Other finite intervals do not allow similarity solutions, unless their end points scale accordingly. We are thus led to FPE with moving boundaries.

In what follows we shall present three classes of exactly solvable FPE of such kind.

III. CLASS I: TWO MOVING BOUNDARIES

We consider a finite domain $x_1(t) \leq x \leq x_2(t)$ with impenetrable moving boundaries at $x_k(t)$ ($k = 1, 2$). We want the transformed FPE in z -space can be exactly solvable. So the simplest choice is such that in the z -space the boundary points of the corresponding domain are static. This implies $z_k = x_k(t)/t^\alpha$ ($k = 1, 2$) are constants. We will assume this choice below. We note here that the fixed domains admitting similarity solutions considered in Ref. 18 correspond to the choice $z = x(t)/t^\alpha = 0, \pm\infty$, which are just the fixed points of the scaling transformation.

Now let us assume the function $f(z)$ to have the following form in the physical domain in the z -space:

$$f(z) = \frac{a_1}{z - z_1} - \frac{a_2}{z_2 - z}, \quad a_1, a_2 > 0, z_1 \leq z \leq z_2. \quad (20)$$

This leads to a choice of $\rho_1(z)$ and $\rho_2(z)$:

$$\rho_2(z) = (z - z_1)(z_2 - z), \quad (21)$$

$$\rho_1(z) = (\alpha - a_1 - a_2 - 2)z + (a_1 + 1)z_2 + (a_2 + 1)z_1 \quad (22)$$

for $z_1 \leq z \leq z_2$, and $\rho_1(z), \rho_2(z) = 0$ otherwise. The function $y(z)$ in the physical domain is

$$y(z) = A(z - z_1)^{a_1}(z_2 - z)^{a_2}. \quad (23)$$

Here the normalization constant A is given by

$$A = [(z_2 - z_1)^{a_1+a_2+1} B(a_1 + 1, a_2 + 1)]^{-1}, \quad (24)$$

where $B(x, y)$ is the Beta function.

The probability density function is

$$W(x, t) = \begin{cases} \frac{A}{t^\alpha} \left(\frac{x}{t^\alpha} - z_1\right)^{a_1} \left(z_2 - \frac{x}{t^\alpha}\right)^{a_2}, & z_1 t^\alpha \leq x \leq z_2 t^\alpha \\ 0, & \text{otherwise} \end{cases}. \quad (25)$$

There are three subclasses:

$$(i) z_1, z_2 > 0 \quad (z_1, z_2 < 0), \quad (26)$$

$$(ii) z_1 = 0, z_2 > 0 \quad (z_1 < 0, z_2 = 0), \quad (27)$$

$$(iii) z_1 < 0, z_2 > 0. \quad (28)$$

The situations given in the brackets correspond to mirror images of the corresponding classes with an appropriate change of parameters. In subclass (ii), $z_1 = 0$ is a fixed point of the scale transformation, and can be considered as a special case of Case II to be discussed below.

In Figs. 1–3 we show figures for subclass (i) and (iii). For Figs. 1 and 3 we set $\alpha > 0$, while for Fig. 2 we have $\alpha < 0$. It is seen that for $\alpha > 0$ (< 0), the boundaries move away from (toward) the

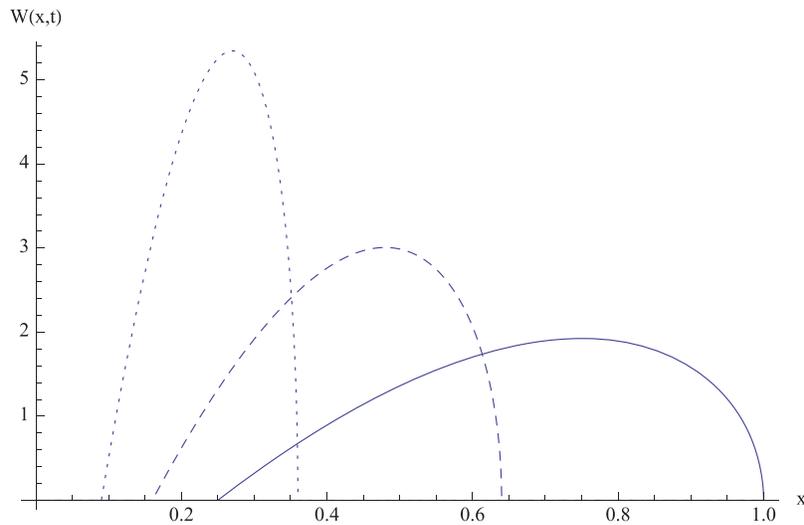


FIG. 1. $W(x, t)$ vs x for Case I(i) with $\alpha = 2$, $z_1 = 1$, $z_2 = 4$, $a_1 = 1$, and $a_2 = 1/2$ for $t = 0.3$ (dotted line), 0.4 (dashed line), and 0.5 (solid line).

origin (except when the boundary is a fixed point). The same pattern is also exhibited in the other classes.

IV. CLASS II: ONE MOVING BOUNDARY WITH $x = 0$ A FIXED POINT

We now consider the case on the positive half-line with $x = 0$ a fixed point. The case with the moving front in the negative half-line is simply the mirror images of the situation discussed here with an appropriate change of parameters.

The function $y(z)$ is given by

$$f(z) = \frac{a_1}{z} - \frac{a_2}{z_2 - z} + \beta, \quad a_1, a_2 > 0, 0 \leq z \leq z_2, \beta : \text{real}. \quad (29)$$

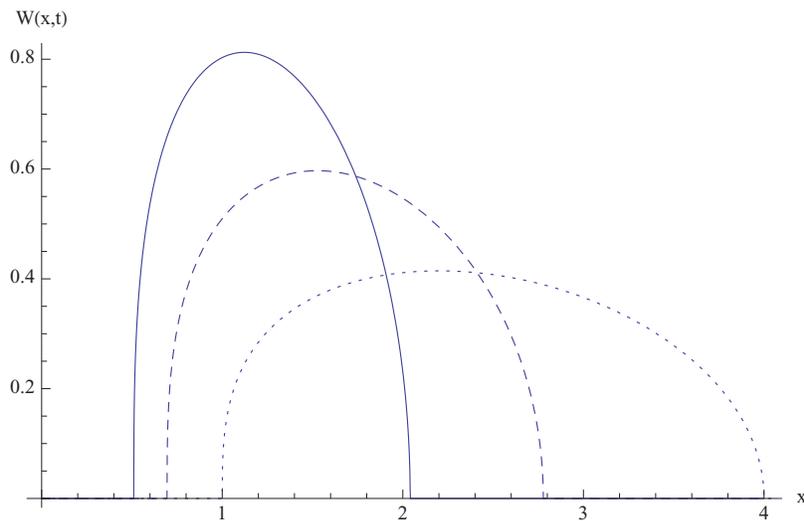


FIG. 2. $W(x, t)$ vs x for Case I(i) with $\alpha = -2$, $z_1 = 1$, $z_2 = 4$, $a_1 = 1/3$, and $a_2 = 1/2$ for $t = 1.0$ (dotted line), 1.2 (dashed line), and 1.4 (solid line).

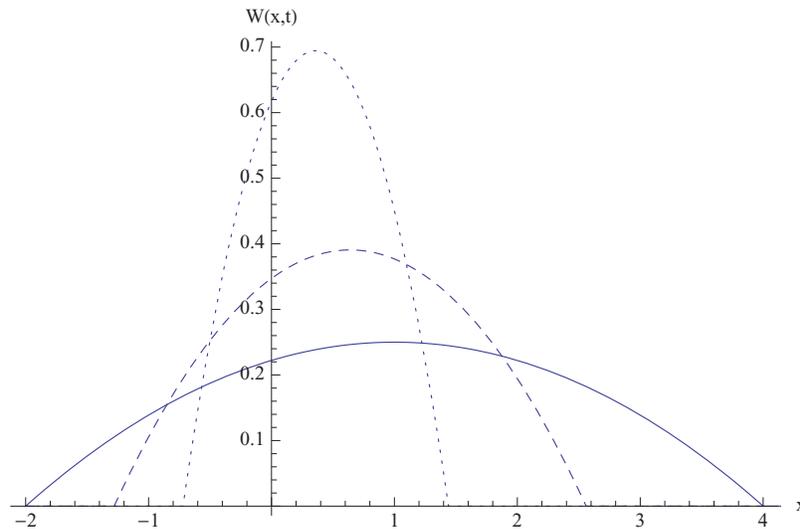


FIG. 3. $W(x, t)$ vs x for Case I(iii) with $\alpha = 2$, $z_1 = -2$, $z_2 = 4$, and $a_1 = a_2 = 1$ for $t = 0.6$ (dotted line), 0.8 (dashed line), and 1.0 (solid line).

This leads to a choice of $\rho_1(z)$ and $\rho_2(z)$:

$$\rho_2(z) = z(z_2 - z), \quad (30)$$

$$\rho_1(z) = -\beta z^2 + (\alpha - a_1 - a_2 - 2 + \beta z_2)z + (a_1 + 1)z_2 \quad (31)$$

for $0 \leq z \leq z_2$, and $\rho_1, \rho_2 = 0$ otherwise. The function $y(z)$ is

$$y(z) = Az^{a_1}(z_2 - z)^{a_2}e^{\beta z}, \quad (32)$$

where the normalization constant A is given by

$$A = [z_2^{a_1+a_2+1} B(a_1 + 1, a_2 + 1) {}_1F_1(a_1 + 1; a_1 + a_2 + 2; \beta z_2)]^{-1}, \quad (33)$$

where ${}_1F_1(\mu; \nu; z)$ is the Kummer confluent hypergeometric function.

The probability density function is

$$W(x, t) = \begin{cases} \frac{A}{t^\alpha} \left(\frac{x}{t^\alpha}\right)^{a_1} \left(z_2 - \frac{x}{t^\alpha}\right)^{a_2} e^{\beta \frac{x}{t^\alpha}}, & 0 \leq x \leq z_2 t^\alpha \\ 0, & \text{otherwise} \end{cases}. \quad (34)$$

A representative situation of this class is depicted in Fig. 4.

V. CLASS III: ONE MOVING BOUNDARY WITH $x = \infty$ A FIXED POINT

This last class represents the case on the positive half-line with $x = \infty$ a fixed point. The situation with $x = -\infty$ a fixed point is the mirror image of the situation discussed here with an appropriate change of parameters.

For this class the function $y(z)$ is taken to be

$$f(z) = \frac{a_1}{z - z_1} - \frac{a_2}{z_2} - \beta, \quad a_1, a_2 > 0, z_1 \leq z, \beta > 0. \quad (35)$$

The corresponding choice of $\rho_1(z)$ and $\rho_2(z)$ is

$$\rho_2(z) = (z - z_1)z, \quad (36)$$

$$\rho_1(z) = -\beta z^2 + (\alpha + a_1 + a_2 + 2 + \beta z_1)z - (a_2 + 1)z_1 \quad (37)$$

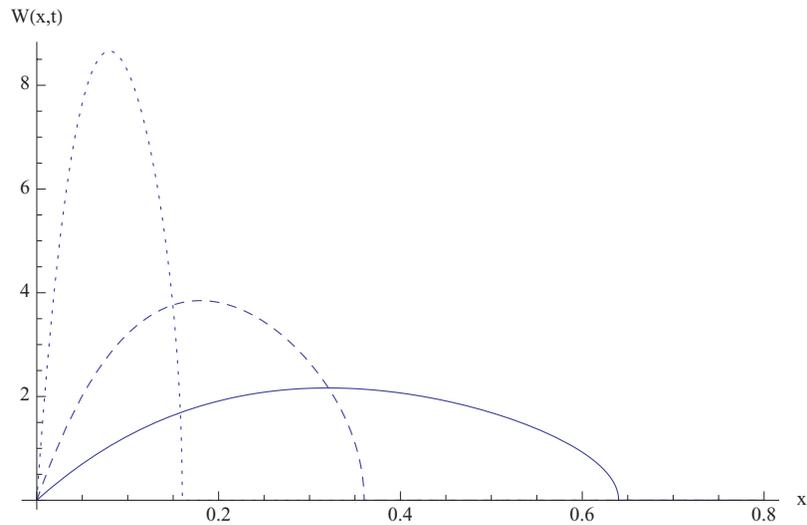


FIG. 4. $W(x, t)$ vs x for Case II with $\alpha = 2, z_2 = 1, a_1 = 1, a_2 = 1/2,$ and $\beta = -1$ for $t = 0.4$ (dotted line), 0.6 (dashed line), and 0.8 (solid line).

for $z_1 \leq z,$ and $\rho_1, \rho_2 = 0$ otherwise. The $y(z)$ is

$$y(z) = A(z - z_1)^{a_1} z^{a_2} e^{-\beta z}. \tag{38}$$

Here the normalization constant A is given by

$$A = [\beta^{-\frac{a_1+a_2+2}{2}} z_1^{\frac{a_1+a_2}{2}} \Gamma(a_1 + 1) e^{-\frac{1}{2}\beta z_1} W_{\frac{a_2-a_1}{2}, -\frac{a_1+a_2+1}{2}}(\beta z_2)]^{-1}, \tag{39}$$

where $\Gamma(z)$ and $W_{\mu, \nu}(z)$ are the Gamma and Whittaker functions, respectively.

The probability density function is

$$W(x, t) = \begin{cases} \frac{A}{t^\alpha} \left(\frac{x}{t^\alpha} - z_1\right)^{a_1} \left(\frac{x}{t^\alpha}\right)^{a_2} e^{-\beta \frac{x}{t^\alpha}}, & z_1 t^\alpha \leq x \leq \infty \\ 0, & \text{otherwise} \end{cases}. \tag{40}$$

Fig. 5 shows the evolution of $W(x, t)$ for some choice of parameters.

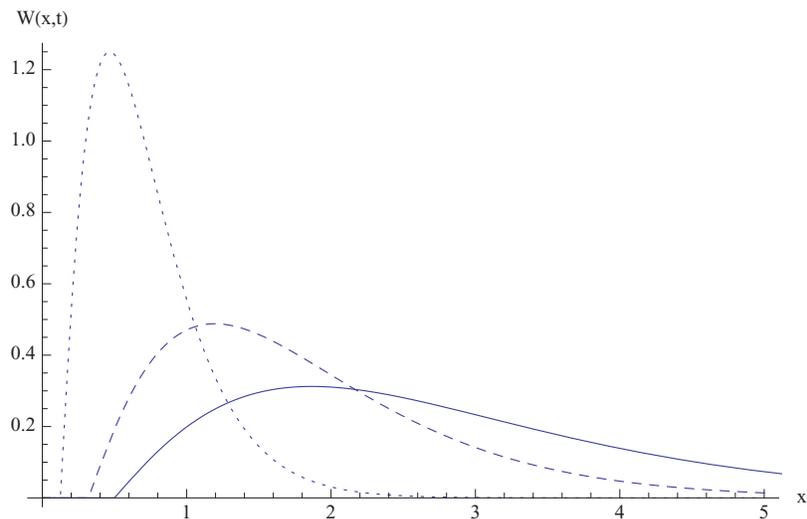


FIG. 5. $W(x, t)$ vs x for Case III with $\alpha = 2, z_1 = 0.5, a_1 = 1, a_2 = 1/2,$ and $\beta = 1$ for $t = 0.5$ (dotted line), 0.8 (dashed line), and 1.0 (solid line).

VI. DISCUSSION

Solving the FPEs with time-dependent drift and/or diffusion coefficient is in general a very difficult task. This explains why the number of papers on such kind of FPE is far less than that on the FPE with time-independent coefficients.

We have explored the solvability of the FPE with both time-dependent drift and diffusion coefficients by means of the similarity method. One advantage of the similarity method is that it allows one to reduce the FPE to an ordinary differential equation which is easier to solve, provided that the FPE possesses proper scaling property under certain scaling transformation of the basic variables. Some interesting exactly solvable cases of such FPE on the real line $x \in (-\infty, \infty)$ and the half lines $x \in [0, \infty)$ and $x \in (-\infty, 0]$ were considered in Ref. 18. These domains admit similarity solutions because their boundary points are the fixed points of the scaling transformation considered.

This implies that similarity solutions are not possible for other finite domains. It is therefore natural to suspect that similarity solutions of FPE on a finite domain may be possible, if its boundary points scale accordingly. We are thus led to consider FPE with moving boundaries. We have found that similarity solutions of the FPE are possible with moving boundaries, provided that the boundaries points scale properly. Three new classes of exactly solvable FPEs with moving boundaries are presented.

So far, we have studied cases where the boundaries are impenetrable. For penetrable boundaries, one expects non-vanishing probability current density across the boundaries. This implies a non-vanishing constant $C \neq 0$ in Eq. (13). Finding exactly solvable systems described by Eq. (13) with $C \neq 0$ is more involved, and is presently under investigation.

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