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Fundamental Constraints on Uncertainty Evolution in Hamiltonian Systems

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***Abstract*— A realization of Gromov's nonsqueezing theorem and its applications to uncertainty analysis in Hamiltonian systems are studied in this paper. Gromov's nonsqueezing theorem describes a fundamental property of symplectic manifolds, however, this theorem is usually started in terms of topology and its physical meaning is vague. In this paper we introduce a physical interpretation of the linear symplectic width, which is the lower bound in the nonsqueezing theorem, given the eigenstructure of a positive-definite, symmetric matrix. Since a positive-definite, symmetric matrix always represents the uncertainty ellipsoid in practical mechanics problems, our study can be applied to uncertainty analysis. We find a fundamental inequality for the evolving uncertainty in a linear dynamical system and provide some numerical examples.**

I. INTRODUCTION

In this paper we study the realization of Gromov's nonsqueezing theorem [4] in Hamiltonian systems and apply our results to uncertainty analysis. An example from orbit determination of spacecraft is given as a potential application. The nonsqueezing theorem was first proved by Gromov using J-holomorphic curves [4]. Other mathematicians have given proofs using different approaches [5], [10]. The theorem was then extended to arbitrary symplectic manifolds by Lalonde and McDuff [6]. Although this theorem has been proven rigorously in a topological sense, its application to practical issues is still vague.

A well known result from Hamiltonian dynamical systems theory is Liouville's Theorem, which says that the "volume" of a phase flow in a nondissipative system is conserved. This result also applies to uncertainty distributions, meaning that the total probability of finding a spacecraft in a given phase-space volume does not change over time under dynamical mapping. In a linear dynamical system, uncertainty distributions are generally formed as an ellipsoid. As time elapses, the ellipsoid may change its shape and orientation due to the dynamics of the system, but its volume must be conserved. An immediate question arises: Is it possible for the precision to increase in some modes by sacrificing precision in certain other modes, as long as the total phase-space volume is conserved? Are there any additional constraints which restrict such manipulations?

The current paper answers this question in the affirmative. Starting with some basic properties of symplectic manifolds and Hamiltonian systems, we introduce the nonsqueezing theorem and define the associated "linear symplectic width". Then several sufficient conditions to establish a specific symplectic transformation which generates a "standard ellipsoid" are given. Provided these nonlinear algebraic equations, we conclude that the transformation and the symplectic width are well defined, which is consistent with the proof given in [7]. Having obtained such a transformation by solving these equations, we are able to compute the symplectic width for certain classes of initial uncertainty distributions, and establish a set of inequalities that constrain a positive-definite, symmetric matrix. This provides a fundamental constraint on the evolution of a positive-definite symmetric matrix under linear mapping in a Hamiltonian dynamical system.

At the end of this paper numerical examples are given which illustrate potential applications of this result to uncertainty analysis. Although our approach is derived for a linear system, the nonsqueezing theorem can also be extended to nonlinear systems.

II. NONSQUEEZING THEOREM

A. Non-squeezing Theorem

One significant result for Hamiltonian systems is volume preservation, stated in Liouville's Theorem [1].

Theorem 1: (Liouville's Theorem) The phase flow of Hamilton's equations preserves phase volume: for any region D we have.

$$\text{volume of} \quad g^t D \quad = \quad \text{volume of} \quad D$$

Actually, this is a result which has been derived from the symplectic structure.

Proposition 1: (for proof see Ref. [8]) A smooth canonical transformation between symplectic manifolds of the same dimension is volume preserving and is a local diffeomorphism.

However, under symplectic transformations, the "shape" of any region D cannot change arbitrarily. This means that we cannot find a symplectic transformation which transforms a ball B onto any subset $U \subset \mathbb{R}^{2n}$ with the same volume. The subset U is constrained by the *Non-squeezing Theorem*. Consider a closed Euclidean ball $B^{2n}(r)$ in $\mathbb{R}^{2n}$ centered in 0 with radius r, and a symplectic cylinder Z^{2n} defined as

$$Z^{2n}(r) \quad = \quad B^2(r) \times \mathbb{R}^{2n-2}$$

We should notice that the disc $B^2(r)$ is symplectic, which means its coordinates are (q_i, p_i), not (q_i, q_j). Let $\varphi : U \rightarrow$

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V be a symplectic embedding, where $U, V \subset \mathbb{R}^{2n}$. Then the nonsqueezing theorem gives a constraint on φ [7]:

Theorem 2: (for proof see Ref. [7]) (Non-squeezing Theorem) If there is a symplectic embedding $B^{2n}(r) \hookrightarrow Z^{2n}(R)$, then $r \leq R$.

A similar result also applies to an "affine symplectomorphism". An affine symplectomorphism of $\mathbb{R}^{2n}$ is a map $\psi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ of the form

$$\psi(\mathbf{z}) \quad = \quad \varphi\mathbf{z} + \mathbf{z}_0 \tag{1}$$

where φ is a symplectomorphism and $\mathbf{z}_0 \in \mathbb{R}^{2n}$.

B. Symplectic Width and The Standard Ellipse

In the preceding section, the transformation of a symplectic ball is constrained by the nonsqueezing theorem or the affine nonsqueezing theorem. In practice, we are often interested in general volume distributions. We generalize the concept of $B^{2n}(r)$ to an arbitrary volume by introducing the "linear symplectic width" [7].

Definition 1: The linear symplectic width of an arbitrary subset $A \subset \mathbb{R}^{2n}$, denoted as $w_L(A)$, is defined as:

$$\begin{aligned} w_L(A) \quad = \quad & \sup_{r \in \mathbb{R}^+} \{\pi r^2 | \psi(B^{2n}(r)) \subset A \quad \text{for some} \\ & \psi \in ASp(\mathbb{R}^{2n})\} \end{aligned} \tag{2}$$

where $ASp(\cdot)$ denotes the group of affine symplectomorphism.

Several properties associated with the linear symplectic width are given in Ref. [7]. From those properties we can see that the symplectic width replaces the role of the symplectic ball in the application of the nonsqueezing theorem to an arbitrary subset in $\mathbb{R}^{2n}$.

A specific example for the application of symplectic width is an arbitrary ellipsoid in $\mathbb{R}^{2n}$, which can be formulated as:

$$E \quad = \quad \left\{ \mathbf{w} \in \mathbb{R}^{2n} | \sum_{i,j=1}^{2n} a_{ij} w_i w_j \leq 1 \right\} \tag{3}$$

According to Ref. [7], this ellipsoid can be symplectically transformed into a "standard ellipsoid", $E_s(r)$, for some n-tuple $r = (r_1, \cdots, r_n)$ with $0 < r_1 \leq \cdots \leq r_n$, where

$$E_s(r) \quad = \quad \left\{ z \in \mathbb{C}^n | \sum_{i=1}^{n} \left| \frac{z_i}{r_i} \right|^2 \leq 1 \right\} \tag{4}$$

Moreover, if E has an empty interior, then $w_L(E) = 0$, otherwise, $w_L(E) = \pi r_1^2$.

III. ESTABLISHING A SYMPLECTIC TRANSFORMATION

From the previous section we know that, given an arbitrary ellipsoid, there exists a linear symplectic transformation that transforms the ellipsoid into a standard ellipsoid in a complex space from which the symplectic width can be computed. In this section, we investigate the practical computation needed to establish this transformation.

A. Symplectic Basis

Before starting, we need to define the term "symplectic basis" for future use.

Definition 2: Let (V, ω) be a symplectic vector space of dimension $2n$. Then there exists a basis $\vec{\alpha}_1, \cdots, \vec{\alpha}_n, \vec{\beta}_1, \cdots, \vec{\beta}_n$ on V such that $\omega(\vec{\alpha}_i, \vec{\alpha}_j) = 0$, $\omega(\vec{\beta}_i, \vec{\beta}_j) = 0$, $\omega(\vec{\alpha}_i, \vec{\beta}_j) = \delta_{ij}$. Such a basis is called a "symplectic basis", or sometimes called an "ω-standard basis".

Moreover, there exists a linear map $\psi : \mathbb{R}^{2n} \rightarrow V$ which is symplectic, i.e., the 2-form is preserved under the transformation:

$$\psi^* \omega \quad = \quad \omega_0 \tag{5}$$

Assume $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ are the coordinates of $\mathbf{z} \in \mathbb{R}^{2n}$. Then we can show that

$$\psi(\mathbf{z}) \quad = \quad \sum_{i=1}^{n} (x_i \vec{\alpha}_i + y_i \vec{\beta}_i) \tag{6}$$

satisfies Eq. (5) [7]

B. Sufficient Conditions

Ref. [7] offers a standard way to establish a symplectic transformation but does not provide the details on how to practically compute such a transformation. Detailing the discussion in Ref. [7], we can obtain a set of sufficient conditions which the symplectic transformation must satisfy. Given any ellipsoid E in Eq. (3), a symplectic transformation, which transforms E into a standard ellipsoid $E_s(r)$ in a complex space, can be established by picking a basis $\vec{\alpha}_1, \cdots, \vec{\alpha}_n, \vec{\beta}_1, \cdots, \vec{\beta}_n$ which is g-orthogonal and ω-standard. Consider the inner product

$$g(\mathbf{v}, \mathbf{w}) \quad = \quad \sum_{i,j=1}^{2n} a_{ij} v_i w_j \tag{7}$$

Then $\vec{\alpha}_i, \vec{\beta}_i, i = 1, \cdots, n$ must satisfy the following equations:

$$\begin{aligned} g(\vec{\alpha}_i, \vec{\alpha}_i) \quad &= \quad g(\vec{\beta}_i, \vec{\beta}_i) \\ &= \quad \frac{1}{r_i^2} \end{aligned} \tag{8}$$

$$J\vec{\alpha}_i \quad = \quad -k_{1i}\vec{\beta}_i \tag{9}$$

$$J\vec{\beta}_i \quad = \quad k_{2i}\vec{\alpha}_i \tag{10}$$

where r_i is determined by the inner product $g(\cdot, \cdot)$, and (k_{1i}, k_{2i}) are scaling factors. Let $\xi_i = \tilde{\alpha}_i + j\tilde{\beta}_i \in \mathbb{C}^{2n}$, and

$$\bar{\xi}_i^T \xi_j \quad = \quad \delta_{ij} \tag{11}$$

where $\tilde{\alpha}_i = m_i \vec{\alpha}_i$ and $\tilde{\beta}_i = n_i \vec{\beta}_i$, m_i and n_i are scaling factors. Actually, if a symplectic transformation is established as in Eq. (6), the matrix representation, Ψ, of ψ will satisfy

$$\Psi \quad = \quad [\vec{\alpha}_1, \cdots, \vec{\alpha}_n, \vec{\beta}_1, \cdots, \vec{\beta}_n] \tag{12}$$

$$\Psi^T J \Psi \quad = \quad J \tag{13}$$

where, given the standard bases,

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

As a result, Eqs. (7)–(13) are the sufficient conditions needed for establishing a symplectic transformation matrix.

However, we note that Eqs. (9) and (10) are equivalent. Moreover, in the Appendix we prove that Eq. (11) is automatically satisfied if Eqs. (9), (12) and (13) hold. Hence, the sufficient conditions degenerate to Eqs. (8), (9), (12) and (13).

1) Practical Computation: We have shown that Eqs. (8), (9), (12) and (13) are the necessary constraints to establish the desired symplectic transformation matrix. In this section we explore the practical computation of every element in the transformation matrix.

First, we write Eq. (7) in the matrix form,

$$\begin{aligned} g(\mathbf{v}, \mathbf{w}) &= \sum_{i,j=1}^{2n} a_{ij} v_i w_j \\ &= \mathbf{v}^T P^{-1} \mathbf{w} \end{aligned} \tag{14}$$

where P is assumed to be a symmetric matrix. Let $(\lambda_i, \mathbf{u}_i), i = 1, \cdots, 2n$ be P's eigenvalues and associated eigenvectors. P can be diagonalized as $P = V \Lambda_p V^T$, where

$$\begin{aligned} V &= [\mathbf{u}_1, \cdots, \mathbf{u}_{2n}] \\ \Lambda_p &= \operatorname{diag}(\lambda_1, \cdots, \lambda_{2n}) \end{aligned}$$

We should notice that V is orthonormal and the $\{\mathbf{u}_i\}$ span the whole $\mathbb{R}^{2n}$ space. Assume $\vec{\alpha}_i = \sum_{j=1}^{2n} a_{ji} \mathbf{u}_j = V \mathbf{a}_i$, and $\vec{\beta}_i = \sum_{j=1}^{2n} b_{ji} \mathbf{u}_j = V \mathbf{b}_i$. Then Eq. (14) can be simplified as $g(\vec{\alpha}_i, \vec{\alpha}_i) = \mathbf{a}_i^T \Lambda_p^{-1} \mathbf{a}_i$ and $g(\vec{\beta}_i, \vec{\beta}_i) = \mathbf{b}_i^T \Lambda_p^{-1} \mathbf{b}_i$. Along with Eq. (8) we obtain the following relation for $\vec{\alpha}_i$ and $\vec{\beta}_i$:

$$\sum_{j=1}^{2n} \frac{a_{ji}^2}{\lambda_j} = \sum_{j=1}^{2n} \frac{b_{ji}^2}{\lambda_j} \tag{15}$$

Also, from Eq. (12) we can write the transformation matrix in terms of the eigenvector matrix of P and the coefficients: $\Psi = VS$, where $S = [\mathbf{a}_1, \cdots, \mathbf{a}_n, \mathbf{b}_1, \cdots, \mathbf{b}_n]$.

By substituting $\Psi^{-1} = -J\Psi^T J$ into $\Psi = VS$ we obtain

$$V^T J V = S J S^T \tag{16}$$

The third relationship between the $\vec{\alpha}$'s and $\vec{\beta}$'s is posed by Eq. (9). Given $\vec{\alpha}_i = (\vec{\alpha}_{1i}^T, \vec{\alpha}_{2i}^T)$, we conclude that $J\vec{\alpha}_i = (-\vec{\alpha}_{2i}^T, \vec{\alpha}_{1i}^T)$. Therefore, the magnitude of $\vec{\alpha}_i$ equals the magnitude of $J\vec{\alpha}_i$. From Eq. (9), $k_i = |J\vec{\alpha}_i|/|\vec{\beta}_i| = |\vec{\alpha}_i|/|\vec{\beta}_i|f$. Since $\vec{\alpha}_i = V\mathbf{a}_i$ and V is orthonormal, which preserves the magnitude of a vector, we obtain $|\vec{\alpha}_i| = |V\mathbf{a}_i| = |\mathbf{a}_i|$, $|\vec{\beta}_i| = |V\mathbf{b}_i| = |\mathbf{b}_i|$. Therefore, $k_i = |\mathbf{a}_i|/|\mathbf{b}_i|$, and

$$J V \mathbf{a}_i = -\frac{|\mathbf{a}_i|}{|\mathbf{b}_i|} V \mathbf{b}_i \tag{17}$$

Here, we have successfully transformed the sufficient conditions into the computation of coefficients for a symplectic basis, as described in Eqs. (15), (16) and (17). By analyzing the numbers of unknowns and equations, we conclude that there exists a consistent solution. There are $2n \times 2n = 4n^2$ unknowns for a_{ij} and b_{ij}. However, there are n equations in Eq. (15), $n(2n-1)$ equations in Eq. (16) (because of the skew symmetric properties), and $2n^2$ equations in Eq. (17). Therefore, in total we have $n + n(2n-1) + 2n^2 = 4n^2$ equations. Since the numbers of unknowns equals the the numbers of equations, a consistent solution is possible. After obtaining the solutions for the sufficient condition, we can determine all the r_i's and the linear symplectic width of the system. We note that the above equations are nonlinear.

We do not need to prove the existence of a solution, since this property has been shown using a different approach in Ref. [7]. However, we do need to discuss the uniqueness of solutions. Assume that a set of α's and β's are solutions to Eqs. (8), (9), (12) and (13), and let

$$\begin{aligned} J' &= \Psi^T J \Psi \\ &= \left[\begin{array}{c|c} J11 & J12 \\ \hline -J21 & J22 \end{array}\right] \end{aligned} \tag{18}$$

Comparing the two sides of Eq. (13), we obtain $J11 = J22 = 0$, and $J12 = J21 = I$. Moreover, if Ψ is constructed as in Eq. (12), then every element in $J11$, $J11_{ij}$, will be $\vec{\alpha}_i^T J \vec{\alpha}_j$. Since $J\vec{\alpha}_i = -k_i \vec{\beta}_i$, we obtain $J11_{ij} = -k_j \vec{\alpha}_i^T \vec{\beta}_j$. Because $J11 = 0$, then

$$\vec{\alpha}_i^T \vec{\beta}_j \equiv 0 \tag{19}$$

A similar analysis can be applied to $J12$ to find

$$\vec{\alpha}_i^T \vec{\alpha}_j = \mu_{ij} \delta_{ij} \tag{20}$$

$$\vec{\beta}_i^T \vec{\beta}_j = \nu_{ij} \delta_{ij} \tag{21}$$

where μ_{ij} and ν_{ij} are scaling factors.

By inspection we note that Eq. (8), (9), (19), (20) and (21) are linear, quadratic or quartic equations. This implies that the signature symmetry may apply to solutions. That is, if α_{ij} is a solution, then $-\alpha_{ij}$ may also be a solution. Therefore, if Ψ is a solution for Eqs. (8), (9), (13), then flipping the signs of a column and its corresponding symplectic column or of a row and its corresponding symplectic row are also solutions. In this way, we can note trivially that the solutions to the conditions are not unique.

IV. APPLICATIONS TO UNCERTAINTY ANALYSIS

Given the position and momentum $(\mathbf{q}, \mathbf{p})$ in a Hamiltonian system, we can define a covariance matrix for the analysis of uncertainty evolution. Let $\mathbf{X} = (q_1, \cdots, q_n, p_1, \cdots, p_n)$ be the coordinates in the phase space. Assume the system is zero-mean, then the covariance matrix P can be defined as [9]

$$P = E[\mathbf{X}\mathbf{X}^T] \tag{22}$$

where P is symmetric and positive-definite, and $E[\cdot]$ denotes the expectation value. The equation defining the uncertainty ellipsoid can be written as

$$\mathbf{X}^T P^{-1} \mathbf{X} \quad \leq \quad 1 \tag{23}$$

Sometimes the matrix P^{-1} is also known as the "information matrix". We should notice that (23) has the same form as (14). Therefore, all the analysis in the preceding section can be applied to the covariance matrix. Since P is symmetric and positive-definite, we define the eigenvalues of P as $(\sigma_1^2, \cdots, \sigma_{2n}^2)$. If there are no correlation terms, i.e., all the off-diagonal terms are identically zero, then the σ's are called the "standard deviation". Moreover, if this is a linear system, i.e.,

$$\begin{aligned} \dot{\mathbf{X}} &= A(t)\mathbf{X} \\ \mathbf{X}(t) &= \Phi(t, t_0)\mathbf{X}_0 \end{aligned}$$

then the covariance is mapped in time as $P = \Phi P_0 \Phi^T$ where $P_0 = E[\mathbf{X}_0 \mathbf{X}_0^T]$

A. General Result

The preceding section gives a set of sufficient conditions from which we can establish a symplectic matrix that transforms an arbitrary ellipsoid in real space into a standard ellipsoid in complex space. Although those equations are highly nonlinear, we may obtain the solutions via analytical or numerical methods. Accordingly, it is reasonable to assume that we can obtain the symplectic width of an arbitrary ellipsoid and make use of it. Having obtained the symplectic width, we can derive a fundamental inequality by application of Gromov's theorem to the evolution of uncertainty ellipsoids in a linear system. This inequality constrains the projected area of uncertainty in all symplectic pairs of the dynamical distribution, defining a minimum area. This holds not only for a dynamically mapped trajectory, but for any canonical transformation of the initial distribution.

Consider a symmetric, $2n \times 2n$ matrix P arranged as:

$$P = \left[\begin{array}{c|c|c} P_{11} & \cdots & P_{1n} \\ \hline \vdots & \ddots & \vdots \\ \hline P_{n1} & \cdots & P_{nn} \end{array}\right] \tag{24}$$

where P_{ij} is a 2×2 sub-matrix of the specific form:

$$P_{ij} = \begin{bmatrix} P_{q_i,q_j} & P_{q_i,p_j} \\ P_{p_i,q_j} & P_{p_i,p_j} \end{bmatrix} \tag{25}$$

where $P_{q_i,q_j} = E[q_i q_j^T]$ etc. Then, the following theorem can be obtained.

Theorem 3: Assume P_0 is symmetric and arranged as (24) and (25) at $t = t_0$. Let $\Phi(t, t_0)$ be an arbitrary linear symplectic transformation, and

$$P(t) = \Phi(t, t_0) P_0 \Phi^T(t, t_0) \tag{26}$$

Then, the following inequality holds:

$$|P_{ii}(t)| \geq \left(\frac{w_L(P_0)}{\pi}\right)^2 \quad \forall i = 1 \cdots n \tag{27}$$

where $w_L(P_0)$ is the linear symplectic width of P_0 defined previously.

<u>Proof</u>

If P_0 is a symmetric matrix in $\mathbb{R}^{2n}$, there exists a corresponding ellipsoid which can be expressed as:

$$\mathbf{X}^T P_0^{-1} \mathbf{X} \quad \leq \quad 1 \tag{28}$$

According to our derivation, there exists a symplectic transformation, Ψ, which transforms P_0 into a standard ellipsoid, Es, from which we can define the linear symplectic width, $w_L(P_0)$. According to the nonsqueezing theorem, among all the possible symplectic transformations of a symplectic disk formed by (q_i, p_i), $w_L(P_0)$ is the smallest area it may reach.

Let $\mathbf{X} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$, where $\mathbf{x}_i = (q_i, p_i)$. Then the equation of the ith symplectic disc would be

$$\mathbf{x}_i^T (P_{ii})^{-1} \mathbf{x}_i \quad \leq \quad 1 \tag{29}$$

We should notice that P_{ii} is also symmetric. As a result, the physical shape of the symplectic disc is an ellipse and the area, A_i, of the symplectic disc is

$$\begin{aligned} A_i &= \pi \sigma_{i1} \sigma_{i2} \\ &= \pi \sqrt{|P_{ii}|} \end{aligned} \tag{30}$$

where $\sigma_{i1}^2, \sigma_{i2}^2$ are the eigenvalues of P_{ii}. Since $P(t)$ is an ellipsoid under arbitrary symplectic transformation, according to the nonsqueezing theorem and (30),

$$\pi \sqrt{|P_{ii}(t)|} \quad \geq \quad w_L(P_0)$$

or,

$$|P_{ii}(t)| \geq \left(\frac{w_L(P_0)}{\pi}\right)^2 \tag{31}$$

B. Specific Orientations to Compute The Symplectic Width

Having shown the importance of the symplectic width in the evolution of an uncertainty ellipsoid, it remains to compute the symplectic width for a given ellipsoid. Equations (15), (16) and (17) are the only equations we need to solve. However, they are highly nonlinear, and it is difficult to obtain a general analytical solution. In the current paper, we present two special cases where the symplectic width can be found analytically.

1) Standard Ellipsoid in Real Space:: First of all, we consider that P is a diagonal matrix, that is, there are no correlation terms. In geometry, this means that all the principle axes of the uncertainty ellipsoid lie along the standard bases, i.e., $V = I$.

Proposition 2: In a system where the coordinates are uncorrelated, the symplectic width is simply the smallest elliptical area defined by the product of the standard deviations associated with each symplectic pair (q_i, p_i).

<u>Proof</u>

Given $V = I$ for an uncorrelated system, (15), (16) and (17) can be re-expressed as

$$\sum_{j=1}^{2n} \frac{a_{ji}^2}{\sigma_j^2} = \sum_{j=1}^{2n} \frac{b_{ji}^2}{\sigma_j^2} \quad (32)$$

$$SJS^T = J \quad (33)$$

$$J\mathbf{a}_i = -\frac{|\mathbf{a}_i|}{|\mathbf{b}_i|}\mathbf{b}_i \quad (34)$$

Going through the computation, we find that the off-diagonal terms of S must be zero, and the diagonal terms satisfy the following relations:

$$a_{ii}b_{i+n,i} = 1 \quad (35)$$

$$\frac{a_{ii}^2}{\sigma_i^2} = \frac{b_{i+n,i}^2}{\sigma_{i+n}^2} \quad (36)$$

where $i = 1, \cdots, n$. Solving the above equations for α_i and β_i we obtain

$$\begin{aligned} r_i &= \sqrt{\frac{1}{g(\vec{\alpha}_i, \vec{\alpha}_i)}} \\ &= \sqrt{\sigma_i \sigma_{i+n}} \end{aligned} \quad (37)$$

Therefore, the linear symplectic width is

$$\begin{aligned} w_L(E_P) &= \min\{\pi r_i^2 | i = 1, \cdots, n\} \\ &= \min\{\pi \sigma_i \sigma_{i+n} | i = 1, \cdots, n\} \end{aligned} \quad (38)$$

2) Special Orientation – Symplectic Eigenstructure:: Now we consider a covariance matrix whose eigenvector matrix is both orthonormal and symplectic. It means that the eigenvector matrix must have a structure as follows:

$$V = \left[\begin{array}{c|c} V_{11} & -V_{21} \\ \hline V_{21} & V_{11} \end{array}\right]$$

and $V^T V = I$. Then a simple conclusion about linear symplectic width can be obtained.

Proposition 3: In a system where the eigenvector matrix of a covariance matrix is both orthonormal and symplectic, the symplectic width is the smallest elliptical area among those defined by the products of symplectic eigenvalues, where the symplectic pair relationship are determined by the relation $\mathbf{u}_{i+n} = J\mathbf{u}_i$, where $\mathbf{u}_i$ is an eigenvector of a covariance matrix P.

Proof

Given V symplectic, we can show that Eqs. (16) and (17) also have the same format as Eqs. (33) – (34). As a result, we conclude that the linear symplectic width is

$$\begin{aligned} w_L(E_P) &= \min\{\pi r_i^2 | i = 1, \cdots, n\} \\ &= \min\{\pi \sigma_i \sigma_{i+n} | i = 1, \cdots, n\} \end{aligned} \quad (39)$$

This result implies that we define symplectic pairs "along the principle axes", instead of considering the physical symplectic pairs, (q_i, p_i). Actually, we can view an uncorrelated system as a special case of this example, where the symplectic pairs happen to agree with the physical symplectic pairs (q_i, p_i).

V. NUMERICAL EXAMPLES AND IMPLICATIONS

Here we consider the two body problem from astrodynamics. The equations of motion are described by Newton's Law of Gravity:

$$\ddot{\mathbf{r}} = -\frac{\mu}{|\mathbf{r}|^3}\mathbf{r} \quad (40)$$

Going through linearization we obtain the equations for the relative motion about a nominal orbit is

$$\begin{aligned} \delta\ddot{\mathbf{r}} &= -\frac{\mu}{|\mathbf{r}|^3}(I - 3\hat{\mathbf{r}}\hat{\mathbf{r}}^T) \cdot \delta\mathbf{r} \\ &= \mathbb{V}(t)\delta\mathbf{r} \end{aligned} \quad (41)$$

where $\delta\mathbf{r} = (x, y)$ and their corresponding momenta are $(\dot{x}, \dot{y})$. We note that $\mathbb{V}(t)$ is symmetric and time varying. Let $\mathbf{X} = (x, y, \dot{x}, \dot{y})$. Equation (41) can be re-expressed as $\dot{\mathbf{X}} = A(t)\mathbf{X}$ and we claim that this is a Hamiltonian system since the dynamics matrix $A(t)$ satisfies the symplectic property:

$$A^T(t)J + JA(t) = 0$$

Thus, our preceding derivations are applicable to this system.

Consider an initial covariance matrix P_0:

$$P_0 = \text{diag}(\sigma_x^2, \sigma_y^2, \sigma_{\dot{x}}^2, \sigma_{\dot{y}}^2) \quad (42)$$

Then we can compute $P(t)$ by

$$P(t) = \Phi(t, t_0)P_0\Phi^T(t, t_0)$$

where $\Phi(t, t_0)$ is the state transition matrix for our linearized system. Arrange $P(t)$ such that

$$P(t) = \left[\begin{array}{c|c} P_{xx} & P_{xy} \\ \hline P_{yx} & P_{yy} \end{array}\right]$$

Here P_{xx} denotes the covariance matrix within the x mode, P_{yy} denotes the covariance matrix within the y mode, and P_{xy} denotes the correlation matrix between the x and y mode. Then the projections of the uncertainty distribution onto the $x - \dot{x}$ and $y - \dot{y}$ planes are defined by

$$X^T(P_{xx})^{-1}(t)X \leq 1 \quad (43)$$

$$Y^T(P_{yy})^{-1}(t)Y \leq 1 \quad (44)$$

where $X = (x, \dot{x})$ and $Y = (y, \dot{y})$. From (31) we have the inequalities:

$$|P_{xx}(t)| \geq c \quad (45)$$

$$|P_{yy}(t)| \geq c, \quad (46)$$

where $c = \min(\sigma_x^2\sigma_{\dot{x}}^2, \sigma_y^2\sigma_{\dot{y}}^2)$. The inequality for the projected area in the ith mode can be computed as

$$\begin{aligned} \mathcal{A}_i &\geq \pi \cdot \min(\sigma_x\sigma_{\dot{x}}, \sigma_y\sigma_{\dot{y}}) \\ &\geq \min(\mathcal{A}_{x0}, \mathcal{A}_{y0}) \end{aligned} \quad (47)$$

Thus the area in which the spacecraft will lie in the $x - \dot{x}$ or $y - \dot{y}$ space will always be greater than the minimum area computed from the symplectic width. This implies that, when starting from a standard diagonal covariance matrix, the distribution of uncertainty over the symplectic pairs of the dynamical space will in general grow larger and can never decrease below the initial minimum. We note that the areas of both symplectic disks, $(x, \dot{x})$ and $(y, \dot{y})$, are always greater than or equal to the symplectic width, as predicted in the preceding theorem.

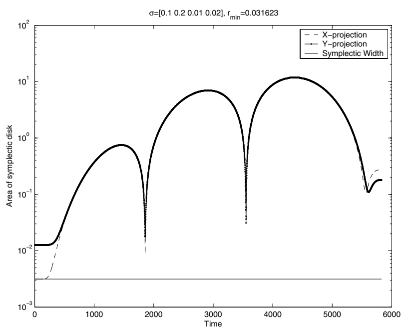

Fig. 1. In this example, a nominal circular orbit about the Earth with radius of 7000 km is selected. This figure shows the projected areas of the ellipsoid onto the $x - \dot{x}$ and $y - \dot{y}$ plane. The original covariance matrix is set as $P_0 = \mathrm{diag}(0.1^2, 0.2^2, 0.01^2, 0.02^2)$ for $(P_{xx}, P_{yy}, P_{\dot{x}\dot{x}}, P_{\dot{y}\dot{y}})$

a) b)

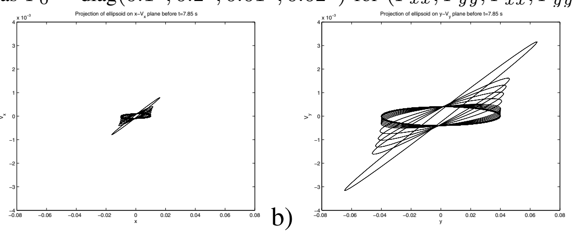

Fig. 2. a) The projected ellipse of the uncertainty ellipsoid on the x-$\dot{x}$ plane. b) The projected ellipse of the uncertainty ellipsoid on the y-$\dot{y}$ plane. All parameters are the same as in Fig. 1. Because the ellipses will become too elongated for visualization, this example stops at $t \approx 7.85s$

Figs. 1 – 2 give some numerical examples. In Figs. 1 and 2, we select a nominal circular orbit about Earth with radius of 7000 km as the example. We can see that the projected elliptical areas on $x - \dot{x}$ and $y - \dot{y}$ plane are always greater than, or equal to the symplectic width as predicted. We note that the inequality holds regardless of the orbit types, including hyperbolic trajectories.

VI. CONCLUSIONS

This paper studies the realization of Gromov's non-squeezing theorem and its applications to uncertainty analysis in a Hamiltonian system. Starting with the establishment of a symplectic transformation matrix, we can show that an arbitrary ellipsoid in real space can be uniquely transformed into a standard ellipsoid in complex space by solving a set of nonlinear algebraic equations. In terms of the eigenvectors of the ellipsoid with certain orientations, we can find a very simple result for the physical meaning of the linear symplectic width: the smallest elliptical area among all symplectic pairs. An immediate application of this result is the constraint on the lower bound of uncertainty. Namely, the projected area of a given uncertainty distribution for every symplectic pair, (q_i, p_i) in a Hamiltonian system cannot be smaller than a certain number defined by the initial uncertainty. This applies to both time mapping of the covariance, and to the introduction of new coordinates defined by a canonical transformation. This implies that we cannot arbitrarily increase the precision of both the position and momentum in one mode by sacrificing that in other modes. This is similar to the Uncertainty Principle in quantum mechanics. A general inequality is proposed for future applications.

VII. APPENDIX

Consider Eq. (11), $\bar{\xi}_i^T \xi_j = \delta_{ij}$, where $\xi_i = \tilde{\alpha}_i + j\tilde{\beta}_i \in \mathbb{C}^{2n}$. Then we can expand Eq. (11) into the real and imaginary terms:

$$\tilde{\alpha}_i^T \tilde{\alpha}_j + \tilde{\beta}_i^T \tilde{\beta}_j = \delta_{ij} \quad (48)$$

$$\tilde{\alpha}_i^T \tilde{\beta}_j + \tilde{\beta}_i^T \tilde{\alpha}_j = 0 \quad (49)$$

Equation (19) leads to the result of Eq. (49) explicitly. Equations (20) and (21) give the result of Eq. (48) by properly choosing scaling factors. Therefore, we conclude that Eq. (11) is redundant if Eqs. (9), (12) and (13) are satisfied.

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