Finding Hamiltonian Cycles on Incrementally Extensible Hypercube Graphs

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Abstract

The existence of a Hamiltonian cycle is the premise of usages in an interconnection network. A novel interconnection network, the Incrementally Extensible Hypercube (IEH) graph, has been proposed recently. The IEH graphs are derived from hypercubes and also retain most parts of properties in hypercubes. Unlike hypercubes without incrementally extensibility, IEH graphs can be constructed in any number of nodes. In this paper, we present an algorithm to find a Hamiltonian cycle or path and prove that there exists a Hamiltonian cycle in all of IEH graphs except for those containing exactly $2^n - 1$ nodes.

1 Introduction

In view of the well-defined properties, hypercubes are extensively studied for the interconnection architectures in parallel machines and parallel computing. Even though hypercube is architecturally defined as a structure with regularity, symmetry and powerful computability, some imperfections still exist in the hypercube for being a general architecture of parallel computation. The primary imperfection is that hypercubes are not incrementally extensible. That is, an n-dimensional hypercube must connect exactly with 2^n nodes. Several generalizations of the hypercube structure have been proposed to overcome the problem such as incomplete hypercube[9], the generalized hypercube[2], supercubes[12], Fibonacci Cubes[10] and so on.

The generalized hypercube is an initial work for overcoming the problem of restricted number of nodes. There are two major drawbacks in the generalized hypercube: 1) the network may result in a completely connected graph whenever the number of computing nodes is a prime number; 2) the significant change of the interconnections is needed whenever a new node is going to be added into the graph. The incomplete hypercube also has a substantial problem for disregarding fault tolerance. Even if a single node is failure in incomplete hypercube, the graph may be disconnected.

In 1989, A. Sen proposed a new structure, known as supercubes, which circumvents problems in the previous designs. Although supercubes are optimal fault-tolerant in terms of connectivity of nodes and its diameter is logarithmic in the number of nodes, they still have some defects. First, supercubes are moderately irregular and if the size of the graph becomes large, then the more irregularity of the graph will be made. Second, for a supercube with N nodes, $2^n < N < 2^{n+1}$, the maximal difference of degrees among nodes is as high as n, and each degree of all nodes is distributed over the range $(n, 2n)$. Another problem of the supercubes is the structure need significantly restructure when we add a new node into a supercube.

There is also another generalization of hypercube, called Fibonacci Cubes[10]. Although it improves the ability of the fault tolerance over the incomplete hypercube, the unavoidable asymmetric structure and relatively sparse interconnections still result in insufficient ability of fault tolerance.

Recently, the *Incrementally Extensible Hypercube* (IEH) graph[13, 14] has been proposed. Unlike the

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hypercube, the IEH graph is incrementally extensible. In other words, IEH graphs can be constructed by any number of nodes. Besides, the structure of IEH graphs have the optimal ability of the fault tolerance and the diameter of logarithmic in the number of nodes. Architecturally, the difference between the maximum and the minimum degree of nodes in an IEH graph is at most 1. Hence it is enough to say that the IEH graph is almost regular. In addition, the IEH graph does not have the drawbacks of the other generalizations of the hypercube.

In the parallel computing, finding a Hamiltonian cycle or path is important for the uses of an interconnection networks. Typically, a Hamiltonian cycle can be treated as a largest ring embedded in a graph, and the broadcasting in an interconnection network can be accomplished by traversing its Hamiltonian cycle. In this paper, we propose an algorithm to find a Hamiltonian cycle for IEH graphs. Due to some IEH graphs do not contain a Hamiltonian cycle, this paper also proves that IEH graphs include a Hamiltonian cycle except those containing $2^n - 1$ nodes. In brief, an IEH graph containing $2^n - 1$ nodes only exists a Hamiltonian path.

In the next section, we introduce the necessary notations and definitions. Section 3 describes how to find a Hamiltonian cycle and path in IEH graphs in an IEH graph and proves the existence of Hamiltonian cycles in an IEH graph. By the procedures of finding the Hamiltonian cycle in IEH graphs, section 4 shows two example in different situation. Section 5 is the conclusion.

2 Preliminaries

An n-dimensional IEH graph containing N nodes, denoted by $G_n(N)$, is defined by connecting m different-sized hypercubes with extra edges, named the Inter-Cube or IC edges, where $1 \leq m \leq n$[13, 14]. $G_n(N)$ contains an i-dimensional hypercube, denoted by H_i, if and only if the ith bit of the binary representation of N is 1.

Definition 1 *Consider two hypercubes H_i and H_j in an IEH graph and without any loss of generality assume $i > j$. Suppose x and y are two nodes in H_j and H_i respectively, then x can be represented by a j-bit binary $b_{j-1}b_{j-2}...b_0$ and y can be represented by an i-bit binary number. For the Architecture of an IEH graph, a node x in H_j have a set of image nodes in H_i. The set is denoted by S_i^x, and the cardinality of the set is $i - j$. The image nodes in the set S_i^x can be defined as follows[13].*

$$i - j \ \textit{image nodes} \begin{cases} 0\,\overbrace{11......11}^{i-j-1}\,b_{j-1}b_{j-2}......b_0 \\ 0\,\overbrace{01......11}^{i-j-1}\,b_{j-1}b_{j-2}......b_0 \\ \quad\ddots \qquad\qquad \vdots \\ 0\,\overbrace{11......10}^{i-j-1}\,b_{j-1}b_{j-2}......b_0 \end{cases}$$

Definition 2 *Let H_i and H_j be two hypercubes in an IEH graph, and assume $i > j$. For each node $x = b_{j-1}b_{j-2}...b_0$ in H_j, the links that are used for connecting x and its image nodes y, $y \in S_j^x$, are called Inter-Cube or IC edges.*

Figure 1 shows the example of $G_3(12)$. The nodes $8, 9, 10$ and 11 are composed to be a 2-subcube(H_2), and nodes $0, 1, 2, 3, 4, 5, 6$ and 7 are the elements of a 3-subcube(H_3). By the definition of IEH graphs, each node in H_2 of $G_3(12)$ has an IC edge connecting with H_3. Consequently, $(0, 8), (1, 9), (2, 10)$ and $(3, 11)$ are IC edges between H_2 and H_3 in $G_3(12)$.

Another example is an IEH graph containing 13 nodes, denoted by $G_3(13)$. $G_3(13)$ consists of three subcubes. The three subcubes are 0-subcube(H_0), 2-subcube(H_2) and 3-subcube(H_3). The node 14 is the single node in H_0. The node numbers of H_2 and H_3 in the $G_3(13)$ are the same as the numbers in the $G_3(12)$. The edges $(8, 14)$ and $(10, 14)$ are IC edges connecting between H_0 and H_2 such that H_0 and H_2 are connected to be an IEH graph containing 5 nodes($G_2(5)$). In addition, the H_3 connects to $G_2(5)$ with the IC edges $(0, 8), (1, 9), (2, 10), (3, 11)$ and $(6, 14)$. The IEH graph containing 13 nodes are shown in Figure 2.

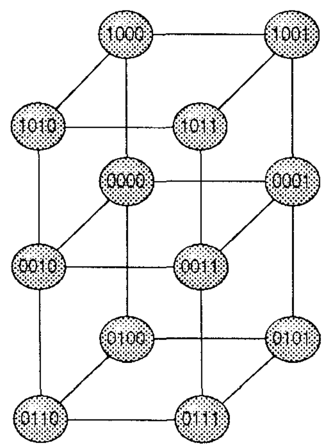

Figure 1. The IEH graph containing 12 nodes

An IEH graph is called a full IEH graph if and only if it has $2^n - 1$ nodes. Intuitively, a full IEH graph must consist of hypercubes H_0, H_1, H_2,..., and H_{n-1}.

The Hamming distance between two nodes $x = x_{n-1}x_{n-2}...x_0$ and $y = y_{n-1}y_{n-2}...y_0$ is defined as $HD(x, y) = \sum_{i=0}^{n-1} x_i \oplus y_i$, where $\oplus$ is defined as the "exclusive-or" operation. In hypercubes or IEH graphs, two nodes are connected by an edge if and only if their Hamming distance is exactly equal to one except IC edges of IEH graphs. The dimension of edge connecting two nodes x and y is defined as $Dim(x, y)$. To be precise, $Dim(x, y) = i$ if and only if $HD(x, y) = 1$ and $x_i \neq y_i$.

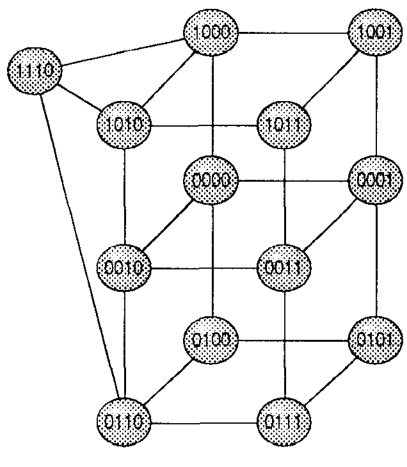

Figure 2. The IEH graph containing 13 nodes

Let $C_n = \{c_{2^n-1}, c_{2^n-2}, ...c_1, c_0\}$ be a sequence of full binary code contains 2^n different binary codewords with n bits, then the reverse sequence of the code is defined as $(C_n)^R = \{c_0, c_1, c_2, ..., c_{2^n-2}, c_{2^n-1}\}$. Normally, the concatenation of two codewords $x = x_{n-1}x_{n-2}...x_0$ and $y = y_{m-1}y_{m-2}...y_0$ is defined as $xy = x_{n-1}x_{n-2}...x_0y_{m-1}y_{m-2}...y_0$. According to the concatenation of two codewords, $0C_n$ can be defined as the sequence of codewords $\{0c_{2^n-1}, 0c_{2^n-2}, ...0c_1, 0c_0\}$ where C_n is a full binary code. Consequently, a full binary code C_n can be defined as $C_n = 0C_{n-1} \cup 1C_{n-1}$.

Definition 3 *The code produced by following procedures are called the Binary Reflective Gray Code (BRGC).*

1. $C_1 = \{0, 1\}$
2. $C_2 = 0C_1 \cup 1(C_1)^R$
3. $C_{n+1} = 0C_n \cup 1(C_n)^R$

For example, a 2-bit Gray Code can be constructed by the sequence as follows: 1) insert a cypher in front of each codeword in C_1; 2) insert a one in front of each codeword in $(C_1)^R$. We get the code C_2 = {00, 01, 11, 10}. Now we can repeat the procedure to built a 3-bit Gray Code, then get the code $C_3 = 0C_2 \cup 1(C_2)^R$ = {000, 001, 011, 010, 110, 111, 101, 100}.

Suppose that H_n is an n-dimensional hypercube, then the binary numbers of all nodes in H_n can be permuted into a sequence of a BRGC). The traversal BRGC sequence of nodes in a hypercube is a Hamiltonian cycle because the hamming distance between last and first node in the sequence is exactly equal to one.

Theorem 1 *Suppose that H_n is an n-dimensional hypercube, then the permutation of nodes in H_n with the sequence in a BRGC C_n is a Hamiltonian path. Consequently, a Hamiltonian cycle exists in a hypercube.*

Proof. Let $C_n = \{a_0, a_1, a_2, ..., a_{2^n-1}\}$ is a BRGC with 2^n different codewords, then these codewords correspond to the label strings of nodes in H_n. Then, for any $a_i \in C_n$, the neighbor nodes of a_i can be defined as a_j and a_k, where $j = (i + 1)\ mod\ 2^n$ and $k = (i - 1)\ mod\ 2^n$. By the definition of BRGC, $HD(a_i, a_j) = HD(a_i, a_k) = 1$. In the construction of hypercube, links exist whenever the hamming distance of two nodes is exactly equal to one, so (a_i, a_j) and (a_i, a_k) are two links exist in H_n. As the consequence, it is sure that the sequence of C_n exactly forms a Hamiltonian path and cycle in an n-dimensional hypercube. □

3 Finding a Hamiltonian cycle

For an IEH graph, hypercubes in the IEH graph are also called subcubes. Each subcube is a hypercube graph, which have complete properties of hypercube. Thus, the way for finding a Hamiltonian cycle in each subcube of an IEH graph is simple except H_0 and H_1. An IEH graph including a Hamiltonian cycle must contain more than 3 nodes since $G_0(1)$, $G_1(2)$,and $G_1(3)$ are not closed graphs.

Lemma 1 *If $G(2^n - 1)$ is a full IEH graph containing $2^n - 1$ nodes, then the IEH graph only contains a Hamiltonian path and no Hamiltonian cycles.*

$G(2^n - 1)$ must be composed by subcubes $H_0, H_1, H_2, ...,$ and H_{n-1}, and the Hamming distance between nodes, connected by IC edges, is exactly 1. Nodes in $G(2^n - 1)$ can be exactly labeled by the number $0, 1, 2, ...,$ and $2^n - 2$. Because the node, labeled by $2^n - 1$, does not exist in $G_n(2^n - 1)$, there is at least one pair of neighboring nodes such that the Hamming distance of the two nodes is greater than 1. Therefore, there is no Hamiltonian cycle in $G_n(2^n - 1)$.

Let $G_n(N)$ be an IEH graph contains N nodes, and $4 \leq N < 2^n - 1$. The steps of finding an Hamiltonian cycle in an IEH graph are listed as follows:

1. Finding a Hamiltonian cycle in each subcube H_i in $G_n(N)$ except subcube H_0 and H_1.
2. Suppose that the Hamiltonian cycles in all subcubes of the IEH graph $G_n(N)$ are C_{H_i} for $i > 1$, then we can combine all of cycles C_{H_i} by IC edges between different-sized subcubes to form a large cycle.
3. Add the nodes of H_1 into the cycle if H_1 exists in the IEH graph.
4. Add the node of H_0 into the cycle if H_0 exists.
5. The final cycle will be a Hamiltonian cycle of the IEH graph.

In the first step of the construction, we can find a Hamiltonian cycle of each subcube containing a Hamiltonian cycle by the sequence of BRGC. In the next step, two cycles in different-sized subcube are linked by the IC edges.

Lemma 2 *Let H_i and H_j are two subcubes in an IEH graph $G_n(N)$ and $2 \leq i < j \leq n$. Suppose $G_n(N)$ has no k-dimensional subcube such that $i < k < j$, then the subcube H_i contains an edge (p,q) and there must exist an edge (r,s) in subcube H_j such that (p,r) and (q,s) are two IC edges in the IEH graph.*

By the definition of IC edges, each node in H_i connects with H_j by $i - j$ IC edges. Suppose that (p,q) is an edge in subcube H_i, then $HD(p,q) = 1$. Two nodes r and s in H_j are respectively the image nodes of p and q if and only if $HD(r,s) = 1$ and $Dim(r,s) = Dim(p,q)$. The edge (r,s) is called an image edge, in H_j, of (p,q).

Lemma 3 *Assume an IEH graph contains two connected subcubes H_i and H_j and the Hamiltonian cycles C_{H_i} and C_{H_j} are found in the subcubes H_i and H_j respectively, where $i < j$. If $(p,q) \in C_{H_i}$, then (r,s) is the image edge of C_{H_j} and $\{(p,r),(q,s)\} \cup C_{H_i} \cup C_{H_j} - \{(p,q),(r,s)\}$ is a Hamiltonian cycle in the graph containing H_i, H_j, and IC-edges between H_i and H_j.*

All of subcubes in an IEH graph have Hamiltonian cycles except H_0 and H_1, and these cycles can be combined into a larger cycle that passes all nodes expect nodes in H_0 and H_1. The following algorithm **Combine** is to combine those subcubes have dimension greater than 1.

Algorithm Combine(C, H_i)
Input: C is a initial cycle, H_i is the i-subcube
Output: a cycle C
Begin

```
    C_{H_i} = the Hamiltonian cycle of H_i
        { found by BRGC };
    Select an edge (p,q) ∈ C;
        { (p,q) must be an edge of the
            maximal-sized subcube in C }
    (r,s) = the image edge of (p,q) in C_{H_i};
    C = {(p,r),(q,s)} ∪ C ∪ C_{H_i} − {(p,q),(r,s)};
    Return C;
```

End.

To find a Hamiltonian cycle in an IEH graph should combine the rest subcube H_0 and H_1 into the previous constructed cycle. Since H_1 includes two nodes with Hamming distance 1 and an edge, it also can be added by the algorithm **Combine.** For the H_0, it is necessary to find two IC edges that connect to two neighboring nodes r and s in a same subcube such that $HD(r,s) = 1$ and (r,s) belongs to the previous constructed cycle. Consider the following cases for an IEH graph contains N nodes:

1. $N = (1(1+0)*01*1)_2$. The IEH graph consists of H_0,H_1,...,H_i, H_{i+k} and so on, where $k \geq 2$. By the construction of an IEH graph, H_0,H_1,... and H_i can be composed into $G_i(2^{i+1} - 1)$, then every node in $G_i(2^{i+1} - 1)$ have k image nodes in H_{i+k}. According to the definition of image nodes, the node in H_0 must contain k image nodes in H_{i+k}. Suppose p is the node of H_0, then the image nodes set S_p^{i+k} contains $k-1$ combinations of two image nodes (r,s) such that $DIM(r,s) = 1$. In the $k-1$ combinations, there must be a pair which exist in the cycle constructed by the algorithm **Combine.** Therefore, the node p of H_0 can be added into the cycle by moving the edge (r,s) out of the cycle and inserting two edges (p,r) and (p,s).
2. $N = (1^+)_2$. The IEH graph is an full IEH graph. It has been proved that there is no Hamiltonian cycle in a full IEH graph.

Conceptually, finding a Hamiltonian cycle in an IEH graph can be constructed by connecting all the Hamiltonian cycles of subcubes with dimension larger than 1. Afterwards, add the noncycle components H_0 and H_1 into the whole cycle if they exist in the IEH graph. The following algorithm **FindHamiltionCycle** is the brief steps of constructing a Hamiltonian cycle of an IEH graph.

Algorithm FindHamiltonianCycle($G_n(N)$)
Input: $G_n(N)$ is an IEH graph contains N nodes
Output: a Hamiltonian cycle C
Begin
    $N = x_{n-1}x_{n-2}\cdots x_1x_0$;
    For $i = 2$ **to** $n-1$ **Do**
        C_{H_i} = the Hamiltonian cycle of H_i;
            {constructed by BRGC};
    $C = \Phi$;
    For $i = 2$ **to** $n-1$ **Do**
        If $x_i = 1$ **then** C =Combine(C, C_{H_i});
    If $x_1 = 1$ **then**
        begin
            (r,s)=an image of (p,q) in H_1;
            $C = \{(p,r),(q,s)\} \cup C \cup C_{H_i}$;
            $C = C - \{(p,q)\}$;
        end
    If $x_0 = 1$ **then**
        begin
            (r,s)= the selected edge;
             {where (p,r) and (p,s) are IC edges
              in H_i and H_0 respectively}
            $C = \{(p,r),(p,s)\} \cup C$;
        end
    Return C;
End.

4 Examples

The first example is to find a Hamiltonian cycle in the IEH with 12 nodes. Because $12 = 1100_2$, the maximal dimension of subcube in the IEH graph is 3. $G_3(12)$ consists of two subcubes H_2 and H_3. Finding the Hamiltonian cycle of $G_3(12)$ are shown in Figure 3 and listed as follows:

1. Using the sequence of the BRGC, we can find the Hamiltonian Cycles C_{H_2} and C_{H_3} in the subcubes H_2 and H_3 respectively. Hence $C_{H_2} = \{(8,9), (9,11), (11,10), (10,8)\}$ and $C_{H_3} = \{(0,1), (1,3), (3,2), (2,6), (6,7), (7,5), (5,4), (4,0)\}$.

2. Initially, $C = \Phi$. Update the cycle by executing *Combine*(C, C_{H_3}), and then execute *Combine*(C, C_{H_2}) to add the cycle C_{H_2} into the whole cycle. The latter combination is described as follows.

    (a) Select an edge in the C_{H_2}, denoted by (p,q), and test if p and q have image nodes in the existing cycle. Suppose $(p,q) = (8,9)$ and the image nodes of 8 and 9 are 0 and 1 respectively, then the image edge of $(8,9)$ is $(r,s) = (0,1)$.

    (b) The new cycle is $C = \{(0,8),(1,9)\} \cup C \cup C_{H_2} - \{(0,1),(8,9)\}$.

3. There is no H_1 and H_0 existing in the IEH graph. Finally, $C = \{(0,8), (1,9), (9,11), (11,10), (10,8), (1,3), (3,2), (2,6), (6,7), (7,5), (5,4), (4,0)\}$ is a Hamiltonian cycle of the IEH graph with 12 nodes.

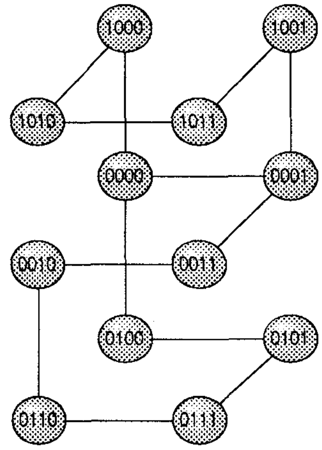

Figure 3. The Hamiltonian cycle of $G_3(12)$

The second example is to find a Hamiltonian cycle in the IEH graph with 13 nodes. For $13 = 1101_2$, $G_3(13)$ is composed by subcubes H_3,H_2, and H_0. The procedures of finding the Hamiltonian cycle are listed as follows:

1. Let C_{H_2} and C_{H_3}be the Hamiltonian Cycle of the subcubes H_3 and H_2 respectively. They are constructed by BRGC. Then $C_{H_2} = \{(8,9), (9,11), (11,10), (10,8)\}$ and $C_{H_3} = \{(0,1), (1,3), (3,2), (2,6), (6,7), (7,5), (5,4), (4,0)\}$.

2. The cycle that traverses all nodes in the two subcubes H_2 and H_3 are constructed as the result in first example.

3. There is no subcube H_1 in the $G_3(13)$, but it contains H_0. The binary string of number 13(1101) is a string of the form $(1(1+0)^*01^*1)_2$. Therefore, H_0 has two IC edges connect to subcube H_2. Let the node in H_0 is $p = 14$. Two nodes connect to H_0 are 8 and 10, then the edge $(r,s) = (8,10)$ is the image edge of H_0 and $(8,10)$ is also an edge in the cycle C. Finally, $C = \{(14,8),(14,10)\} \cup C - \{(8,10)\}$ is a Hamiltonian cycle.

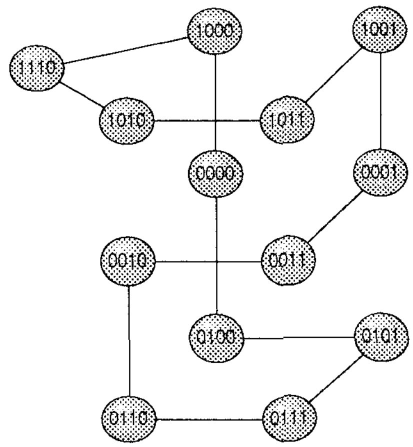

Figure 4. The Hamiltonian cycle of $G_3(13)$

The Hamiltonian cycle found by above steps is shown in Figure 4.

5 Conclusion

This paper discussed the existence of Hamiltonian cycles and paths. And the methods for finding a Hamiltonian cycle in an IEH graph are developed. If an IEH graph is a full IEH graph, then there is no Hamiltonian cycle in the graph. This algorithm is proceeded by combining all Hamiltonian cycles of the subcubes in an IEH graph. The result will be used for the embedding of rings in the IEH graphs in our further researches.

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