Hasofer-Lind's Reliability Based Optimization for Multiobjective Fuzzy and Stochastic Design Problem

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Abstract

This paper introduces a design method using fuzzy logic to find the best stochastic design by maximizing Hasofer-Lind's (H-L's) reliability and simultaneously optimizing design goals. The H-L's reliability of the system is represented by the shortest distance from the origin to the failure surface. The formulation of the problem involves random parameters and fuzzy probabilistic constraints. The objective weighting strategy in multiobjective fuzzy formulation is adopted to represent the importance among the design goals. The paper presents the computation of H-L's reliability index, the formulation of multiobjective fuzzy probabilistic constraints and the optimization process by an engineering design problem that has random loads and random parameters.

1. Introduction

A recognized means of handling the uncertain information existing in the real-world engineering problem is to deal with them as random parameters or random variables. In the field of optimization, stochastic or probabilistic programming [1] deal with such circumstance where some parameters and variables are random or probabilistic. A typical stochastic programming problem consists of probabilistic constraints where the safety constraint as a whole has to be greater than or equal to a specified probability. Another type of uncertainty exists in the real-world problem as vagueness or fuzziness recognized by people but unsolved until Zadeh [2] proposed the fuzzy set theory [3]. Many areas such as humanistic system, expert systems, natural languages and decision analysis [4-7] were expanded by the role of fuzzy logic. Especially during the last decade, some researchers developed optimization methods based on fuzzy formulation to work out the problem in which goals and constraints were often represented by the vague linguistic form [8-10]. Thus the fuzzy single or multiobjective optimum design problem [11-13] even in discrete design space [14] appears to be more fascinating than ever before.

The earliest effort to do the probabilistic optimization in engineering design may be the chance programming technique proposed by Rao [1] who consider the engineering structural security by the probabilistic constraints. After that several works of probabilistic optimum design were studied and presented in the engineering [15,16]. The recent literature was written by Wang, Grandhi and Hopkins [17] who developed an algorithm to work on the probabilistic single-objective optimization problems. Although this method can guarantee a constraint of the certain strength below a predetermined probability of failure, however, it lacks the enhancement of the structural reliability. Also, the importance of structural reliability on a same level with the other design objectives is not adequately considered. As one knows that a precision engineering system such as the aerospace applications demand the lowest weight, the highest performance and reliability. Nevertheless, no literature has presented a design problem and method containing the relative importance existing between the multiple design objectives and reliability, nor the fuzzy and random information existing in it.

This paper presents a multiobjective fuzzy optimization process for maximizing the Hasofer and Lind's (H-L) reliability index [18] of an engineering system and other design goals at the same design level simultaneously. The computations of H-L's reliability index of a problem containing several limited state functions are also presented

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in the paper. The other features of the presented paper contain random, fuzzy, probabilistic and even fuzzy-probabilistic information in the design phase. A concerning paper presented in FUZZ-IEEE/IFES'95 by the leading author [19] had given a background of such fuzzy and probabilistic design environment. An objective weighting technique [20] is used in the multiobjective fuzzy optimization that can generate a design representing the relative important degree of individual objective function and reliability. We illustrate the proposed method by applying it to design a symmetric structure with random loading and other stochastic parameters.

2. Multiobjective Fuzzy Optimization Problem with Maximizing H-L's Reliability

The mathematical formulation for a multiobjective and reliability allocation optimization problem with fuzzy and random information is stated as follows:

Find **X** that minimized

$$[f_1(\mathbf{X},\mathbf{Y}), f_2(\mathbf{X},\mathbf{Y}), \ldots, f_N(\mathbf{X},\mathbf{Y}), R_{system}(\mathbf{X},\mathbf{Y},\mathbf{R})]^T \quad (1)$$

subject to

$$g_i(\mathbf{X},\mathbf{Y}) \leq (C_{fuzzy})_j, \quad i=1,2,\ldots,m_1 \quad (2)$$

$$P[g_j(\mathbf{X},\mathbf{Y}) \leq 0] \geq p_j, \quad j=1,2,\ldots,m_2 \quad (3)$$

$$P[g_k(\mathbf{X},\mathbf{Y}) \leq 0] \geq (p_{fuzzy})_j, \quad k=1,2,\ldots,m_3 \quad (4)$$

$$R_{system}(\mathbf{X},\mathbf{Y},\mathbf{R}) \geq R_{specified} \quad (5)$$

where **X** is a vector of random design variables, $\mathbf{X}=[x_1,x_2,\ldots x_{nx}]^T$, required to be found out; **Y** represents a vector of random parameters, $\mathbf{Y}=[y_1,y_2,\ldots y_{ny}]^T$. C_{fuzzy} and p_{fuzzy} represent the allowable fuzzy limited value and the fuzzy probability in a constrained function, respectively. Eq. (3) shows the probability of satisfying the jth constraint, $g_j(\mathbf{X},\mathbf{Y}) \leq 0$, must be equal to or greater than an expected value p_j corresponding to the jth constraint. For obtaining the maximized system reliability, we adopt the strategy that maximizes the minimum H-L's based reliability index β corresponding to each failure constrained function as follows:

$$\text{maximize } (\text{min. } (R_1, R_2, \ldots R_{nR})) \quad (6)$$

where the reliability R_i corresponding to the ith failure function is obtained by the following equation:

$$R_i = 1 - P_f = 1 - \phi(-\beta_i), \quad i=1,nR \quad (7)$$

where ϕ is the standardized normal distribution function. Normally, one directly uses the value of H-L's reliability index β to measure the reliability. Thus β is equal to $-\phi^{-1}(P_f)$. The construction of the membership function requires solving the worst case or the relaxing condition of every single objective optimization problem. Consequently, a general λ-formulation of the multiobjective fuzzy optimization can be stated as:

$$\text{maximize } \lambda \quad (8)$$

subject to

$$\lambda - \mu_{fi(X,Y)} \leq 0, \quad i=1,2,\ldots,N \quad (9)$$

$$\lambda - \mu_{Rsystem(X,Y,R)} \leq 0, \quad i=1,2,\ldots,nR \quad (10)$$

$$\lambda - \mu_{gi(X,Y)} \leq 0, \quad i=1,2,\ldots,m_1 \quad (11)$$

$$g_j(\mathbf{X},\mathbf{Y}) + \phi_j(p_j)\left[\sum_{i=1}^{nx}\left(\frac{\partial g_j}{\partial x_i}\Big|_X\right)^2 \sigma_{xi}^2 + \sum_{i=1}^{ny}\left(\frac{\partial g_j}{\partial y_i}\Big|_Y\right)^2 \sigma_{Yi}^2\right]^{1/2} \leq 0, \quad j=1,2,\ldots,m_2 \quad (12)$$

$$\lambda - \mu_{P[gi(X,Y) \leq 0]} \leq 0, \quad i=1,2,\ldots,m_3 \quad (13)$$

where λ is a scalar between zero and one. It is treated as an objective function as well as a design variable. $\mu_{(i)}$ here represents a membership function corresponding to the ith function which is constructed by solving individual single objective optimization problem with tight and relaxed conditions. $\mu_{Rsystem(X,Y,R)}$ represents the membership function of the system reliability obtained by the similar way. If the relative important rank among the multiobjective function is predetermined, the mathematical formulation is increased by the following equality constraints:

$$\omega_i\left(\frac{f_i(\mathbf{X},\mathbf{Y}) - f_i^{id}}{f_i^{id}}\right) = \omega_{i+1}\left(\frac{f_{i+1}(\mathbf{X},\mathbf{Y}) - f_{i+1}^{id}}{f_{i+1}^{id}}\right), \quad i=1,2,\ldots,N-1 \quad (14)$$

$$\omega_N\left(\frac{f_N(\mathbf{X},\mathbf{Y}) - f_N^{id}}{f_N^{id}}\right) = \omega_R\left(\frac{R_{system}(\mathbf{X},\mathbf{Y}) - R^{id}_{system}}{R^{id}_{system}}\right) \quad (15)$$

$$\omega_1 + \omega_2 + \ldots + \omega_N + \omega_R = 1 \quad (16)$$

where ω_i represents the relative important degree of the ith objective function. Once the fuzzy optimization problem with objective weighting strategy is formulated as here, one can solve this problem by any preffered and reliable nonlinear optimization algorithm.

3. Hasofer and Lind's Reliability Index

One normalizes a set of uncorrelated design variables of $\mathbf{X}=[x_1,x_2,...x_{nx}]^T$, the new set of variables $\mathbf{Z}=[z_1,z_2,...z_{nx}]^T$ is defined by the follows:

$$z_i = \frac{x_i - \mu_{xi}}{\sigma_{xi}}, \quad i=1,2,...,nx \qquad (17)$$

where μ_{xi} and σ_{xi} are the mean and the standard deviation of the random variable x_i, respectively. Note that $\mu_{zi}=0$ and $\sigma_{zi}=1$, $i=1,2,...,nx$. Hasofer and Lind's reliability index β is defined as the shortest distance from the origin to the failure surface in the normalized z-coordinate system. For a two-dimensional case in Fig. 1, β is equal to the distance OA. The point of A is called the design point.

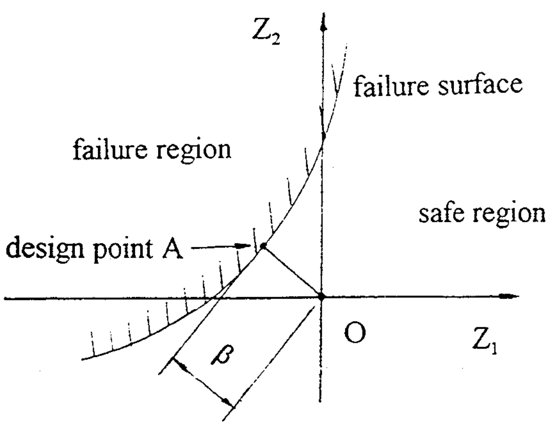

Fig. 1 Two dimensional representation of the Hasofer and Lind's reliability index.

The definition of the reliability index β by Hasofer and Lind can be formulated in the following way:

$$\beta = \min_{\partial\omega} \left(\sum_{i=1}^{nx} z_i^2 \right)^{1/2} \qquad (18)$$

where $\partial\omega$ is the failure surface in the z-coordinate system. The distance β is given by the vector $\mathbf{OA} = \beta\,\boldsymbol{\alpha}$ where $\boldsymbol{\alpha}$ is the unit vector represented as $[\alpha_1,\alpha_2,\alpha_3,...\alpha_n]^T$.

Thus, for a failure function, there exists a failure surface. Each failure surface has a distance β corresponding to a unit vector of $\boldsymbol{\alpha}$. One can apply the above description to rewrite the form of each design variable. For example, one can transform a random design variable x_i to the form of $\mu_x + \sigma_{xi} z_i$ from Eq. (17). z_i can further be represented as $\beta\alpha_i$. Therefore, the optimum design variables are obtained by the optimization with intermediate variables of μ_{xi}, β_j and $\boldsymbol{\alpha}$.

4. Illustrative Example

A symmetric three-bar structural design is popular to use as an illustrative example for an optimization algorithm and process. The configuration and a 40000 lb of loading P is shown in Fig. 2 where θ is 45 degree. The detailed information can be obtained from the book of Arora [21]. Each member has a cross-sectional area A_i (i=1,2,3), where $A_1=A_3$.

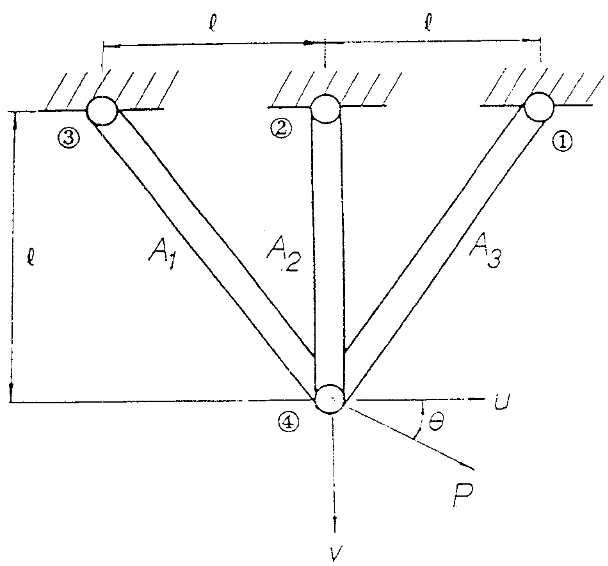

Fig. 2 A symmetric three-bar structure with random loading and parameters.

Each member has two failure modes of yielding and buckling. We modify the problem and formulate it as a multiobjective fuzzy optimization problem with the maximization of reliability in an environment of stochastic parameters and variables. The problem is to find the cross-sectional area of each member by minimizing the structural weight and maximizing the Hasofer and Lind's reliability, simultaneously. The stochastic data of the problem are given in Table 1.

Table 1 Design data for a symmetric three-bar structure.

Design variables	$A_1 = A_3 = N(\mu_{A1}, 0.01\mu_{A1})$ in^2
	$A_2 = N(\mu_{A2}, 0.01\mu_{A2})$ in^2
Horizontal load	$P_u = N(20000\sqrt{2}, 4000\sqrt{2})$ lb
Vertical load	$P_v = N(20000\sqrt{2}, 4000\sqrt{2})$ lb
Length parameter	$\ell = N(1, 0.01)$ in
Young's modulus	$E = N(1.0, 0.1)\times 10^7$ psi
Allowable stress	$\sigma_{1a} = N(5000, 500)$ psi $= \sigma_{3a}$
	$\sigma_{2a} = N(20000, 2000)$ psi
Allowable displacements	$u_a = N(0.005, 0.0005)$ in
	$v_a = N(0.005, 0.0005)$ in
Mass density	$\rho = N(3.125, 0.3125)\times 10^{-3}$ lb$_m$/in^3
Lower limit on frequency	$\omega_0 = 2500$ Hz

The multiobjective optimum design with reliability allocation is expressed by minimizing the structural volume of V(**X**) and maximizing system reliability of β_{system} as follows:

Find $\mathbf{X} = [A_1, A_2]^T$ that minimize $F(\mathbf{X}) = [V(\mathbf{X}), -\beta_{system}(\mathbf{X},\mathbf{Y})]^T$ (19)

where

$$V(\mathbf{X}) = \ell\,(2\sqrt{2}\,A_1 + A_2) \qquad (20)$$

$$\beta_{system}(\mathbf{X},\mathbf{Y}) = \text{minimum}\ \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\} \qquad (21)$$

subject to

$$s_i(\beta_i, \boldsymbol{\alpha}_i) - \sigma_{ia} \le 0\ ,\ i=1,2\ \ (\text{for } g_1(\mathbf{X}) \text{ and } g_2(\mathbf{X})) \qquad (22)$$

$$-s_{bi}(\beta_i, \boldsymbol{\alpha}_i) - \pi^2 E I_i / \ell_i^2 \le 0\ ,\ i=1,2,3\ \ (\text{for } g_3(\mathbf{X}), g_4(\mathbf{X}) \text{ and } g_5(\mathbf{X})) \qquad (23)$$

$$\sum_{k=1}^{4} (\alpha^2_k)_{gi} = 1\ ,\ i=1,2,3,4,5 \qquad (24)$$

$$P[u(\mathbf{X},\mathbf{Y}) - u_a \le 0] \ge 0.95 \sim 0.999 \qquad (25)$$

$$P[v(\mathbf{X},\mathbf{Y}) - v_a \le 0] \ge 0.95 \sim 0.999 \qquad (26)$$

$$P[(2\pi\omega_o)^2 - \zeta(\mathbf{X},\mathbf{Y}) \le 0] \ge 0.98 \qquad (27)$$

The variable of A_i (i=1,2) has to design between 0.1 and 100. in^2. The vector of **Y** contains random parameters such as those in Table 1 excluding design variables of A_1 and A_2. $s_i(\beta_i, \boldsymbol{\alpha}_i)$ is the representation of stress under the load P for members 1 and 2. $-s_{bi}(\beta_i, \boldsymbol{\alpha}_i)$ representes the buckling stress of members 1,2 and 3. In this problem, Eq. (22) and (23) are considered as the five failure functions of $g_1(\mathbf{X})$ to $g_5(\mathbf{X})$. Thus, each of the failure function corresponds to a H-L's reliability index β_i (i=1,...,5). It includes the stochastic variable of A_i(i=1,2) and random parameters of P_u and P_v. The formulations for the horizontal displacement of u(**X**,**Y**), the vertical displacement of v(**X**,**Y**) and the natural frequency of ζ(**X**,**Y**) can be found in the book of Arora [21].

To tackle this problem, we solve each single-objective optimization first in the most relaxing condition. The computing algorithm and solution process are presented as following for a better illustration for this optimization problem.

Step 1: Initiate the pseudo variables of μ_{A1}, μ_{A2}, α_{ij}, i=1,5, j=1,4 and β_1 to β_5.

Step 2: Replace the stochastic variables and parameters of A_1, A_2, P_u and P_v by their mean, H-L's reliability index, component of unit vector and deviation. For example, $A_1 = \mu_{A1}+\beta_i\alpha_1\sigma_{A1}$, $A_2 = \mu_{A2}+\beta_i\alpha_2\sigma_{A2}$, $P_u = \mu_{Pu}+\beta_i\alpha_3\sigma_{Pu}$, and $P_v = \mu_{Pv}+\beta_i\alpha_4\sigma_{Pv}$ where i=1,5 to match the five failure functions.

Step 3: Formulate the objective function as Eq. (20). Formulate the thirteen constrained functions in which consists of ten constraints concerning H-L's reliability (22-24) and three probabilistic constraints (25-27).

Step 4: Apply Eq. (20) and set up a range of reliability index. For minimizing the volume, reliability is between 1.645 and 10. Table 2 shows the optimum design of the single-objective design in the worst conditions.

Step 5: Using the λ-formulation of fuzzy optimization to construct the mathematical model with minimizing the total volume and maximizing the H-L's reliability. In general, the linear membership function is adopted for the solutions.

Step 6: For the known important rank of each design goal, Eq. (14) to (16) are used to solve the multiple fuzzy optimization problem.

Step 7: Design engineer select the optimum design.

Table 2 The results of single-objective design in the worst conditions for 3-bar problem.

Minimize V(X)			
V^*	β_{min}	A_1	A_2
18.966 in^3	1.654	4.62 in^2	5.86 in^2
Maximize $\beta_{min}(X)$			
V	β^*_{min}	A_1	A_2
34.274 in^3	5.0	9.50 in^2	7.39 in^2

The results of fuzzy formulation with weighting strategy are listed in Table 3. The design without relative weighting coefficients has a highest satisfaction and it is different from the result of equal importance.

Table 3 Results of multiobjective design with weighting technique for the 3-bar.

(ω_V, ω_β)	V(X)	β_{min}	A_1	A_2
Without weighting	20.2931	4.7378	6.239	2.648
(0.1 , 0.9)	29.6511	4.5677	7.769	7.676
(0.3 , 0.7)	21.7210	4.645	6.288	3.936
(0.5 , 0.5)	20.3263	4.642	5.959	3.470
(0.7 , 0.3)	19.7179	4.642	5.852	3.166
(0.9 , 0.1)	19.2426	4.642	5.746	4.643

This illustrative design example has a compromise result between the values of single objective optimization. The weighing technique combined with the multiobjective optimization design generates the sets of Pareto optimum solutions where a larger weighting rank corresponding to a higher satisfaction of an objective function.

5. Conclusion

In this paper, we present a multiobjective fuzzy and stochastic optimization process by fuzzy formulation to optimize the fuzzy multiple design goals and to maximize the Hasofer and Lind's reliability derived from the constraints of failure functions. This optimization problem contains stochastic parameters and random design variables. In the computation, pseudo variables such as the components of the unit vector are generated in the solution process. The constrained functions consist of fuzzy, probabilistic and fuzzy probabilistic constraints. Each constraint has to be transformed to a deterministic constraint, then one can solve the problem by any reliable optimization algorithm. The optimum result without the weighting rank is a design set with the highest design satisfaction. The expression of the relative importance for the individual objective function can be done by adding the extra constraints described in the paper. Each design corresponding to different weighting coefficient is in the Pareto solutions set. The presented design model and process is simple and successfully built up; however, the solution technique of reducing the pseudo design variables requires a further study.

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