Progressively Interval-censored Life Test with Acceptance Sampling

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Abstract

Considering the producer and consumer risks, the paper develops acceptance sampling procedures under the progressively interval-censored test with intermittent inspections for the exponential lifetime model. The proposed approach allows removing surviving items during the life test such that some extreme lifetimes can be sought, or the test facilities can be freed up for other tests. A reduction in testing effort and administrative convenience can be achieved by employing the proposed approach. One example is introduced for illustration.

1 Introduction

Acceptance sampling is a set of rules concerning with inspection and decision making regarding items. In life-testing applications, the quality characteristic of an item often is measured by its lifetime. Acceptance sampling plan used to determine the acceptability of items with respect to their lifetimes are also called the life test plan (LTP). Since engineering technologies have highly developed nowadays, most products have been designed with high reliabilities. Thus, how to develop acceptance sampling procedures for high reliability items becomes an important problem to manufacturers.

In life-testing applications, it may take a long time to collect complete lifetime data. For saving test time, LTPs often are developed under the type I censoring or the type II censoring schemes. In a type I censoring scheme, life tests are run over a fixed time period in such a way that an item's lifetime will be known exactly only if it is less than or equal to some predetermined values. Alternatively, in a type II censoring scheme, the experimenter terminates the life test when the r smallest observations in a random sample of n items are observed.

Traditionally, a LTP consists of the sample size required in the life test and the corresponding critical value such that the quality personnel can make a decision to the testing lot of items based on the critical value. For example, the LTP can be designed to protect both the produce and consumer such that the produce risk and consumer risk are lower than or equal to the required levels. The produce risk denotes the probability that a good lot of items will be rejected. The consumer risk denotes the probability of accepting a lot of items with poor quality. Based on the producer and consumer risks, some LTPs have been developed. Epstein and Sobel [8] constructed LTPs for exponential distribution with type II censoring. Epstein [7] proposed a hybrid LTP which combines the type I censoring and the type II censoring. Jeong and Yum [10] developed LTPs for exponential distribution with type I censoring. Kim and Yum [11] did comparisons of different designs of the exponential LTP with intermittent inspections. Wu and Tsai [18] developed the design of truncated LTP for Birnbaum-Saunders distribution. Tsai and Wu [14] proposed the design of truncated LTP for generalized Rayleigh distribution.

The type I and type II censoring schemes, however, do not allow removing surviving items at the times other than the termination time of the life test. This allowance, however, may be desirable when a compromise between reducing test time and an expectation of some extreme lifetimes in life test can be sought. This reason motivates the quality personnel to adopt a progressively censoring scheme, for example, the progressively type I censoring scheme and the progressively type II censoring scheme. The progressively type I censoring scheme allows removing surviving items at some fixed times before the termination time of life test, and the progressively type II censoring scheme allows removing surviving items at the time of each failure of the $r-1$ smallest observations are observed if r failures are required among n test items. A comprehensive reference regarding the progressive censoring tests, its applications and mathematical results can be found in Balakrishnan and Aggaarwala [3]. The parameter estimations of the exponential, lognormal and Weibull lifetime models with progressive type I censoring have been discussed by Cohen [6], and Wong [17], respectively. The problems of simulation, estimation, calculation of moments and the construction of

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a LTP with progressive type II censoring have been discussed by Aggarwala and Balakrishnan [2], Balakrishnan and Sandhu [4], Balakrishnan and Saw [5], Tse and Yuen [13], Viverps and Balakrishnan [16] and Yuen and Tse [19], respectively.

2 Motivation

It is easier to develop the statistical inferences with type II censoring data than using the type I censoring data. But the termination time of life test under a type II censoring scheme is randomly and cannot be predetermined. Moreover, the quality personnel may inspect the test items only at some specific times during the life test so that the exact failure times cannot be observed. These reasons encourage quality personnel to conduct a life test with an interval-censored scheme which only records the failure numbers in some predetermined time intervals and not record the exact lifetimes of items continuously. Based on the administrative convenience and a compromise between reducing test time and the allowance to remove surviving items during the life test, the study is thus motivated to develop LTPs under a progressively interval-censored scheme with intermittent inspections. Basically, the construction of the proposed LTP is easy for practitioners.

3 Progressively Interval-censored Test

Assume that the lifetimes of items $Y_1, Y_2, \cdots$ are independent and identically distributed as an exponentially distributed with the following cumulative density function (c.d.f.)

$$F(y|\eta) = 1 - exp\left(-\frac{y}{\eta}\right),\ y > 0,\ \eta > 0, \qquad (1)$$

where η is the scale parameter and the mean lifetime of items. Assume that a lot of items taken from (1) are submitted to an inspection of acceptance sampling. The lot is accepted if the mean lifetime η is longer than or equal to η_0, and rejected if the mean lifetime reduces to $\eta_a(< \eta_0)$ or less, where η_0 and η_a are two predetermined level. In the development of the life test, we can transform the lifetime data by letting $T_i = Y_i/\eta_0$, $i = 1, 2, \cdots$ so that the transformed data do not depend on the measuring scale. Let $\theta = \eta/\eta_0$, it can be shown that $T_1, T_2, \cdots$ are independent and identically exponentially distributed with parameter θ.

The progressively interval-censored test can be conducted as follows: Assume that $m(\geq 2)$ inspection times $0 < \tau_1 < \tau_2 < \cdots < \tau_m < \infty$ are predetermined and n items are drawn randomly from the lot and put on a life test at the initial time 0. At the time τ_1, X_1 failure items in the interval $(0, \tau_1]$ are recorded, and R_1 of the $n - X_1$ surviving items are selected randomly and removed. Continuing on the test, at the time τ_2, X_2 failure items in the interval $(\tau_1, \tau_2]$ are recorded, and R_2 of the $n - X_1 - X_2 - R_1$ surviving items are selected randomly and removed. Finally, at the termination time τ_m, X_m failure items in the interval $(\tau_{m-1}, \tau_m]$ are recorded, and all surviving numbers $R_m = n - \sum_{j=1}^{m} X_j - \sum_{j=1}^{m-1} R_j$ are removed. Then the test is stopped. In the life test, either the proposed values of $R_1, R_2, \cdots, R_{m-1}$, or probabilities of units removed $p_1, p_2, \cdots, p_{m-1}$ are predetermined such that $p_m = 1$ and $R_i = [(n - \sum_{j=1}^{i} X_j - \sum_{j=1}^{i-1} R_j)p_i]$, $i = 1, 2, \cdots, m$, where $[s]$ is the largest integer smaller than or equal to s.

4 Ordinary and Approximate Life Test Plans

Assume that a random sample of size n are drawn from a lot and put on a progressively interval-censored test with given numbers of removals $\boldsymbol{R} = (R_1, , R_2, \cdots, R_m)$. The failure numbers in the given time intervals are collected and denoted by $\boldsymbol{X} = (X_1, X_2, \cdots, X_m)$, respectively. Let $B(a, b)$ denote the binomial distribution with the number of individuals a and the probability of success b. Let $\nu_1 = n$, $\delta_1(\theta) = F(\tau_1|\theta)$, $\nu_i = n - \sum_{j=1}^{i-1}(X_j + R_j)$ and $\delta_i(\theta) = \frac{F(\tau_i|\theta) - F(\tau_{i-1}|\theta)}{1 - F(\tau_{i-1}|\theta)} = 1 - exp(-\frac{\tau_i - \tau_{i-1}}{\theta})$, $i = 2, 3, \cdots, m$. It can be shown that $X_1 \sim B(\nu_1, \delta_1(\theta))$ and for $i = 2, 3, \cdots, m$, $X_i|X_{i-1}, ..., X_1, R_{i-1}, ..., R_1 \sim B\left(\nu_i, \delta_i(\theta)\right)$.

The statistical hypotheses regarding the acceptance sampling of the life test can be described as $\mathrm{H}_0 : \theta = 1$ vs. $\mathrm{H}_1 : \theta = \theta_a(< 1)$. The θ_a is so-called the discrimination ratio. In particular, the life test is conducted with intermittent inspections, that is, m inspections are equally-spaced such that the length of the i-th time interval $\tau_i - \tau_{i-1} = \tau$ and $\delta_i(\theta) = \delta(\theta) = 1 - exp(-\tau/\theta)$, $i = 1, 2, \cdots, m$. If $X_1 = n$ then the log-likelihood is maximized when θ approaches to zero, otherwise, if $R_m = n$, then log-likelihood is maximized when θ approaches to infinity, otherwise the maximum likelihood (ML) estimator of θ can be obtained. According to the computation procedure of Aggarwala [1], the likelihood estimator of θ can be found and denoted by $\hat{\theta}$ (see Tsai *et al.*[15]). The producer risk and the consumer risk are defined respectively as $P_R = P(\hat{\theta} < c|\theta = 1) = \sum_{R_e} P(\boldsymbol{x}|\boldsymbol{R}, \theta = 1)$ and $P_C = P(\hat{\theta} \geq c|\theta = \theta_a) = 1 - \sum_{R_e} P(\boldsymbol{x}|\boldsymbol{R}, \theta = \theta_a)$, where c is a constant called the critical value, $R_e = \{\boldsymbol{x}| \sum_{i=1}^{m}(x_i + R_i) = n, \hat{\theta} < c\}$, and $P(\boldsymbol{x}|\boldsymbol{R}, \theta)$ is the joint probability of $\boldsymbol{X} = \boldsymbol{x}$. Based on the ML estimate of $\hat{\theta}$, the ordinary LTP (n, c) can be developed such that the producer and consumer risks as follows are satisfied:

$$P_R \leq \alpha \ \text{ and } \ 1 - P_C \geq 1 - \beta. \qquad (2)$$

Assume that the number of removals are given by $R_i = [(n - \sum_{j=1}^{i} X_j - \sum_{j=1}^{i-1} R_j)p_i]$, $i = 1, 2, \cdots, m-1$, where

$R_0 = 0$. A searching procedure with the following steps is suggested to establish the ordinary LTP:

Step 1: Specify the values of m, τ, α, β, θ_a and $p_1, p_2, \cdots, p_m$.

Step 2: Let $n = 3$.

Step 3: Generate all possible combinations of $\boldsymbol{x}$ and $\boldsymbol{R}$ based on the process of $X_1 \sim B(\nu_1, \delta_1(\theta))$ and for $i = 2, 3, \cdots, m$, $X_i | X_{i-1}, ..., X_1, R_{i-1}, ..., R_1 \sim B(\nu_i, \delta_i(\theta))$, and compute the ML estimate of θ for each combination of $\boldsymbol{x}$ and $\boldsymbol{R}$, and denoted by $\hat{\theta}$.

Step 4: The sets of $\{\hat{\theta}, P(\boldsymbol{x}|\boldsymbol{R}, \theta = 1)\}$ and the $\{\hat{\theta}, P(\boldsymbol{x}|\boldsymbol{R}, \theta = \theta_a)\}$ for all possible combinations of $\boldsymbol{x}$ constitute the p.d.f.'s of $\hat{\theta}$ under $\theta = 1$ and $\theta = \theta_a$, respectively. Hence, the c.d.f.'s of $\hat{\theta}$ under $\theta = 1$ and $\theta = \theta_a$ can be determined by accumulating the joint probabilities of $P(\boldsymbol{x}|\boldsymbol{R}, \theta = 1)$ and $P(\boldsymbol{x}|\boldsymbol{R}, \theta = \theta_a)$, respectively.

Step 5: If there exist c's for which equations in (2) are satisfied, then select the largest one among those c's and stop. The desired ordinary LTP is determined. Otherwise, go to Step 6.

Step 6: Let $n = n + 1$, go to Step 3.

The ordinary LTPs are exact, however, the method may fail due to a memory overflow error especially for large m and n. The computation is complicated and time consuming. For get over this difficulty, an approximate method is proposed to find the approximate LTPs. Based on asymptotic theory of ML estimator, we can approximate the sampling distribution of $\hat{\theta}$ by a normal distribution with mean θ and variance $I^{-1}(\theta)$, where $I(\theta)$ is the Fisher information (see Hogg and Craig [9]). Tsai *et al.* [15] prove the following results: Let $\phi_1(\theta) = \delta(\theta)$, and for $i = 2, 3, \cdots, m$, let $\phi_i(\theta) = \delta(\theta)\left[1 - \sum_{j=1}^{i-1} p_j (1-\delta(\theta))^{2i-j-1} \prod_{h=1}^{i-1}(1-p_h)\right]$,

$$G_1(\theta) = \left[\frac{\tau(1-\delta(\theta))}{\theta^3 \delta(\theta)}\right]\left[\frac{\tau}{\theta\delta(\theta)} - 2\right]\sum_{i=1}^{m}\phi_i(\theta),$$

$$G_2(\theta) = \frac{2\tau}{\theta^3}\sum_{i=1}^{m}(m-i+1)\phi_i(\theta);$$

if $m = 2$, let $G_3(\theta) = \frac{2\tau}{\theta^3}[2 - (1-\delta(\theta))p_1]$; if $m = 3$, let

$$G_3(\theta) = \frac{2\tau}{\theta^3}\left[m - (m-1)(1-\delta(\theta))p_1 - \sum_{i=2}^{m-1}(m-i)p_i(1-\delta(\theta))^i \prod_{j=1}^{i-1}(1-p_j)\right].$$

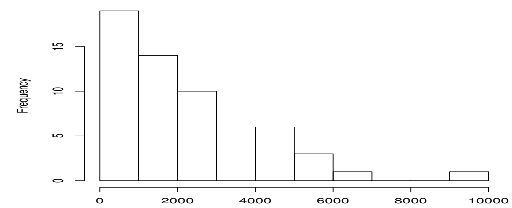

Figure 1. Histogram of the number of cycles to failure data.

Let $G(\theta) = G_1(\theta) + G_2(\theta) + G_3(\theta)$. It can be shown that $I(\theta) \simeq nG(\theta)$, and the required sample size can be determined by

$$n \simeq \left[\frac{z_{1-\beta}/\sqrt{G(\theta_a)} + z_{1-\alpha}/\sqrt{G(1)}}{1-\theta_a}\right]^2,$$

$$c = 1 - \frac{z_{1-\alpha}}{\sqrt{n}\sqrt{G(1)}} \text{ or } c = \theta_a + \frac{z_{1-\beta}}{\sqrt{n}\sqrt{G(\theta_a)}}.$$

5 Illustrative Example

A sample of the number of cycles to failure for a group of 60 electrical appliances in a life test are reported on P.112 of Lawless [12]. Figure 1 indicates that the histogram of the data is exponentially decayed so that it is reasonable to roughly characterize the data by an exponential distribution. Assume that we treat the data as a random sample drawn from a lot of such items and put on a progressively interval-censored test with two equally-spaced inspection times and the termination time 450. Let the produce risk $\alpha = 0.05$, the consumer risk $\beta = 0.1$, $\eta_0 = 4500$, $\eta_a = 1500$, $p_1 = 0.5$ and $p_2 = 1$. It can be shown that $\theta_0 = 1$, $\theta_a = 1/3$, $\tau = 1/20$. At the time 225, there are 10 failure items are found in the interval $(0, 225]$ or $X_1 = 10$, and $R_1 = (60 - 10) \times 0.5 = 25$ surviving items are selected randomly and removed. Continuing on the test, at the time 450, we find a item fails in the interval $(225, 450]$ or $X_2 = 1$, and all surviving items $R_2 = 60 - 10 - 25 = 25$ are removed. Then the test is stopped. The ordinary LTP can be found as $(n, c) = (60, 0.5183)$. The ML estimate of θ can be determined as $\hat{\theta} = 0.3608$. Since $\hat{\theta} < 0.5183$, we reject the lot.

6 Conclusions

In this paper, the design of progressively interval-censored sampling plans with equally-spaced inspection times is developed based on the ordinary and approximate methods. the use of the proposed method is illustrated by an example. The ordinary life test plans are exact, however, the computation is complicated and time consuming if the number of inspection and the sample size are large. Moreover, the search procedure may fail due to a memory overflow error on computer. If the computer facilities are excellent for practitioners, the ordinary life test plans are suggested. Otherwise, the approximate life test plans are suggested to replace the ordinary life test plans.

The proposed sampling plans can be applied to the Weibull lifetime model with known shape parameter. The practitioners can transform the Weibull lifetime data into the exponential data, then use the proposed approach. Extending the proposed study to the case of two-parameter lifetime models may be a fruitful area of future research.

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