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Article *in* Social Choice and Welfare · November 2021

DOI: 10.1007/s00355-021-01323-0

Making Socioeconomic Health Inequality Comparisons When Health Concentration Curves Intersect*

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March 2, 2021

*This work was financially supported by the Center for Research in Econometric Theory and Applications [Grant 109L9002] from the Featured Areas Research Center Program within the framework of the Higher Edu-

Abstract

Among the various methods adopted to compare health inequality, Makdissi and Yazbeck (2014) developed positional stochastic dominance conditions to identify an ordering. To reach a conclusion, their rules require that the (generalized) health concentration curve of the dominant distribution lie above that of the dominated one. However, it is frequently observed in practice that these curves intersect. Our paper proposes new criteria to cope with this problem by allowing a relatively small violation of the condition proposed by Makdissi and Yazbeck (2014). We characterize our conditions by linking them with some ethical constraints of the weight functions. We further use individual data for Côte d'Ivoire and Guinea from the Demographic and Health Survey to demonstrate the usefulness of our newly-proposed method.

Keywords: positional stochastic dominance; socioeconomic health inequality; health achievement; almost stochastic dominance; sustainable development goals

JEL classification: D63; I10; I14

cation Sprout Project by the Ministry of Education in Taiwan. R. J. Huang and L. Y. Tzeng gratefully acknowledge financial support from the Ministry of Science and Technology in Taiwan [MOST107-3017-F-002-004] and [MOST107-2410-H-008 -012 -MY3].

1 Introduction

The measurement of health inequality is an important issue not only in economics, but also in public health and epidemiology (see Wagstaff, Paci and Van Doorslaer, 1991; Mackenbach and Kunst, 1997). The United Nations (2015) in its 2030 agenda for sustainable development launched in 2015 promotes the enhancement of health equity, which requires the continuous monitoring of health inequalities. When countries commit themselves to improving health in this era of pursuing sustainable development goals, monitoring health inequalities becomes a priority and appropriately identifying health inequalities is fundamental to addressing health inequities when generating evidence to advise on equity-oriented policies. Over the past several decades, issues surrounding health inequality have increasingly become a focus of attention in the domains of policy-making and academic research, initially in high-income countries, but increasingly too in low- and middle-income countries (Hosseinpoor, Bergen, Schlotheuber and Grove, 2018). Contributions from multiple academic disciplines, including social welfare, health economics, and the social sciences, continue to improve the methodologies for monitoring health inequality worldwide.[1]

To evaluate health inequality based on income or some other measure of socioeconomic status, i.e., socioeconomic health inequality, Wagstaff, Van Doorslaer and Paci (1989) proposed using the concentration index, which takes into consideration a specific weight function that represents the aversion to socioeconomic health inequality. The concentration index could be viewed as an extension of the Gini index, which is widely adopted in the income inequality literature. Since their seminal contribution, several alternative indices based on concentration curves have been established by employing different weight functions that represent different judgements of inequality aversion.[2]

Instead of considering a specific weight function, Makdissi and Yazbeck (2014) adopted the concept of positional stochastic dominance and introduced higher orders of (generalized) health

[1] For example, see Jones, Roemer and Rosa Dias (2014), Gravel, Magdalou and Moyes (2019) and Van de Gaer and Ramos (2020).

[2] For example, see Wagstaff, Paci and Van Doorslaer (1991), Kakwani, Wagstaff and Van Doorslaer (1997), Wagstaff (2002 and 2005), Clarke, Gerdtham, Johannesson, Bingefors and Smith (2002), Allison and Foster (2004), Erreygers (2009a and 2009b), Alkire and Foster (2011), Erreygers and Van Ourti (2011) and Zheng (2011).

concentration curves.[3] They demonstrated how these curves can be used to identify an ordering of health distributions for all weight functions which exhibit the same ethical judgement of inequality aversion.[4] For example, they showed that all policy-makers with decreasing weight functions (i.e., satisfying the second-order ethical principle) would prefer one health distribution to another one if and only if the (generalized) health concentration curve of the former distribution lies above that of the other one.[5]

Although Makdissi and Yazbeck (2014) successfully established the conditions to identify an ordering of health achievement and socioeconomic health inequality, their conditions are strict. For example, suppose that there exists a situation where the health of 1 million persons substantially increases together with a slight decrease in the health of the poorest person (let's say a decrease of 0.001%). This change results in an intersection of the (generalized) health concentration curves. If the health status of the poorest person happens to be in the very top category, most policy-makers would agree that it is an improvement in health achievement. However, this case will never be ranked as an improvement using the rules proposed by Makdissi and Yazbeck (2014) since their rule requires that the dominant (generalized) health concentration curve should lie above the dominated one. Moreover, even if one uses their higher-order rules, e.g., the rule for all decreasing and convex weight functions, this change still cannot be identified as an improvement. It is because the socioeconomic health inequality weights are "blind" to the health statuses and are only based on income ranks (Makdissi and Yazbeck, 2016). Increasing the order of dominance cannot help in this case.[6]

The goal of our paper is to propose new criteria that complement Makdissi and Yazbeck (2014), so that even if the concentration curves intersect each other, our rules can still be applied. One reason why the rules in Makdissi and Yazbeck (2014) are rigid is that they seek

[3] The concept of higher order stochastic dominance has been applied to perform inequality comparisons with Lorenz curves. For example, see Davies and Hoy (1995), Chiu (2007), Aaberge (2009) and Chiu (2020).

[4] Complementing Makdissi and Yazbeck (2014), Khaled, Makdissi and Yazbeck (2018) introduced generalized health range curves to correspond to the principle of symmetry around the median introduced by Erreygers, Clarke and Van Ourti (2012).

[5] The decreasing weight functions indicate that the policy-makers' attitudes towards inequality satisfy the second-order ethical principle. This principle states that a mean-preserving transfer of health from a person with a lower rank in terms of socioeconomic status to another person with a higher rank in terms of socioeconomic status results in an increase in health inequality.

[6] We thank an anonymous reviewer for pointing out this example as well as for the insights gained form it.

to apply the socioeconomic health inequality ranking criteria to all weight functions, including some extreme ones. For example, in Makdissi and Yazbeck (2014), it is permissible to have a weight function such that the weight for the group with the lowest socioeconomic status is one and zero otherwise. With this type of weight function, the policy-maker can not tolerate a slight decrease of the health of the poorest person regardless of how small the decrease is. Thus, this policy-maker will not value at all a substantial increase in the health of 1 million persons in the above case. However, a weight function which only reflects care for the group with the lowest socioeconomic status and does not care for other groups at all may be too extreme for most of the policy-makers.

To derive a more applicable condition, we adopt the same framework as Makdissi and Yazbeck (2014) but exclude some weight functions which are too extreme. Specifically, to exclude an extreme weight function, we first require that the weight function have a positive and non-zero weight for each group. In other words, policy-makers care about each group, with some groups having a higher weight and others a lower weight. In addition, we require that the ratios of the weights between two socioeconomic groups not be too large, i.e., the ratio of the maximum weight to the minimum weight should be bounded. Furthermore, when considering socioeconomic health inequality, the literature commonly assumes that the marginal weight is negative. By the same token, we further require that the ratio of the maximum to the minimum of the absolute amount of the marginal weight also be bounded.

With additional constraints on the weight function, we first derive a new notion of positional stochastic dominance conditions and refer to it as "generalized almost positional stochastic dominance" which includes positional stochastic dominance rules proposed by Makdissi and Yazbeck (2014) as special cases.[7] Note that our newly-developed rules should be considered as a complement rather than a substitute for those rules in Makdissi and Yazbeck (2014). To deal with the cases where the concentration curves intersect, Makdissi and Yazbeck (2014) eliminated some weight functions by requiring convexity of the weight functions, while we place additional

[7] To be specific, we include the second-order rules proposed by Makdissi and Yazbeck (2014) as a special case. The rules proposed by Makdissi and Yazbeck (2014) have different orders. The second order places conditions on the sign of the weight function and the first derivative of the weight function, while the higher orders have further assumptions regarding the sign of the higher derivatives of the weight function. Our rules do not impose any condition on the second or higher derivatives of the weight function.

conditions on the "magnitude" of the weight functions, as well as that of the marginal weight functions.[8] Whether one adopts our new rules or employs the higher order rules suggested by Makdissi and Yazbeck (2014) depends on what kind of information we have regarding the preferences of the policy-makers.

The idea of generalized almost positional stochastic dominance is inspired by the concept of almost stochastic dominance first initiated by Leshno and Levy (2002) and further developed by Tzeng, Huang and Shih (2013) and Tsetlin, Winkler, Huang and Tzeng (2015). Almost stochastic dominance has been demonstrated to be useful in examining several finance and economics issues.[9] Our paper is the first to extend this line of research to the field of socioeconomic health inequality. In addition, although our rules consider the joint distribution of income and health, our rules are different from the bivariate almost stochastic dominance rules proposed by Denuit, Huang and Tzeng (2014). The major difference is that the joint distribution in Denuit, Huang and Tzeng (2014) is constructed based on the "level" of two variables, whereas the "position" or the order of income is adopted in our rules. For example, if the income of the poorest person decreases but the health score remains the same, then the (generalized) health concentration curves do not change at all. Therefore, the conclusion regarding the ranking of the health distribution remains the same. However, bivariate almost stochastic dominance would view this change as a shift in the joint distribution of income and health and thus could result in a change in the ranking of the joint distributions of income and health.

Our paper is close to that in Zheng (2018) who employed a similar concept to define almost Lorenz dominance. Our paper, however, differs from Zheng (2018) in two ways. First, he defined almost Lorenz dominance to rank income inequality. We by contrast propose generalized almost positional stochastic dominance to evaluate both health achievement and socioeconomic health inequality. Second, Zheng (2018) placed conditions on the ratio of the maximum weight to the minimum weight while calculating the Gini-type inequality indices. In other words, he followed the concept of almost first-degree stochastic dominance proposed by Leshno and Levy (2002) and Tzeng, Huang and Shih (2013). Our paper not only confines the ratio of the maximum to the

[8] Section 3.1 provides a detailed comparison.

[9] For example, see Bali, Demirtas, Levy and Wolf (2009), Bali, Brown and Demirtas (2013) and Levy (2016).

minimum weight, but also places constraints on that of the marginal weight while calculating the health achievement and socioeconomic health inequality indices. Our methodology is similar to that of the generalized almost second-degree stochastic dominance proposed by Tsetlin, Winkler, Huang and Tzeng (2015).[10]

We further employ data from the Demographic and Health Survey (DHS) to demonstrate the applications of our newly-proposed rules. The prevalence of child malnutrition is a major health problem in many countries. To address this issue, we examine the cases of two countries, taking Côte d'Ivoire (2011-2012) and Guinea (2012) as our example. We employ stunting as a measure of child undernutrition and wealth as a measure of socioeconomic status. The objective is to compare health achievements and socioeconomic health inequalities in stunting between these two countries. When using the approach proposed by Makdissi and Yazbeck (2014), we find that neither Côte d'Ivoire nor Guinea dominates the other country, mainly because the (generalized) health concentration curves intersect each other. We then apply our rules to compare the (generalized) health concentration curves between Côte d'Ivoire and Guinea. We could conclude that there is a higher level of health achievement in Côte d'Ivoire than in Guinea, but that there is a lower level of socioeconomic health inequality in stunting in Côte d'Ivoire than in Guinea.

The remainder of this paper is organized as follows. Section 2 describes the model setup, and provides ethical principles for our proposed constraints on the weight functions. Section 3 derives new notions of almost positional stochastic dominance conditions to rank the (generalized) health concentration curves. Section 4 provides an empirical illustration. Section 5 concludes the paper. All the proofs are included in the appendix.

[10] If a distribution F dominates another distribution G in terms of almost first-degree stochastic dominance, then the distribution F dominates the distribution G in terms of generalized almost second-degree stochastic dominance, but not vice versa.

2 The Framework of Generalized Almost Positional Stochastic Dominance

2.1 Model Setting

Let $F(y)$ be a cumulative distribution function of income y and let $p = F(y)$ be the socioeconomic status of an individual whose income is y. Furthermore, let $H(p)$ represent the health (or ill-health) status for a given individual with socioeconomic status p.

Following Wagstaff (2002) and Makdissi and Yazbeck (2014), the weighted average level of health of the society could be viewed as the achievement in health. A health achievement index can be defined as

$$A(H) = \int_0^1 w(p) H(p)\, dp, \tag{1}$$

where $w(p)$ denotes a weight function on status p and $w(p) \geq 0$. Assume that $w(p)$ is continuous and differentiable everywhere and $\int_0^1 w(p)\, dp = 1$. For a given $w(p)$, a higher $A(H)$ means a better health achievement.[11]

Furthermore, the relative index of socioeconomic health inequality can be defined as

$$\begin{aligned} I(H) &= \frac{1}{\mu} \int_0^1 [1 - w(p)] H(p)\, dp \\ &= 1 - \frac{A(H)}{\mu}, \end{aligned} \tag{2}$$

where $\mu = \int_0^1 H(p)\, dp$ denotes the average health status. For a given weight function, a higher $I(H)$ represents a higher degree of socioeconomic health inequality.

2.2 Ethical Principles of Weight Functions

Following the literature, we assume that $w(p) > 0$ and $w'(p) < 0$, where $w'(p)$ denotes the first derivative of the weight function $w(p)$. The condition $w(p) > 0$ indicates that $w(p)$ is a weight function for each socioeconomic group exhibiting the first-order ethical principle as

[11] For an ill-health variable, there is an opposite conclusion.

proposed by Makdissi and Yazbeck (2014). Note that we do not allow $w(p) = 0$. This means that policy-makers cannot completely ignore any group. The condition $w'(p) < 0$ satisfies the second-order ethical principle in Makdissi and Yazbeck (2014), and indicates that the weight function exhibits an aversion to socioeconomic health inequality.[12] We also assume that $w'(p)$ cannot be zero. In other words, the weights for each group should at least have some differences.

To further exclude some extreme weights, we adopt the concept of generalized almost second-degree stochastic dominance defined by Tsetlin, Winkler, Huang and Tzeng (2015).[13] Let ε_1 and ε_2 be constants within the range $(0, 0.5)$. We focus on the set of weights $w(p)$, $W_2(\varepsilon_1, \varepsilon_2)$ as follows:

$$\begin{aligned} W_2(\varepsilon_1, \varepsilon_2) = & \left\{ w(p) | \, w(p) > 0, \, w'(p) < 0, \, \frac{\sup\{w(p)\}}{\inf\{w(p)\}} \leq \frac{1}{\varepsilon_1} - 1 \text{ and} \right. \\ & \left. \frac{\sup\{-w'(p)\}}{\inf\{-w'(p)\}} \leq \frac{1}{\varepsilon_2} - 1 \right\}. \end{aligned} \tag{3}$$

In this set, on the one hand, the constraint $\frac{\sup\{w(p)\}}{\inf\{w(p)\}} \leq \frac{1}{\varepsilon_1} - 1$ limits the ratio of any two weights. It requires that the ratio of the maximum weight to the minimum weight have an upper bound. If ε_1 approaches zero, then the upper bound is infinity. The constraint will never be binding. If ε_1 approaches 0.5, then in the set of $W_2(\varepsilon_1, \varepsilon_2)$ the weight function of all policy-makers approaches that which attaches equal weights to all p. On the other hand, the constraint $\frac{\sup\{-w'(p)\}}{\inf\{-w'(p)\}} \leq \frac{1}{\varepsilon_2} - 1$ limits the ratio of any two marginal weights. If ε_2 approaches zero, this condition is always satisfied. If ε_2 approaches 0.5, then $W_2(\varepsilon_1, \varepsilon_2)$ only contains linear weight functions.

Note that Makdissi and Yazbeck (2014) consider the set of weight functions, W_2^{MY}, as follows:

$$W_2^{MY} = \left\{ w(p) | \, w(p) > 0, \, w'(p) < 0 \text{ and } w(1) = 0 \right\}.$$

Our set of the weight functions $W_2(\varepsilon_1, \varepsilon_2)$ differs from W_2^{MY} in two ways. First, Makdissi and Yazbeck (2014) require that $w(1) = 0$. This condition is modified in our analysis. From

[12] The second-order ethical principle has been described and named the "principle of income-related health transfer" by Bleichrodt and Van Doorslaer (2006).

[13] Tsetlin, Winkler, Huang and Tzeng (2015) placed constraints on the utility functions, whereas in this paper we add constraints to the weight functions.

Equation (3), we require that $w(p)$ not be zero for any status p. Secondly, we require that $\frac{\sup\{w(p)\}}{\inf\{w(p)\}} \leq \frac{1}{\varepsilon_1} - 1$ and $\frac{\sup\{-w'(p)\}}{\inf\{-w'(p)\}} \leq \frac{1}{\varepsilon_2} - 1$ to exclude some extreme preferences.

Let us further use two cases of the transfer in health scores to characterize the underlying ethical principle for the new conditions: $\frac{\sup\{w(p)\}}{\inf\{w(p)\}} \leq \frac{1}{\varepsilon_1} - 1$ and $\frac{\sup\{-w'(p)\}}{\inf\{-w'(p)\}} \leq \frac{1}{\varepsilon_2} - 1$. In the following discussion, we always employ a discrete case. That is, the society can be classified into n socioeconomic groups. $w(p_i)$ denotes the weight for the ith group. The health achievement index is then defined as $\sum_{i=1}^{n} w(p_i) H_i$, where H_i is the health score for the ith group.

First, suppose that the health score for the p_1 group decreases by δ_1, while the health score for the p_2 group increases by δ_2, where $p_2 > p_1$, $\delta_1 > 0$ and $\delta_2 > 0$. Due to $w(p) > 0$, the health achievement index rises (falls) with respect to an increase (a decrease) in the health score for any group. However, the above transfers aim at a trade-off between an increase in the health score for one group and a decrease in the health score for another group.

In this case, the net increase in the health achievement index is equal to

$$-\delta_1 w(p_1) + \delta_2 w(p_2).$$

If the above net increase is positive, we then have

$$\frac{\delta_2}{\delta_1} \geq \frac{w(p_1)}{w(p_2)}.$$

If $\frac{\sup\{w(p)\}}{\inf\{w(p)\}} \leq \frac{1}{\varepsilon_1} - 1$, then $\frac{w(p_1)}{w(p_2)} \leq \frac{1}{\varepsilon_1} - 1$. Therefore, if $\frac{\delta_2}{\delta_1} \geq \frac{1}{\varepsilon_1} - 1$, i.e., δ_2 is greater than $\delta_1(\frac{1}{\varepsilon_1} - 1)$, then we can conclude that a policy-maker with $\frac{\sup\{w(p)\}}{\inf\{w(p)\}} \leq \frac{1}{\varepsilon_1} - 1$ would prefer this transformation of the health score. Thus, $\frac{\sup\{w(p)\}}{\inf\{w(p)\}} \leq \frac{1}{\varepsilon_1} - 1$ characterizes the trade-off principle between an increase in the health score for one group and a decrease in the health score for another group.

Second, suppose that there is a mean-preserving transfer k of health from a person at any rank p_1 to a person at p_2, where $p_2 > p_1$ and $k > 0$. This transformation is regarded as a deterioration in socioeconomic health inequality due to $w'(p) < 0$. If simultaneously there is also a mean-preserving transfer l of health from a person at any rank p_4 to a person at p_3, where

$p_4 > p_3$ and $l > 0$, this is considered to be an improvement in socioeconomic health inequality. Now, we are facing a trade-off between a mean-preserving deterioration and a mean-preserving improvement in socioeconomic health inequality.

The net increase in the health achievement index is equal to

$$-kw\left(p_1\right) + kw\left(p_2\right) + lw\left(p_3\right) - lw\left(p_4\right).$$

If the net increase is positive, then we have

$$k\left(p_2 - p_1\right)\left[\frac{w\left(p_2\right) - w\left(p_1\right)}{p_2 - p_1}\right] \geq l\left(p_4 - p_3\right)\left[\frac{w\left(p_4\right) - w\left(p_3\right)}{p_4 - p_3}\right].$$

Suppose that the weight function is continuous and differentiable. Thus, there exists a $p_1^+ \in [p_1, p_2]$ and a $p_3^+ \in [p_3, p_4]$ such that

$$k\left(p_2 - p_1\right) w'\left(p_1^+\right) \geq l\left(p_4 - p_3\right) w'\left(p_3^+\right).$$

Rewriting the above condition yields

$$\frac{l\left(p_4 - p_3\right)}{k\left(p_2 - p_1\right)} \geq \frac{-w'\left(p_1^+\right)}{-w'\left(p_3^+\right)}.$$

If $\frac{\sup\{-w'(p)\}}{\inf\{-w'(p)\}} \leq \frac{1}{\varepsilon_2} - 1$, then $\frac{-w'\left(p_1^+\right)}{-w'\left(p_3^+\right)} \leq \frac{1}{\varepsilon_2} - 1$. Thus, if $\frac{l(p_4-p_3)}{k(p_2-p_1)} \geq \frac{1}{\varepsilon_2} - 1$, i.e., l is greater than $k\frac{(p_2-p_1)}{(p_4-p_3)}(\frac{1}{\varepsilon_2} - 1)$, then we can conclude that a policy-maker with $\frac{\sup\{-w'(p)\}}{\inf\{-w'(p)\}} \leq \frac{1}{\varepsilon_2} - 1$ would prefer this combination of the above two transfers of the health score. Thus, $\frac{\sup\{-w'(p)\}}{\inf\{-w'(p)\}} \leq \frac{1}{\varepsilon_2} - 1$ characterizes the trade-off principle between a mean-preserving deterioration on the one hand and a mean-preserving improvement in socioeconomic health inequality on the other.

3 Criteria for Generalized Almost Positional Stochastic Dominance

3.1 Health Achievement

The following formally defines $(\varepsilon_1, \varepsilon_2)$-generalized almost positional second-degree stochastic dominance for health achievement ($(\varepsilon_1, \varepsilon_2)$-GAPSSD$^{\mathrm{A}}$):

Definition 1 *Let $\varepsilon_1, \varepsilon_2 \in (0, 0.5)$. A health distribution $\widetilde{H}$ dominates another distribution H in terms of $(\varepsilon_1, \varepsilon_2)$-GAPSSD^A if*

$$A(\widetilde{H}) \geq A(H) \tag{4}$$

for all weight functions in $W_2(\varepsilon_1, \varepsilon_2)$.

To determine the ordering of health distributions, Makdissi and Yazbeck (2014) defined the following generalized concentration curve:

Definition 2 *The second-order generalized health concentration curve is defined as*

$$GC_H^2(p) = \int_0^p H(t)\,dt, \forall p \in [0,1], \tag{5}$$

where $H(p)$ denotes the health score for individuals with socioeconomic status p.

Using the second-order generalized health concentration curve, Makdissi and Yazbeck (2014) proposed ranking health distributions based on the achievement index as shown in the following theorems:

Theorem 1 (*Makdissi and Yazbeck, 2014*). *$A(\widetilde{H}) \geq A(H)$ for all $w(p)$ in the set of W_2^{MY} if and only if*

$$GC_{\widetilde{H}}^2(p) \geq GC_H^2(p), \forall p \in [0,1]. \tag{6}$$

Proof. See Makdissi and Yazbeck (2014). ■

When the generalized health concentration curves $GC^2_{\widetilde{H}}$ and GC^2_H cross, neither the health distribution $\widetilde{H}$ nor H dominate the other based on Theorem 1. To deal with the cases where generalized health concentration curves intersect each other, Makdissi and Yazbeck (2014) suggested using the higher order rules with the following set of weight functions:

$$W_3^{MY} = \left\{ w(p) | \, w(p) > 0, \, w'(p) < 0, \, w''(p) > 0, \, w(1) = 0 \text{ and } w'(1) = 0 \right\}.$$

Note that in W_3^{MY}, $w''(p) > 0$ and $w'(1) = 0$ play a role like $\frac{\sup\{-w'(p)\}}{\inf\{-w'(p)\}} \leq \frac{1}{\varepsilon_2} - 1$ in our $W_2(\varepsilon_1, \varepsilon_2)$ to add constraints on $w'(p)$. The constraint on $w''(p) > 0$ requires that the marginal weight be an increasing function whereas $\frac{\sup\{-w'(p)\}}{\inf\{-w'(p)\}} \leq \frac{1}{\varepsilon_2} - 1$ requires that the ratio of the maximum to the minimum of the absolute amount of the marginal weight also be bounded regardless of whether the marginal weight is a decreasing or increasing function. Thus, how to confine the marginal weight seems to be up to the preferences of policy-makers. However, and moreover, in our $W_2(\varepsilon_1, \varepsilon_2)$, we require $\frac{\sup\{w(p)\}}{\inf\{w(p)\}} \leq \frac{1}{\varepsilon_1} - 1$ which serves as the main driving force of our paper that allows the policy-makers to value a substantial increase in the health of 1 million persons more than a slight decrease in the health of the poorest person as mentioned in the example in Section 1.

The following theorem provides an unambiguous ranking for the cases where the generalized health concentration curves may intersect each other:

Theorem 2 (($\varepsilon_1, \varepsilon_2$)-$GAPSSD^A$) *Let $\varepsilon_1, \varepsilon_2 \in (0, 0.5)$. $A(\widetilde{H}) \geq A(H)$ for all $w(p)$ in the set of $W_2(\varepsilon_1, \varepsilon_2)$ if and only if*

$$GC^2_{\widetilde{H}}(1) \geq GC^2_H(1) \tag{7}$$

and

$$\frac{\max_D \left\{ \frac{(1-2\varepsilon_2)\int_D \left[GC^2_H(p) - GC^2_{\widetilde{H}}(p)\right] dp + \varepsilon_2 \int_0^1 \left[GC^2_H(p) - GC^2_{\widetilde{H}}(p)\right] dp}{(1-2\varepsilon_2)|D| + \varepsilon_2} \right\}}{GC^2_{\widetilde{H}}(1) - GC^2_H(1)} \leq \frac{\varepsilon_1}{1 - 2\varepsilon_1}, \tag{8}$$

where $D \subset [0,1]$ *and* $|D| = \int_D dp$.

Proof. See Appendix A.1. ■

In Equation (7), $GC^2_{\widetilde{H}}(1) \geq GC^2_H(1)$ can be written as $\int_0^1 \widetilde{H}(p)\, dp \geq \int_0^1 H(p)\, dp$, i.e., the average health status of $\widetilde{H}(p)$ is greater than that of $H(p)$. Thus, in Equation (8), the denominator of the left-hand side (LHS), $GC^2_{\widetilde{H}}(1) - GC^2_H(1)$ can be treated as the winning part of $\widetilde{H}$ compared with H evaluated by the mean. Furthermore, the numerator of the LHS in Equation (8) represents the maximum loss parts of $\widetilde{H}$ compared with H in the set of $W_2(\varepsilon_1, \varepsilon_2)$. Therefore, the LHS of Equation (8) could be interpreted as the ratio of the loss parts to the winning parts when comparing $\widetilde{H}$ with H. On the other hand, the right-hand side (RHS) in Equation (8) represents a threshold. So, unlike Theorem 1 derived by Makdissi and Yazbeck (2014), Equation (8) allows $GC^2_{\widetilde{H}}(p) < GC^2_H(p)$ at some p, as long as the ratio of the loss parts to the winning parts when comparing $\widetilde{H}$ with H cannot be too large.

The following case could further demonstrate the usefulness of $(\varepsilon_1, \varepsilon_2)$-GAPSSD$^{\mathrm{A}}$. From Theorem 2, the necessary and sufficient condition for $(\varepsilon_1, 0)$-GAPSSD$^{\mathrm{A}}$ can be expressed as follows:

Corollary 1 *($(\varepsilon_1, 0)$-GAPSSD^A). Let $\varepsilon_1 \in (0, 0.5)$ and $\varepsilon_2 = 0$. $A(\widetilde{H}) \geq A(H)$ for all $w(p)$ in the set of $W_2(\varepsilon_1, 0)$ if and only if*

$$GC^2_{\widetilde{H}}(1) \geq GC^2_H(1) \tag{9}$$

and

$$\frac{\max_p \left[GC^2_H(p) - GC^2_{\widetilde{H}}(p) \right]}{GC^2_{\widetilde{H}}(1) - GC^2_H(1)} \leq \frac{\varepsilon_1}{1 - 2\varepsilon_1}. \tag{10}$$

Proof. See Appendix A.2. ■

In this case, the maximum loss parts of $\widetilde{H}$ compared with H in the set of $W_2(\varepsilon_1, \varepsilon_2)$ become much easier to calculate since there is a closed-form solution in the optimization problem. Furthermore, if $\varepsilon_1 = \varepsilon_2 = 0$ and there is also an additional constraint on $w(1) = 0$, from this corollary, we could find that the conditions for Theorem 2 coincide with those for Theorem 1 since, under these constraints, the sets of $W_2(\varepsilon_1, \varepsilon_2)$ and W_2^{MY} are equivalent.

3.2 Socioeconomic Health Inequality

To provide a ranking for health distributions based on the socioeconomic health inequality index, let us define $(\varepsilon_1, \varepsilon_2)$-generalized almost positional second-degree stochastic dominance for socioeconomic health inequality ($(\varepsilon_1, \varepsilon_2)$-GAPSSD$^{\mathrm{I}}$) as:

Definition 3 *Let $\varepsilon_1, \varepsilon_2 \in (0, 0.5)$. A relative health distribution $\widetilde{H}$ dominates another relative distribution H in terms of $(\varepsilon_1, \varepsilon_2)$-GAPSSD^I if*

$$I(\widetilde{H}) \leq I(H) \tag{11}$$

for all weight functions in $W_2(\varepsilon_1, \varepsilon_2)$.

Define the health concentration curve as follows:

Definition 4 *The second-order health concentration curve is defined as*

$$C_H^2(p) = \frac{1}{\mu} \int_0^p H(t)\, dt, \tag{12}$$

where $H(p)$ denotes the health score for individuals with socioeconomic status p and $\mu = \int_0^1 H(p)\, dp$.

Using second-order health concentration curves, Makdissi and Yazbeck (2014) ranked health distributions based on the socioeconomic health inequality index as shown in the following theorem:

Theorem 3 (*Makdissi and Yazbeck, 2014*). *$I(\widetilde{H}) \leq I(H)$ for all $w(p)$ in the set of W_2^{MY} if and only if*

$$C_{\widetilde{H}}^2(p) \geq C_H^2(p), \forall p \in [0, 1]. \tag{13}$$

Proof. See Makdissi and Yazbeck (2014). ■

The following theorem provides an unambiguous ranking for the cases where the health concentration curves may intersect each other.

Theorem 4 (($\varepsilon_1, \varepsilon_2$)-*GAPSSD*^{I}). *Let* $\varepsilon_1, \varepsilon_2 \in (0, 0.5)$. $I(\widetilde{H}) \leq I(H)$ *for all* $w(p)$ *in the set of* $W_2(\varepsilon_1, \varepsilon_2)$ *if and only if*

$$\frac{\int_{C^2_{\widetilde{H}}(p) < C^2_H(p)} \left[C^2_H(p) - C^2_{\widetilde{H}}(p) \right] dp}{\int \left| C^2_{\widetilde{H}}(p) - C^2_H(p) \right| dp} \leq \varepsilon_2 \tag{14}$$

Proof. See Appendix A.3. ■

The numerator of the LHS in Equation (14) represents the loss parts of $\widetilde{H}$ compared with H when considering relative socioeconomic health inequality measures. On the other hand, the denominator of the LHS in Equation (14) represents the sum of the loss parts and the winning parts of $\widetilde{H}$ compared with H when considering the relative socioeconomic health inequality. Thus, Equation (14) allows $C^2_{\widetilde{H}}(p) < C^2_H(p)$ at some p, as long as the ratio of the loss parts to the sum of the loss parts and the winning parts when comparing $\widetilde{H}$ with H is not too large.

Note that Equation (14) is independent of the parameter ε_1. Since $C^2_{\widetilde{H}}(1) = C^2_H(1)$, the constraint $\frac{\sup\{w(p)\}}{\inf\{w(p)\}} \leq \frac{1}{\varepsilon_1} - 1$ is redundant in Theorem 4. Similarly, the conditions of Theorems 3 and 4 coincide if $\varepsilon_1 = \varepsilon_2 = 0$ and $w(1) = 0$. In this setting, $W_2(\varepsilon_1, \varepsilon_2)$ is equivalent to W_2^{MY}.

4 An Empirical Illustration

This section illustrates how to apply our rules in comparing the health distributions across socioeconomic groups between two countries. The prevalence of malnutrition among children less than five years old is an important health concern in many countries. To address this issue, we employ stunting (height-for-age) as a measure of child undernutrition and wealth as a measure of socioeconomic status. We use individual data from two Demographic and Health Surveys (DHS) conducted in Côte d'Ivoire in 2011-2012 ($N = 3,286$) and in Guinea in 2012

($N = 3,221$).[14] The data are publicly available from the DHS website.

Let us define an ill-health score as $h = \max(0, -2 - z)$, where z denotes the height-for-age z-score. A higher value of h indicates a greater level of ill-health.[15] Furthermore, we use the wealth index factor score to measure the socioeconomic status, which is denoted by y. A higher value of y indicates a higher level of socioeconomic status. To analyze child malnutrition under the age of five years by socioeconomic status, we construct the (generalized) health concentration curves from a sample (y_i, h_i) for $i = 1, ..., N$. Following Khaled, Makdissi and Yazbeck (2018), the estimators of $GC_H^2(p)$ and $C_H^2(p)$ are respectively calculated by

$$\widehat{GC}_H^2(p) = \frac{1}{N}\sum_{i=1}^{N} h_i \cdot I\left(y_i \le \widehat{F}_Y^{-1}(p)\right)$$

and

$$\widehat{C}_H^2(p) = \frac{1}{N}\frac{1}{\overline{h}}\sum_{i=1}^{N} h_i \cdot I\left(y_i \le \widehat{F}_Y^{-1}(p)\right),$$

where N is the sample size, $\overline{h}$ is a sample average of the ill-health scores, $I(\cdot)$ is an indicator function and $\widehat{F}_Y^{-1}(\cdot)$ is an estimator of the inverse of the cumulative distribution function of the wealth index factor scores. Note that $\widehat{GC}_H^2(p)$ and $\widehat{C}_H^2(p)$ can be described as a step function with a discontinuity at each p.

Figure 1 shows the second-order generalized health concentration curves, $\widehat{GC}_H^2(p)$, for Côte d'Ivoire and Guinea. It can be seen in the left panel in Figure 1 that the two $\widehat{GC}_H^2(p)$ curves are close to each other at the low socioeconomic status p. In order to distinguish between these two $\widehat{GC}_H^2(p)$ curves, we present the difference in $\widehat{GC}_H^2(p)$ between Côte d'Ivoire and Guinea in the right panel of the same figure. Since the difference in $\widehat{GC}_H^2(p)$ changes sign from positive to negative, the two $\widehat{GC}_H^2(p)$ curves cross each other in this case. Thus, when using Theorem 1 proposed by Makdissi and Yazbeck (2014), we find that neither Côte d'Ivoire nor Guinea

[14] From the children's recode file, we select the following variables: V191 (wealth index factor score), HW1 (age in months) and HW70 (height-for-age z-score). After filtering out the missing data, we obtain $3,286$ and $3,221$ children less than 60 months of age in Côte d'Ivoire and Guinea, respectively.

[15] Following Khaled, Makdissi and Yazbeck (2018), the health status of the individual is based on an ill-health variable.

stochastically dominates the other.

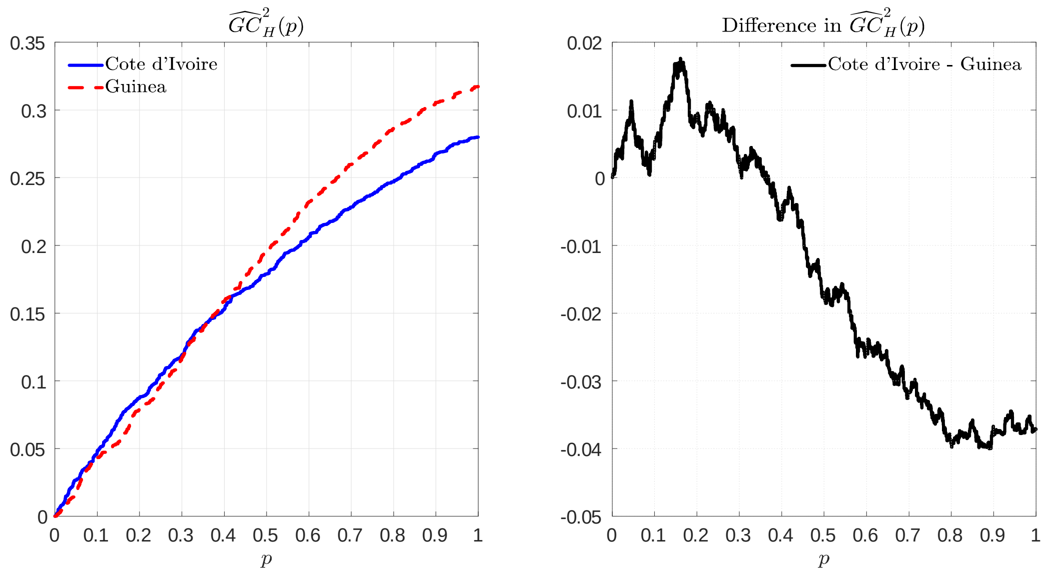

Figure 1: The second-order generalized health concentration curves for Côte d'Ivoire and Guinea

Using these two $\widehat{GC}_H^2(p)$ curves, Theorem 2 can help us to rank health distributions when there is a small deterioration for the group with the lower socioeconomic status but a large improvement for all other groups. When using the $(\varepsilon_1, \varepsilon_2)$-GAPSSD$^{\mathrm{A}}$ rule, we could conclude that Guinea has a higher ill-health level and hence a lower health achievement than Côte d'Ivoire. Table 1 presents the critical values of ε_1 for comparisons within Côte d'Ivoire and Guinea. The first three columns in Table 1 present the results from estimating Equation (8). The remaining columns of Table 1 provide the constraints of the weight functions, $w(p)$ and $w'(p)$, which are considered in the set of $W_2(\varepsilon_1, \varepsilon_2)$ in Equation (3).

Table 1: The critical values of ε_1

ε_2	Left-hand side of Equation (8)	Critical value of ε_1 obtained from Equation (8)	Upper bounds of $\frac{\sup\{w(p)\}}{\inf\{w(p)\}}$	Upper bounds of $\frac{\sup\{-w'(p)\}}{\inf\{-w'(p)\}}$
0.00	0.47	0.24	3.11	∞
0.02	0.21	0.15	5.78	49.00
0.04	0.15	0.11	7.73	24.00
0.06	0.10	0.09	10.55	15.67
0.08	0.07	0.06	15.49	11.50
0.10	0.04	0.04	26.88	9.00

Given ε_2 in the first column of Table 1, we first calculate the values of the LHS of Equation (8) and then we find the threshold values of ε_1 such that the condition in Equation (8) is satisfied.[16] As shown in the third column of Table 1, for the levels of ε_2 of 0, 0.02, 0.04, 0.06, 0.08 and 0.10[17], the critical values of ε_1 are 0.24, 0.15, 0.11, 0.09, 0.06 and 0.04[18], respectively. For example, given $\varepsilon_2 = 0.10$, when $\varepsilon_1 \geq 0.04$, there is a GAPSSD$^{\mathrm{A}}$ of Guinea over Côte d'Ivoire. Note that we are dealing with an ill-health variable. When there is a GAPSSD$^{\mathrm{A}}$ in terms of ill-health, the dominant country has a higher ill-health level and hence a lower health achievement than the dominated one. In other words, given $\varepsilon_2 = 0.10$, if $\frac{\sup\{-w'(p)\}}{\inf\{-w'(p)\}} \leq 9$, all policy-makers whose $\frac{\sup\{w(p)\}}{\inf\{w(p)\}} \leq 26.88$ would consider the health achievement in Côte d'Ivoire to be better than that in Guinea.

In general, the critical values of ε_1 are decreasing as the levels of ε_2 are increasing in Table 1. This finding can be explained by the constraints on the weight functions. Note that the parameters ε_1 and ε_2 control the restrictions on $w(p)$ and $w'(p)$, respectively. If we impose a tighter constraint on the marginal weight function, $w'(p)$, then the weight function, $w(p)$, could be less constrained by the GAPSSD$^{\mathrm{A}}$ rules. These results can be seen in the last two columns

[16] In Appendix A.4, we provide the details on how to solve the optimization problem in Equation (8).

[17] From Equation (3), the constraint $\frac{\sup\{-w'(p)\}}{\inf\{-w'(p)\}} \leq \frac{1}{\varepsilon_2} - 1$ limits the ratio of any two marginal weights. If ε_2 is assumed to be 0, 0.02, 0.04, 0.06, 0.08 and 0.10, then the upper bound of $\frac{\sup\{-w'(p)\}}{\inf\{-w'(p)\}}$ is associated with ∞, 49, 24, 15.67, 11.50 and 9, respectively. These results are reported in the last column of Table 1.

[18] From Equation (3), the constraint $\frac{\sup\{w(p)\}}{\inf\{w(p)\}} \leq \frac{1}{\varepsilon_1} - 1$ limits the ratio of any two weights. If the critical values of ε_1 are determined as 0.24, 0.15, 0.11, 0.09, 0.06 and 0.04, then the upper bounds of $\frac{\sup\{w(p)\}}{\inf\{w(p)\}}$ are 3.11, 5.78, 7.73, 10.55, 15.49 and 26.88, respectively. These results are reported in the fourth column of Table 1.

of Table 1.

We further generate the second-order health concentration curves, $\widehat{C}_H^2(p)$, for Côte d'Ivoire and Guinea. As shown in Figure 2, the left panel shows the $\widehat{C}_H^2(p)$ curves, and the right panel shows the difference in $\widehat{C}_H^2(p)$ between Côte d'Ivoire and Guinea. It is shown that the two $\widehat{C}_H^2(p)$ curves cross each other many times in this case. Thus, Theorem 3 proposed by Makdissi and Yazbeck (2014) fails to determine the socioeconomic health inequality ordering between Côte d'Ivoire and Guinea.

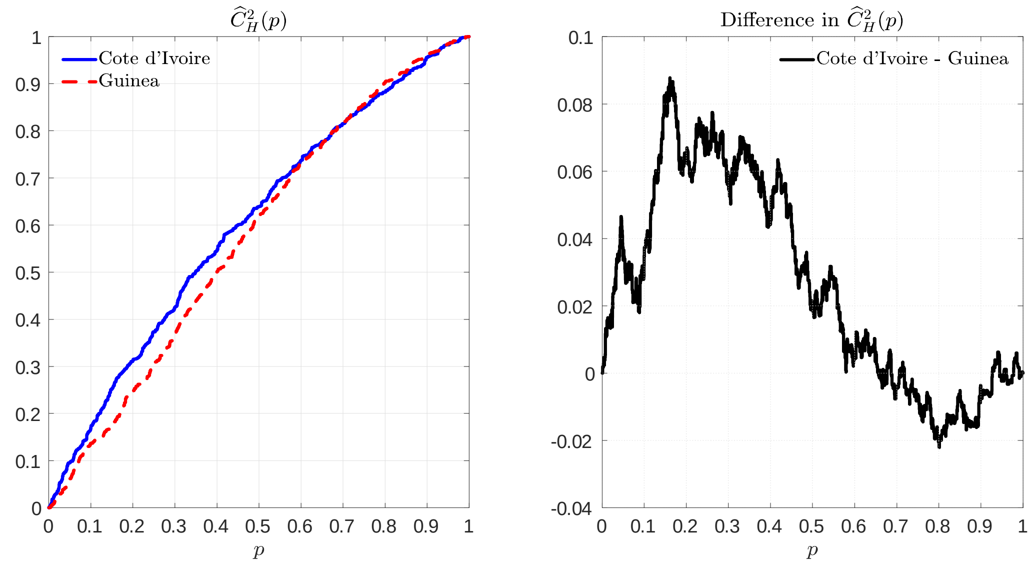

Figure 2: The second-order health concentration curves of Côte d'Ivoire and Guinea

Using these two $\widehat{C}_H^2(p)$ curves, Theorem 4 can help us to rank health distributions based on the socioeconomic health inequality index. When using the $(\varepsilon_1, \varepsilon_2)$-GAPSSD$^{\mathrm{I}}$ rule, we first calculate the values of the LHS of Equation (14) and then examine what levels of ε_2 are required that makes Equation (14) hold. Note that the levels of ε_1 are independent of Equation (14) in Theorem 4. The critical value of ε_2 is then about 0.08 (i.e., $\frac{\sup\{-w'(p)\}}{\inf\{-w'(p)\}} \leq 11.37$). This indicates that, when ε_2 is higher than 0.08 (but lower than 0.5), Côte d'Ivoire has a lower level of socioeconomic health inequality in stunting than Guinea.

As a final remark, firstly, there are always personal preferences while making a decision. However, every category or group should be kept in the policy decision makers' minds. Therefore, with this new proposed model, one decision maker can not assign a zero weight in any one category or group when measuring the health inequality. This ethical concern is included in this revised measurement. Secondly, our new proposed model can be applied to compare cases where two (generalized) health concentration curves intersect each other. Thus, this model provides a more general and wider application zone in comparison, and is an extension of Makdissi and Yazbeck (2014).

5 Conclusions

Parallel to Zheng (2018) who proposes a new way to rank income distributions when Lorenz curves intersect each other, we propose a new ranking criterion to rank socioeconomic health inequality when the (generalized) health concentration curves intersect. Our approach allows for a relatively small violation of the condition proposed by Makdissi and Yazbeck (2014). Moreover, we characterize our conditions by linking them with some ethical constraints of the weight functions. An example with individual data is provided to demonstrate the usefulness of our newly-proposed method.

In this paper, we emphasize constraints on $w(p)$ and $w'(p)$ since they frequently give rise to concerns in the literature. Indeed, our approach could generate some more new criteria if we were to extend it to higher orders, although the resulting optimization problems become increasingly complex. Furthermore, in this paper, we just employ one dataset to demonstrate how to use our method when the health concentration curves intersect. Large-scale applications of our newly-proposed method could prove fruitful in the future.

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Appendix

A.1 Proof of Theorem 2

(1) "If" part: We show that if

$$GC^2_{\widetilde{H}}(1) \geq GC^2_H(1) \tag{A.1}$$

and

$$\begin{aligned} & \max_D \left\{ \frac{(1-2\varepsilon_2)\int_D \left[GC^2_H(p) - GC^2_{\widetilde{H}}(p)\right] dp + \varepsilon_2 \int_0^1 \left[GC^2_H(p) - GC^2_{\widetilde{H}}(p)\right] dp}{(1-2\varepsilon_2)|D| + \varepsilon_2} \right\} \\ \leq \;& \frac{\varepsilon_1}{1-2\varepsilon_1}\left[GC^2_{\widetilde{H}}(1) - GC^2_H(1)\right], \end{aligned} \tag{A.2}$$

then $A(\widetilde{H}) - A(H) \geq 0 \; \forall w \in W_2(\varepsilon_1, \varepsilon_2)$.

By integration by parts, we have

$$\begin{aligned} & A(\widetilde{H}) - A(H) \\ = \;& \int_0^1 w(p)\left[\widetilde{H}(p) - H(p)\right] dp \\ = \;& w(1)\left[GC^2_{\widetilde{H}}(1) - GC^2_H(1)\right] - \int_0^1 w'(p)\left[GC^2_{\widetilde{H}}(p) - GC^2_H(p)\right] dp \\ = \;& w(1)\left\{\left[GC^2_{\widetilde{H}}(1) - GC^2_H(1)\right] - \int_0^1 \left[-\frac{w'(p)}{w(1)}\right]\left[GC^2_H(p) - GC^2_{\widetilde{H}}(p)\right] dp\right\}. \end{aligned} \tag{A.3}$$

By definition, $w'(p) < 0$ and $\frac{\sup\{w(p)\}}{\inf\{w(p)\}} \leq \frac{1}{\varepsilon_1} - 1$ for all $p \in [0,1]$. Since $\frac{w(0)}{w(1)}$ is bounded by $\frac{1}{\varepsilon_1} - 1$, it follows that $\int_0^1 \left[-\frac{w'(p)}{w(1)}\right] dp = \frac{w(0)}{w(1)} - 1 \leq \frac{1}{\varepsilon_1} - 2$.

Let $k(p) = -\frac{w'(p)}{w(1)}\left(\frac{\varepsilon_1}{1-2\varepsilon_1}\right)$. Thus, $\int_0^1 k(p)\,dp \leq 1$ and $\frac{\sup\{k(p)\}}{\inf\{k(p)\}} \leq \frac{1}{\varepsilon_2} - 1$. Equation (A.3) can be rewritten as

$$A(\widetilde{H}) - A(H) = w(1)\left(\frac{1-2\varepsilon_1}{\varepsilon_1}\right)\left\{\frac{\varepsilon_1}{1-2\varepsilon_1}\left[GC^2_{\widetilde{H}}(1) - GC^2_H(1)\right] - \int_0^1 k(p)\left[GC^2_H(p) - GC^2_{\widetilde{H}}(p)\right] dp\right\}. \tag{A.4}$$

According to Equation (A.4), if

$$\int_0^1 k(p)\left[GC_H^2(p) - GC_{\widetilde{H}}^2(p)\right]dp \leq \frac{\varepsilon_1}{1-2\varepsilon_1}\left[GC_{\widetilde{H}}^2(1) - GC_H^2(1)\right], \tag{A.5}$$

then $A(\widetilde{H}) - A(H) \geq 0$.

If $\int_0^1 k(p)\left[GC_H^2(p) - GC_{\widetilde{H}}^2(p)\right]dp \leq 0$, then the above condition holds. On the other hand, if $\int_0^1 k(p)\left[GC_H^2(p) - GC_{\widetilde{H}}^2(p)\right]dp > 0$ and $\int_0^1 k(p)\,dp < 1$, then

$$\begin{aligned}
\int_0^1 k(p)\left[GC_H^2(p) - GC_{\widetilde{H}}^2(p)\right]dp &= \int_0^1 \left(\int_0^1 k(p)\,dp\right)\frac{k(p)}{\left(\int_0^1 k(p)\,dp\right)}\left[GC_H^2(p) - GC_{\widetilde{H}}^2(p)\right]dp \\
&\leq \int_0^1 \frac{k(p)}{\left(\int_0^1 k(p)\,dp\right)}\left[GC_H^2(p) - GC_{\widetilde{H}}^2(p)\right]dp \\
&= \int_0^1 k^*(p)\left[GC_H^2(p) - GC_{\widetilde{H}}^2(p)\right]dp,
\end{aligned}$$

where $k^*(p) = \frac{k(p)}{\int_0^1 k(p)dp} \geq 0$, $\int_0^1 k^*(p)\,dp = 1$ and $\frac{\sup\{k^*(p)\}}{\inf\{k^*(p)\}} \leq \frac{1}{\varepsilon_2} - 1$.

The maximum of $\int_0^1 k^*(p)\left[GC_H^2(p) - GC_{\widetilde{H}}^2(p)\right]dp$ can be written as

$$\max_D\left\{\frac{(1-2\varepsilon_2)\int_D\left[GC_H^2(p) - GC_{\widetilde{H}}^2(p)\right]dp + \varepsilon_2\int_0^1\left[GC_H^2(p) - GC_{\widetilde{H}}^2(p)\right]dp}{(1-2\varepsilon_2)|D| + \varepsilon_2}\right\}, \tag{A.6}$$

where $D \subset [0,1]$ and $|D| = \int_D dp$. To see it, let

$$k^*(p) = \begin{cases} 1-\varepsilon_2 & \text{, if } p \in D \\ \varepsilon_2 & \text{, if } p \notin D \end{cases}$$

and therefore the term $(1-2\varepsilon_2)|D| + \varepsilon_2$ in the denominator of Equation (A.6) is the normalization factor which ensures that $\int_0^1 k^*(p)\,dp = 1$.

Thus, according to Equation (A.6), if

$$\begin{aligned}
&\max_D\left\{\frac{(1-2\varepsilon_2)\int_D\left[GC_H^2(p) - GC_{\widetilde{H}}^2(p)\right]dp + \varepsilon_2\int_0^1\left[GC_H^2(p) - GC_{\widetilde{H}}^2(p)\right]dp}{(1-2\varepsilon_2)|D| + \varepsilon_2}\right\} \\
\leq\ &\frac{\varepsilon_1}{1-2\varepsilon_1}\left[GC_{\widetilde{H}}^2(1) - GC_H^2(1)\right],
\end{aligned}$$

then $A(\widetilde{H}) - A(H) \geq 0$. The sufficient condition for the Theorem is proven.

(2) "Only if" part: We show that if

$$GC^2_{\widetilde{H}}(1) < GC^2_H(1) \tag{A.7}$$

or

$$\max_D \left\{ \frac{(1-2\varepsilon_2)\int_D \left[GC^2_H(p) - GC^2_{\widetilde{H}}(p)\right] dp + \varepsilon_2 \int_0^1 \left[GC^2_H(p) - GC^2_{\widetilde{H}}(p)\right] dp}{(1-2\varepsilon_2)|D| + \varepsilon_2} \right\} > \frac{\varepsilon_1}{1-2\varepsilon_1}\left[GC^2_{\widetilde{H}}(1) - GC^2_H(1)\right], \tag{A.8}$$

then there exists a $w \in W_2(\varepsilon_1, \varepsilon_2)$ such that $A(\widetilde{H}) - A(H) < 0$.

We first show that if Equation (A.7) holds, then $\exists w \in W_2(\varepsilon_1, \varepsilon_2)$ such that $A(\widetilde{H}) - A(H) < 0$. Let θ be a constant, and define a weight function $w \in W_2(\varepsilon_1, \varepsilon_2)$ such that $w(0) = \frac{1}{1-\frac{\theta}{2}}$, $w(1) = \frac{1-\theta}{1-\frac{\theta}{2}}$, and $w'(p) = -\theta$. To guarantee $w(0) > w(1) > 0$ and $w'(p) < 0$, we require that θ lie between 0 and 1. By integration by parts, we have

$$\begin{aligned}
A(\widetilde{H}) - A(H) &= w(1)\left[GC^2_{\widetilde{H}}(1) - GC^2_H(1)\right] - \int_0^1 w'(p)\left[GC^2_{\widetilde{H}}(p) - GC^2_H(p)\right] dp \\
&= \left(\frac{1-\theta}{1-\frac{\theta}{2}}\right)\left[GC^2_{\widetilde{H}}(1) - GC^2_H(1)\right] + \theta \int_0^1 \left[GC^2_{\widetilde{H}}(p) - GC^2_H(p)\right] dp \\
&= \left(\frac{1}{1-\frac{\theta}{2}}\right)\left[GC^2_{\widetilde{H}}(1) - GC^2_H(1)\right] \\
&\quad + \theta\left\{\int_0^1 \left[GC^2_{\widetilde{H}}(p) - GC^2_H(p)\right] dp - \left(\frac{1}{1-\frac{\theta}{2}}\right)\left[GC^2_{\widetilde{H}}(1) - GC^2_H(1)\right]\right\}.
\end{aligned}$$

We assume that $GC^2_{\widetilde{H}}(1) - GC^2_H(1) < 0$, and thus we have

$$A(\widetilde{H}) - A(H) < \int_0^1 \left[GC^2_{\widetilde{H}}(p) - GC^2_H(p)\right] dp - \left(\frac{1}{1-\frac{\theta}{2}}\right)\left[GC^2_{\widetilde{H}}(1) - GC^2_H(1)\right].$$

Since $GC^2_{\widetilde{H}}(1) - GC^2_H(1) < 0$, if

$$\theta > 2\left\{1 - \frac{\left[GC^2_{\widetilde{H}}(1) - GC^2_H(1)\right]}{\int_0^1 \left[GC^2_{\widetilde{H}}(p) - GC^2_H(p)\right] dp}\right\},$$

then $A(\widetilde{H}) - A(H) < 0$.

Next, we show that if Equation (A.8) holds, then $\exists w \in W_2(\varepsilon_1, \varepsilon_2)$ such that $A(\widetilde{H}) - A(H) < 0$. If the LHS of Equation (A.8) is positive, then Equation (A.8) holds since Equation (A.7) holds. On the other hand, if the LHS of Equation (A.8) is nonpositive, let

$$D^* = \arg\max_D \left\{\frac{(1-2\varepsilon_2)\int_D \left[GC^2_H(p) - GC^2_{\widetilde{H}}(p)\right] dp + \varepsilon_2 \int_0^1 \left[GC^2_H(p) - GC^2_{\widetilde{H}}(p)\right] dp}{(1-2\varepsilon_2)|D| + \varepsilon_2}\right\}, \tag{A.9}$$

then we consider a weight function $w \in W_2(\varepsilon_1, \varepsilon_2)$ such that $w(0) = \frac{1-\varepsilon_1}{4}$, $w(1) = \frac{\varepsilon_1}{4}$, and

$$w'(p) = -\frac{1-2\varepsilon_1}{4}\frac{1}{(1-2\varepsilon_2)D^* + \varepsilon_2} \cdot \begin{cases} (1-\varepsilon_2) & \text{, if } p \in D^* \\ \varepsilon_2 & \text{, if } p \notin D^* \end{cases}.$$

This weight function belongs to $W_2(\varepsilon_1, \varepsilon_2)$. By integration by parts, we have

$$\begin{aligned} A(\widetilde{H}) - A(H) &= w(1)\left[GC^2_{\widetilde{H}}(1) - GC^2_H(1)\right] - \int_0^1 w'(p)\left[GC^2_{\widetilde{H}}(p) - GC^2_H(p)\right] dp \\ &= \frac{\varepsilon_1}{4}\left[GC^2_{\widetilde{H}}(1) - GC^2_H(1)\right] \\ &\quad -\frac{1-2\varepsilon_1}{4}\frac{1}{(1-2\varepsilon_2)|D^*| + \varepsilon_2}\left\{(1-2\varepsilon_2)\int_{D^*}\left[GC^2_H(p) - GC^2_{\widetilde{H}}(p)\right] dp\right. \\ &\quad \left.+\varepsilon_2 \int_0^1 \left[GC^2_H(p) - GC^2_{\widetilde{H}}(p)\right] dp\right\}, \end{aligned}$$

where $|D^*| = \int_{D^*} dp$. Thus, by the definition of D^*, if

$$\begin{aligned} &\max_D \left\{\frac{(1-2\varepsilon_2)\int_D \left[GC^2_H(p) - GC^2_{\widetilde{H}}(p)\right] dp + \varepsilon_2 \int_0^1 \left[GC^2_H(p) - GC^2_{\widetilde{H}}(p)\right] dp}{(1-2\varepsilon_2)|D| + \varepsilon_2}\right\} \\ > &\ \frac{\varepsilon_1}{1-2\varepsilon_1}\left[GC^2_{\widetilde{H}}(1) - GC^2_H(1)\right], \end{aligned}$$

then $A(\widetilde{H}) - A(H) < 0$. The necessary condition for the Theorem is proven.

A.2 Proof of Corollary 1

If $\varepsilon_2 = 0$, by Theorem 2, the numerator of the LHS of Equation (8) becomes

$$\max_D \left\{ \frac{\int_D \left[GC_H^2(p) - GC_{\widetilde{H}}^2(p) \right] dp}{|D|} \right\}.$$

By attaching all the weight to $\max \left\{ GC_H^2(p) - GC_{\widetilde{H}}^2(p) \right\}$, we obtain the results.

A.3 Proof of Theorem 4

(1) "If" part: We show that if $\frac{\int_{C_{\widetilde{H}}^2(p) < C_H^2(p)} \left[C_H^2(p) - C_{\widetilde{H}}^2(p) \right] dp}{\int \left| C_{\widetilde{H}}^2(p) - C_H^2(p) \right| dp} \leq \varepsilon_2$, then $I(\widetilde{H}) - I(H) \leq 0$ $\forall w \in W_2(\varepsilon_1, \varepsilon_2)$. By integration by parts, we have

$$\begin{aligned} I(\widetilde{H}) - I(H) &= \int_0^1 [1 - w(p)] \left[\frac{\widetilde{H}(p)}{\mu_{\widetilde{H}}} - \frac{H(p)}{\mu_H} \right] dp \\ &= [1 - w(1)] \left[C_{\widetilde{H}}^2(1) - C_H^2(1) \right] - \int_0^1 [-w'(p)] \left[C_{\widetilde{H}}^2(p) - C_H^2(p) \right] dp \\ &= \int_0^1 [-w'(p)] \left[C_H^2(p) - C_{\widetilde{H}}^2(p) \right] dp. \end{aligned} \tag{A.10}$$

We then divide the integral into two sets. The first set is defined over ranges where $C_{\widetilde{H}}^2(p) < C_H^2(p)$. The second set is defined over ranges where $C_{\widetilde{H}}^2(p) \geq C_H^2(p)$. Equation

(A.10) can be written as

$$\begin{aligned}
I(\widetilde{H}) - I(H) &= \int_{C^2_{\widetilde{H}}(p) < C^2_H(p)} \left[-w'(p)\right] \left[C^2_H(p) - C^2_{\widetilde{H}}(p)\right] dp \\
&\quad + \int_{C^2_{\widetilde{H}}(p) \geq C^2_H(p)} \left[-w'(p)\right] \left[C^2_H(p) - C^2_{\widetilde{H}}(p)\right] dp \\
&\leq \sup\left\{-w'(p)\right\} \int_{C^2_{\widetilde{H}}(p) < C^2_H(p)} \left[C^2_H(p) - C^2_{\widetilde{H}}(p)\right] dp \\
&\quad + \inf\left\{-w'(p)\right\} \int_{C^2_{\widetilde{H}}(p) \geq C^2_H(p)} \left[C^2_H(p) - C^2_{\widetilde{H}}(p)\right] dp \\
&= \inf\left\{-w'(p)\right\} \left\{ \frac{\sup\{-w'(p)\}}{\inf\{-w'(p)\}} \int_{C^2_{\widetilde{H}}(p) < C^2_H(p)} \left[C^2_H(p) - C^2_{\widetilde{H}}(p)\right] dp \right. \\
&\quad \left. + \int_{C^2_{\widetilde{H}}(p) \geq C^2_H(p)} \left[C^2_H(p) - C^2_{\widetilde{H}}(p)\right] dp \right\}. \qquad \text{(A.11)}
\end{aligned}$$

Since $w \in W_2(\varepsilon_1, \varepsilon_2)$, by definition, we have $\frac{\sup\{-w'(p)\}}{\inf\{-w'(p)\}} \leq \frac{1}{\varepsilon_2} - 1$. Therefore,

$$\begin{aligned}
I(\widetilde{H}) - I(H) &\leq \inf\left\{-w'(p)\right\} \left\{ \left(\frac{1}{\varepsilon_2} - 1\right) \int_{C^2_{\widetilde{H}}(p) < C^2_H(p)} \left[C^2_H(p) - C^2_{\widetilde{H}}(p)\right] dp \right. \\
&\quad \left. + \int_{C^2_{\widetilde{H}}(p) \geq C^2_H(p)} \left[C^2_H(p) - C^2_{\widetilde{H}}(p)\right] dp \right\}. \qquad \text{(A.12)}
\end{aligned}$$

Thus, according to Equation (A.12), if

$$\left(\frac{1}{\varepsilon_2} - 1\right) \int_{C^2_{\widetilde{H}}(p) < C^2_H(p)} \left[C^2_H(p) - C^2_{\widetilde{H}}(p)\right] dp + \int_{C^2_{\widetilde{H}}(p) \geq C^2_H(p)} \left[C^2_H(p) - C^2_{\widetilde{H}}(p)\right] dp \leq 0,$$

or,

$$\frac{\int_{C^2_{\widetilde{H}}(p) < C^2_H(p)} \left[C^2_H(p) - C^2_{\widetilde{H}}(p)\right] dp}{\int \left|C^2_{\widetilde{H}}(p) - C^2_H(p)\right| dp} \leq \varepsilon_2,$$

then $I(\widetilde{H}) - I(H) \leq 0$. The sufficient condition for the Theorem is proven.

(2) "Only if" part: We show that if $\frac{\int_{C^2_{\widetilde{H}}(p) < C^2_H(p)} \left[C^2_H(p) - C^2_{\widetilde{H}}(p)\right] dp}{\int \left|C^2_{\widetilde{H}}(p) - C^2_H(p)\right| dp} > \varepsilon_2$, then there exists a

$w \in W_2(\varepsilon_1, \varepsilon_2)$ such that $I(\widetilde{H}) - I(H) > 0$. Take a weight function $w \in W_2(\varepsilon_1, \varepsilon_2)$ such that $w(0) = \frac{1-\varepsilon_1}{4}$, $w(1) = \frac{\varepsilon_1}{4}$, and

$$w'(p) = \begin{cases} -(1-\varepsilon_2) & \text{, if } C^2_{\widetilde{H}}(p) < C^2_H(p) \\ -\varepsilon_2 & \text{, if } C^2_{\widetilde{H}}(p) \geq C^2_H(p) \end{cases}.$$

From Equation (A.10), we have

$$\begin{aligned} I(\widetilde{H}) - I(H) &= \int_0^1 \left[-w'(p)\right] \left[C^2_H(p) - C^2_{\widetilde{H}}(p)\right] dp \\ &= (1-\varepsilon_2) \int_{C^2_{\widetilde{H}}(p) < C^2_H(p)} \left[C^2_H(p) - C^2_{\widetilde{H}}(p)\right] dp \\ &\quad + \varepsilon_2 \int_{C^2_{\widetilde{H}}(p) \geq C^2_H(p)} \left[C^2_H(p) - C^2_{\widetilde{H}}(p)\right] dp \\ &= \int_{C^2_{\widetilde{H}}(p) < C^2_H(p)} \left[C^2_H(p) - C^2_{\widetilde{H}}(p)\right] dp \\ &\quad - \varepsilon_2 \int_0^1 \left|C^2_H(p) - C^2_{\widetilde{H}}(p)\right| dp. \end{aligned}$$

We assume that $\int_{C^2_{\widetilde{H}}(p) < C^2_H(p)} \left[C^2_H(p) - C^2_{\widetilde{H}}(p)\right] dp > \varepsilon_2 \int \left|C^2_{\widetilde{H}}(p) - C^2_H(p)\right| dp$, and thus we have $I(\widetilde{H}) - I(H) > 0$. The necessary condition for the Theorem is proven.

A.4 Method of Implementing Equation (8)

We proceed to maximize the LHS of Equation (8). To implement the scheme, we need to evaluate the integral numerically. The idea is to divide the integration interval $[0, 1]$ into a large number of small intervals, calculate $GC^2_H(p) - GC^2_{\widetilde{H}}(p)$ for each, and determine which one (or combination), D, is optimal when the LHS reaches its maximum. We propose three main steps to implement this framework as follows.

First, the interval $[0, 1]$ can be partitioned into N subintervals with increment $\Delta p = 1/N$. Let us define $p_i = i/N$ for $i = 0, 1, ..., N$. There are $N + 1$ points between 0 and 1. Note that a good approximation is achieved by making N sufficiently large. In our paper, we assume that $N = 10,000$.

Second, for each point p_i, we calculate the difference between two generalized concentration curves. From this we obtain $GC_H^2(p_i) - GC_{\widetilde{H}}^2(p_i)$ for each i. What kind of subset $D \in \{p_0, p_1, ..., p_N\}$ should be chosen for optimization? Since there will be 2^{N+1} different combinations from $N+1$ points, we need to deal with this problem more effectively when N is extremely large.

Third, we develop an integer linear programming technique to solve for the optimal solution D. Let x_i be a binary variable with value 1 if p_i is selected, and 0 otherwise. The integer programming problem can be written as

$$\max_{x_1, x_2, \ldots, x_N} \frac{(1-2\varepsilon_2)\left\{\sum\limits_{i=1}^{N} x_i \left[GC_H^2(p_i) - GC_{\widetilde{H}}^2(p_i)\right]\right\} + \varepsilon_2 \left\{\sum\limits_{i=1}^{N} \left[GC_H^2(p_i) - GC_{\widetilde{H}}^2(p_i)\right]\right\}}{(1-2\varepsilon_2)\frac{\sum\limits_{i=1}^{N} x_i}{N} + \varepsilon_2}, \quad \text{(A.13)}$$

where x_i is assigned a value of 0 or 1. However, the resulting problem is a nonlinear integer programming problem. It is complex and hard to solve.

In order to overcome this difficulty, we show how the objective function of Equation (A.13) can be rewritten as a linear two-stage optimization problem. A constrained linear integer programming problem can be described as follows:

$$\max_{x_1, x_2, \ldots, x_N} \mathcal{L} = (1-2\varepsilon_2)\left\{\sum_{i=1}^{N} x_i \left[GC_H^2(p_i) - GC_{\widetilde{H}}^2(p_i)\right]\right\} + \varepsilon_2 \left\{\sum_{i=1}^{N} \left[GC_H^2(p_i) - GC_{\widetilde{H}}^2(p_i)\right]\right\}$$

$$\text{s.t.} \quad (1-2\varepsilon_2)\frac{\sum\limits_{i=1}^{N} x_i}{N} + \varepsilon_2 = \lambda\,, \qquad \text{(A.14)}$$

where $\lambda \in [\varepsilon_2, 1-\varepsilon_2]$. In the first stage, we maximize $\mathcal{L}$ by setting λ. Since the values of $\sum\limits_{i=1}^{N} x_i$ range from 0 to N, the candidate of λ can be represented as $\frac{j+(N-2j)\varepsilon_2}{N}$ for $j = 0, 1, ..., N$. By repeating the procedure $N+1$ times at this stage, a sample of $N+1$ optimal objective values is generated. In the second stage, based on the optimal objective values $\mathcal{L}_\lambda$ determined in the previous stage for each λ, we maximize $\frac{\mathcal{L}_\lambda}{\lambda}$ over λ. In general, when considering all possible λ, this two-stage procedure is equivalent to solving the nonlinear problem in Equation (A.13).

Note that, alternatively, there is a trick that makes the problem much easier to handle in the first stage. By sorting $GC^2_H(p_i) - GC^2_{\widetilde{H}}(p_i)$ in descending order, the summation of the first j sorted values denotes the optimal value of $\max_{x_1, x_2, ..., x_N} \sum_{i=1}^{N} x_i \left[GC^2_H(p_i) - GC^2_{\widetilde{H}}(p_i)\right]$ subject to $\sum_{i=1}^{N} x_i = j$, where $j = 0$, 1, ..., N. In this case we could directly identify the optimal objective values $\mathcal{L}_\lambda$ through the sorted values for each λ. Note that the second stage is the same as before. Thus we just need to determine which j is the best for optimization. This technique could require less computation time when N is large.