Thus:

1) The proof of this point is evident, because if a set of values of the variables satisfies the two inequalities

$$\sum_{1}^{n}{}_i\, a_{ih}x_i \geq s_h, \qquad \sum_{1}^{n}{}_i\, a_{ik}x_i \geq s_k$$

(a_{ij} is the entry of the P-table in row r_i and column c_j), it satisfies also any linear combination

$$\sum_{1}^{n}{}_i(a_{ih} + Ka_{ik})x_i \geq s_h + Ks_k \qquad (K \geq 0).$$

2) The constant term of the linear combination is

$$s_h + Ks_k = s_h + \frac{\sum \overline{N}_{h,k} - s_h + 1}{s_k} s_k = \sum \overline{N}_{h,k} + 1 \qquad \text{if } K > 0$$

$$s_h + Ks_k = s_h \geq \sum \overline{N}_{h,k} + 1 \qquad \text{if } K = 0.$$

It is not possible to cover $c_h + Kc_k$ by means of rows with nonzero entries in $\overline{N}_{h,k}$ only, because the sum of their contributions could be at most $\sum \overline{N}_{h,k} < s_h + Ks_k$.

Then, at least one row with a nonzero entry in $N_{h,k}$ (and therefore, in c_k) must be contained in each cover of $c_h + Kc_k$. c_k also will be covered, because it is an s-column.

3) Let a cover of $c_h + Kc_k$ be $\Gamma = \{r_\alpha, r_\beta, \cdots, r_\tau\}$. Setting $x_\alpha = x_\beta = \cdots = x_\tau = 1$ and the remaining $x_i = 0$ in the inequality corresponding to column $c_h + Kc_k$, the relation (satisfied by hypothesis) is obtained:

$$a_{\alpha h} + a_{\beta h} + \cdots + a_{\tau h} + K(a_{\alpha k} + a_{\beta k} + \cdots + a_{\tau k}) \geq s_h + Ks_k. \qquad (9)$$

At least one entry among $a_{\alpha k}, a_{\beta k}, \cdots, a_{\tau k}$ must be different from 0 (Point 2). If one of the above entries only is different from 0 (and then equal to s_k, by Definition 1), (9) becomes

$$a_{\alpha h} + a_{\beta h} + \cdots + a_{\tau h} + Ks_k \geq s_h + Ks_k;$$

that is

$$a_{\alpha h} + a_{\beta h} + \cdots a_{\tau h} \geq s_h. \qquad (10)$$

Relation (10) shows that Γ is a cover for c_h. If two (or more) entries among $a_{\alpha k}, a_{\beta k}, \cdots, a_{\tau k}$, say $a_{\epsilon k}$ and $a_{\eta k}$, are different from 0, column c_h is covered by r_ϵ and r_η, because

$$a_{\epsilon h} + a_{\eta h} \geq s_h$$

by Definition 3.

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Testing and Realization of Threshold Functions by Successive Higher Ordering of Incremental Weights

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Abstract—In this paper, a modification or generalization of Sheng's secondary ordering method for testing and realization of threshold functions is presented. Instead of assigning integral values to the incremental weights according to secondary ordering, a search for successively higher ordering is made, and incremental weights of higher orders are successively substituted back into the inequalities until finally no more higher ordering can be found. If the given function is a threshold function, it will turn out that the sum of the coefficients of all the terms on the left side of each of the inequalities will be greater than the sum of the coefficients of all the terms on the right side. Then a minimal integral assignment can be made by assigning unity to every incremental weight of any order appearing in the final set of inequalities. If the given function is not a threshold function, a contradiction will be revealed. Some theorems are proved to justify the method. A complete procedure for testing and realization is given. An example is worked out in detail to illustrate this method.

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I. Introduction

QUITE A FEW methods for testing and realization of threshold functions [1]–[12] have been developed. The method presented in this paper is closely related to the secondary ordering method developed by Sheng [11], and can be considered as a modification or generalization of that method. Sheng's original secondary ordering method has certain advantages and also certain disadvantages. Its chief advantage is that, when the given function is a threshold function, this method is likely to arrive directly at a solution of minimal integral assignment, often without the necessity of cut and trial or adjustment. Its chief disadvantage is that this method is not rigorous enough mathematically, as there is no proof that this method will work for every function, although it works practically for most functions when the number of variables n is not very large.

As suggested by Sheng, the possibility remains that there may exist some orderings among the differences of incremental weights, or higher orderings. In this paper, we shall develop a method based on the higher orderings of weights. Instead of stopping at any particular higher ordering, we continue on right to the end before any assignment of weights is made. In other words, we shall find the highest ordering that exists among the weights by successively substituting the incremental weights of all orders back to the inequalities until finally no more higher ordering can be obtained. If the given function is a threshold function, this fact will become obvious from the sum of coefficients of the terms in the inequalities. If the function is not a threshold function, a contradiction will be revealed. Thus, this modified method is more general, more systematic, and more rigorous mathematically than Sheng's secondary ordering method, as will be shown in the following.

II. Explanation of the Method

We assume the reader is familiar with the basic theory of threshold functions and their testing and realization methods. We shall start with a set of $p'+q'$ irredundant inequalities.

Using Sheng's notation, we have

$$g_i(A) \geq T \quad \text{for } i = 1, \cdots, p'$$
$$g_j(A) < T \quad \text{for } j = p'+1, \cdots, p'+q'.$$

Let the primary ordering be

$$a_1 > a_2 > \cdots > a_n.$$

By setting

$$a_1 = \Delta a_1 + \Delta a_2 + \cdots + \Delta a_{n-1} + a_n$$
$$\vdots$$
$$a_{n-1} = \Delta a_{n-1} + a_n$$
$$a_n = a_n$$

we change the set of $p'+q'$ inequalities in the n unknowns $a_1, \cdots, a_n$, into a set of inequalities in the n unknowns $\Delta a_1, \cdots, \Delta a_{n-1}, a_n$, as follows.

$$I_{p'}: \begin{matrix} g_1(\Delta a_1, \cdots, a_n) \\ g_{p'}(\Delta a_1, \cdots, a_n) \end{matrix} \geq T - C$$
$$I_{q'}: \begin{matrix} g_{p'+1}(\Delta a_1, \cdots, a_n) \\ g_{p'+q'}(\Delta a_1, \cdots, a_n) \end{matrix} < T - C$$

where C is the sum of all terms common to all inequalities.

Now we can expand the $p'+q'$ inequalities into $p' \times q'$ inequalities. To save space, the expanded $p' \times q'$ inequalities can be written with the p' rows of $I_{p'}$ on the left side and with the q' rows of $I_{q'}$ on the right side and with a $>$ sign between them to indicate that each row on the left side is greater than each row on the right side, as shown in inequalities (1). For clarity, $g_1, \cdots, g_{p'}$ are relabelled as $g_{L1}, \cdots, g_{Lp'}$, respectively, and $g_{p'+1}, \cdots, g_{p'+q'}$ are relabelled as $g_{R1}, \cdots, g_{Rq'}$, respectively.

$$\begin{matrix} g_{L1}(\Delta a_1, \cdots, a_n) \\ g_{Lp'}(\Delta a_1, \cdots, a_n) \end{matrix} > \begin{matrix} g_{R1}(\Delta a_1, \cdots, a_n) \\ g_{Rq'}(\Delta a_1, \cdots, a_n) \end{matrix}. \tag{1}$$

Let S_{Li} = sum of coefficients of all the terms in g_{Li}, where $i = 1, \cdots, p'$, and S_{Rj} = sum of coefficients of all the terms in g_{Rj}, where $j = 1, \cdots, q'$.

Consider inequalities (1). Some have the coefficients such that

$$S_{Li} > S_{Rj}.$$

These inequalities do not give any useful information as to higher ordering. Thus, we simply ignore them.

Some of the inequalities have the coefficients such that

$$S_{Li} \leq S_{Rj}.$$

These inequalities are useful, and we shall consider them in detail.

By canceling out common terms on both sides of each inequality, we can obtain simpler inequalities of the following two types.

Type 1: $S_{Li} = 1$, and $S_{Rj} \geq 1$. For instance,

$$\Delta a_1 > \Delta a_3 \tag{2}$$
$$\Delta a_1 > \Delta a_3 + \Delta a_4. \tag{3}$$

For inequality (2), we set

$$\Delta a_1 = \Delta^2 a_1 + \Delta a_3. \tag{4}$$

For inequality (3), we set

$$\Delta a_1 = \Delta^2 a_1 + \Delta a_3 + \Delta a_4. \tag{5}$$

By substituting (4) and (5) into inequalities (2) and (3), respectively, we obtain

$$\Delta^2 a_1 + \Delta a_3 > \Delta a_3$$

or

$$\Delta^2 a_1 > 0 \tag{6}$$

$$\Delta^2 a_1 + \Delta a_3 + \Delta a_4 > \Delta a_3 + \Delta a_4$$

or

$$\Delta^2 a_1 > 0. \tag{7}$$

Inequalities (6) and (7) are redundant, because we assume that every weight and every incremental weight of any order is positive. Now by substituting (4) and (5) back into inequalities (1), owing to the redundant nature of inequalities (6) and (7), and owing to the fact that we add at least one to the sum of coefficients of certain rows having Δa_1 (which are more likely to be on the left side than on the right side) we decrease the number of inequalities with $S_{Li} \leq S_{Rj}$.

This is exactly our new philosophy in this method. Instead of assigning minimum integral values to the incremental weights, as was done by Sheng, we search for higher-order incremental weights and substitute them back into the inequalities.

There are definite advantages to this method over the original one. First, the higher-order incremental weights are not assigned any fixed values until the end of the search process so that they are open to still higher ordering. Secondly, there may be incremental weights which have no direct comparison with others. In that case, in Sheng's method such incremental weights are arbitrarily assigned the weight unity. Although this may work for many functions, there is no assurance that no further adjustment is required. Furthermore, even if all the incremental weights have a complete secondary ordering, higher ordering is still not taken into consideration, and there may still exist the necessity of finding the higher-order incremental weights. Now in this modified method, even if we find only one relation like (4) or (5), we may still substitute it back into inequalities (1) and continue the process of searching for such relations.

We shall call relations given by inequalities (2) and (3) *explicit relations.*

Type 2: $S_{Li} = k \geq 2$, *and* $S_{Rj} \geq k$. For instance,

$$\Delta a_1 + \Delta a_2 > \Delta a_4 + \Delta a_5 \tag{8}$$

$$\Delta a_1 + \Delta a_2 > \Delta a_4 + \Delta a_5 + \Delta a_6. \tag{9}$$

We shall call relations given by inequalities (8) and (9) *implicit relations.*

In an implicit relation, there is at least one incremental weight on the left side that is greater than a certain incremental weight on the right side, although this relation may not appear as an explicit relation. For instance, if inequality (8) is correct, then at least one of the following four relations must be correct.

$$\Delta a_1 > \Delta a_4, \ \ \Delta a_1 > \Delta a_5, \ \ \Delta a_2 > \Delta a_4, \ \text{ or } \ \Delta a_2 > \Delta a_5. \tag{10}$$

We shall call an explicit relation extracted from an implicit relation such as shown by inequality (10) an *extracted explicit relation.*

In using an implicit relation, it is important to choose the *correct* extracted explicit relation. By correct extracted explicit relation, we mean a relation which necessarily must exist in the minimal integral realization of the threshold function. There is no systematic method to determine which is the correct extracted explicit relation. Fortunately, in the beginning of the testing process, there are always some explicit relations available. As long as there are explicit relations available, we do not use implicit relations. We resort to implicit relations only when there is no longer any explicit relation left. In practice, there is little difficulty in choosing a correct extracted explicit relation. As a general rule, we can choose an extracted explicit relation $\Delta a_i > \Delta a_j$ in such a way that Δa_i appears most frequently on the left side of the inequalities and Δa_j appears most frequently on the right side of the inequalities.

Once the extracted explicit relation is chosen, the setting of higher-order incremental weight and the substitution of this higher-order incremental weight back into the inequalities are the same as those for explicit relations.

If we continue this process of finding higher ordering incremental weights and substituting back into inequalities (1), eventually we shall arrive at either one of the following two results.

1) $$S_{Li} > S_{Rj} \quad \text{for all } i = 1, \cdots, p' \quad \text{and} \quad j = 1, \cdots, q'.$$

Then the given function is a threshold function, because if we assign any same integer to every incremental weight of any order in the final set of inequalities, all the inequalities will be satisfied. Thus, the minimal integral assignment is simply to assign unity to every incremental weight of any order in the final set of inequalities.

2) There is some contradiction or contradictions manifested as

$$0 > \text{a sum of one or more incremental weights of some orders.}$$

Then the given function is not a threshold function.

III. Symmetric Variables

In certain Boolean functions there is total or partial symmetry. When a threshold function is symmetric in one or more subsets of the variables, there exists at least one realization in which the variables in a symmetric subset all have the same weight. It seems natural and reasonable to assume that the variables in a symmetric subset all have the same weight. However, the minimal integral assignment is not necessarily the one with the same weight assigned to all variables in a symmetric subset. Winder produced an example of this nature [8]. Thus, for functions symmetric in one or more subsets of the variables, we have two approaches in applying this method.

If strictly minimal integral assignment is required, the weights of the variables in a symmetric subset should be denoted by different symbols, and among the weights of those variables there will be no primary ordering. If the minimal integral assignment is one with different weights for certain symmetric variables, it can be obtained from this method, provided that correct extracted explicit relations are used.

If strictly minimal integral assignment is not required, we can assume all the variables in a symmetric subset to have the same weight, or the same symbol a_i can denote the weight of each variable in a symmetric subset. Following Sheng's notation, a function symmetric in some subsets of variables can be expressed in a neater and simpler form with fewer terms than the original expression. The number of unknowns will be reduced, and the process of testing and realization will become much shorter and simplified. In fact, this approach often still leads to the minimal integral assignment. Even if the assignment is not minimal, it is never far from being minimal. Therefore, it seems to be unjustified to seek the minimal integral assignment with different weights for variables in a symmetric subset, even if such a minimal integral assignment does exist.

IV. Theorems

Theorem 1

A necessary and sufficient condition for a given function F to be a threshold function is that, if only explicit relations or correct extracted explicit relations from implicit relations are used, the successive substitution of higher-order incremental weights will result in a set of inequalities such that

$$S_{Li} > S_{Rj} \quad \text{for all } i \text{ and } j.$$

Proof: We shall prove the necessity first. If the given function F is a threshold function, there must exist a minimal integral assignment of weights. So, let us consider integral weights only. For the minimal integral assignment, there must exist integral incremental weights up to certain orders.

Since inequalities (1) must be satisfied, the higher-order incremental weight obtained from an explicit relation or from a correct extracted explicit relation taken from an implicit relation must be a positive integer, at least being unity.

Since F is a threshold function, there exists no contradiction. Therefore, at any stage of the testing process, there are only two possibilities:

1) $S_{Li} > S_{Rj}$ for all i and j

2) $S_{Li} \leq S_{Rj}$ for some i and j.

For Case 1, there are no explicit and implicit relations, but Theorem 1 is already satisfied.

For Case 2, since $S_{Li} \leq S_{Rj}$ for some i and j, there must be at least one explicit or implicit relation, and therefore the process of successive higher ordering can be carried on further.

If the situation $S_{Li} \leq S_{Rj}$ for some i and j continues to the end when there is only one inequality left, then by setting the left side equal to the right side plus a higher ordering incremental weight, we obtain $S_{Li} > S_{Rj}$ for all i and j. Thus, Theorem 1 is also satisfied.

Now we shall prove that the process of successive higher ordering is convergent.

For $\Delta a_i > \Delta a_j$, we set $\Delta a_i = \Delta^2 a_i + \Delta a_j$. Since all the incremental weights are positive integers,

$$\Delta a_i > \Delta^2 a_i$$

or, in general,

$$\Delta^k a_i > \Delta^{k+1} a_i.$$

Thus, each higher-order incremental weight is smaller than the corresponding lower-order incremental weight, or the successive higher-order incremental weights will decrease monotonically. Since there exists an upper bound for each weight and for the sum of the weights, for a minimal integral assignment each weight must be finite. The successive higher-order incremental weights will all decrease to unity in a finite number of steps. Therefore, the process is convergent.

We shall next prove the sufficiency. If $S_{Li} > S_{Rj}$ for all i and j, by setting every incremental weight in the final set of inequalities to unity, all the inequalities are reduced to $S_{Li} > S_{Rj}$, and thus are all satisfied. Therefore, F must be a threshold function.

Corollary 1-1

A necessary and sufficient condition for a given function F to not be a threshold function is that any successive substitution of higher-order incremental weights will eventually result in a contradiction of the form

$$0 > \text{a sum of one or more incremental weights of some orders.}$$

Proof: We shall prove the necessity first. It can be shown that any contradiction can be reduced to the form given in Corollary 1-1. If the given function F is not a threshold function, and if at any stage of the process no contradiction appears, then either 1) $S_{Li} > S_{Rj}$ for all i and j, or 2) $S_{Li} \leq S_{Rj}$ for some i and j. For Case 1, according to Theorem 1, F is a threshold function. This contradicts the assumption. For Case 2, the process can be carried further on. Again, according to Theorem 1, when there is only one inequality left, we can obtain $S_{Li} > S_{Rj}$ for all i and j, and F is also a threshold function, again contradicting the assumption.

The sufficiency of this corollary is obvious and thus completes the proof of Corollary 1-1.

In spite of Theorem 1, there is, theoretically, still some difficulty in applying this method. The difficulty lies essentially in the implicit relations. If at each stage there is at least one explicit relation available, then we can use explicit relations exclusively and simply ignore the implicit relations. Since each explicit relation represents a condition that must be satisfied, whenever we encounter a contradiction it is implied that the given function is not a threshold function. When there is no explicit relation available, we must extract an explicit relation from an implicit relation. If a correct extraction is made, the situation is similar to that when only explicit relations are used. But a wrong extraction may lead to a contradiction even if the given function is a threshold function. For instance, suppose

$$\Delta a_1 + \Delta a_3 \geq \Delta a_2 + \Delta a_4$$

and

$$\Delta a_1 = 1, \quad \Delta a_2 = 2, \quad \Delta a_3 = 3, \quad \Delta a_4 = 1.$$

If we choose $\Delta a_1 > \Delta a_2$ and set $\Delta a_1 = \Delta^2 a_1 + \Delta a_2$, then $\Delta^2 a_1$ would be negative, and a contradiction will result. Therefore, once we extract explicit relations from implicit relations, a contradiction does not imply that the given function F is not a threshold function. F is not a threshold function if every extracted explicit relation from an implicit relation eventually results in a contradiction. Theoretically, we have to try all the possible extracted explicit relations. In this sense cut and trial is still not completely avoidable.

However, it is, practically, seldom necessary to resort to implicit relations, except in very unusual cases. An example of 18 variables, symmetric in several subsets of variables and not symmetric in some variables, has been worked out, without the necessity of using implicit relations at all.

Theorem 2

If a given function F is a threshold function, the assignment made by setting each incremental weight of any order to unity in the set of inequalities where $S_{Li} > S_{Rj}$ for all i and j is the minimal integral assignment.

Proof: For integral assignment, each incremental weight should be at least unity. Before the set of inequalities with $S_{Li} > S_{Rj}$ for all i and j is reached, such an assignment does not satisfy all the inequalities, because for some inequalities $S_{Li} \leq S_{Rj}$. When such a set of inequalities is reached, there are no more explicit and/or implicit relations left. Therefore, once this set of inequalities is reached, no more higher ordering is possible, or this set of inequalities is unique insofar as the requirement of higher ordering of weights is concerned. Since unity is the smallest integer that can be assigned to any incremental weight of any order, this assignment must be minimal integral.

V. Procedure

Now we can formulate a procedure for testing and realization of threshold functions. The first seven steps are similar to those of Sheng's procedure. We shall just list these steps without explanation.

Step 1: Checking the given function F for unateness.

Step 2: Finding the complementary function $\overline{F}$.

Step 3: Reducing F and $\overline{F}$ to minimum sum-of-product form (or INDF).

Step 4: Primary ordering.

$$a_1 > a_2 > \cdots > a_n.$$

Step 5: Inequalities in A.

$$I_p:\ g_i(A) \geq T \quad \text{for } i = 1, \cdots, p$$
$$I_q:\ g_i(A) < T \quad \text{for } i = p+1, \cdots, p+q.$$

Step 6: Deleting redundant inequalities to reduce the number of inequalities.

$$I_{p'}:\ g_i(A) \geq T \quad \text{for } i = 1, \cdots, p'$$
$$I_{q'}:\ g_i(A) < T \quad \text{for } i = p'+1, \cdots, p'+q'. \tag{11}$$

Step 7: Substituting incremental weights and reduction. Set

$$\begin{aligned} a_1 &= \Delta a_2 + \Delta a_2 + \cdots + \Delta a_{n-1} + a_n \\ a_2 &= \Delta a_2 + \cdots \Delta a_{n-1} + a_n \\ &\vdots \\ a_n &= a_n. \end{aligned} \tag{12}$$

Substitute (12) into the $p'+q'$ inequalities. Find C, the sum of all the terms common to all inequalities. Subtract C from each inequality. Then the inequalities become

$$I_{p'}:\ \begin{matrix} g_1(\Delta a_1, \Delta a_2, \cdots, a_n) \\ g_{p'}(\Delta a_1, \Delta a_2, \cdots, a_n) \end{matrix} \geq T - C$$
$$I_{q'}:\ \begin{matrix} g_{p'+1}(\Delta a_1, \Delta a_2, \cdots, a_n) \\ g_{p'+q'}(\Delta a_1, \Delta a_2, \cdots, a_n) \end{matrix} < T - C.$$

Step 8: "Expanding" of inequalities in tabular form.

$$\begin{matrix} g_{L1}(\Delta a_1, \cdots, a_n) \\ g_{Lp'}(\Delta a_1, \cdots, a_n) \end{matrix} > \begin{matrix} g_{R1}(\Delta a_1, \cdots, a_n) \\ g_{Rq'}(\Delta a_1, \cdots, a_n) \end{matrix}. \tag{1}$$

Find S_{Li} and S_{Rj} for the inequalities and list them in the last column on the left and the right sides of the table, respectively.

Step 9: Comparing of S_{Li} and S_{Rj} for all the rows in inequalities (1).

a) If $S_{Li} > S_{Rj}$ for all i and j, the given function is a threshold function. Proceed to Step 12.

b) If $S_{Li} \leq S_{Rj}$ for some i and j, proceed to Step 10.

Step 10: Checking for contradiction. Check each inequality with $S_{Li} \leq S_{Rj}$ to see if it can be reduced to the form

0 > a sum of one or more incremental weights of some order.

a) If one or more inequalities can be reduced to the above form, the given function is not a threshold function. Stop.

b) If no inequality can be reduced to the above form, proceed to Step 11.

Step 11: Higher ordering of weights. Of the $p' \times q'$ inequalities, ignore those with $S_{Li} > S_{Rj}$. Consider only those inequalities with $S_{Li} \leq S_{Rj}$. Compare every left side row with every right side row.

a) If there are one or more explicit relations, express the left side terms as follows:

$$\text{If} \quad \Delta a_i > \Delta a_j, \text{ set } \Delta a_i = \Delta^2 a_i + \Delta a_j \tag{13}$$

$$\text{If} \quad \Delta a_i > \Delta a_j + \Delta a_k + \cdots$$

$$\text{set} \quad \Delta a_i = \Delta^2 a_i + \Delta a_j + \Delta a_k + \cdots. \tag{14}$$

Substitute (13), (14), etc., back into inequalities (1). Subtract all new common terms, if any. Delete all new redundant inequalities, if any. Go back to Step 9.

b) If there are no explicit relations, choose extracted explicit relation or relations of the form $\Delta a_i > \Delta a_j$ from implicit relations. Extracted explicit relations are to be chosen in such a way that Δa_i appears most frequently on the left side and Δa_j appears most frequently on the right side of the inequalities. Set

$$\Delta a_i = \Delta^2 a_i + \Delta a_j. \tag{13}$$

Substitute (13) back into inequalities (1). Subtract all new common terms, if any. Delete all new redundant inequalities, if any. Go back to Step 9.

Step 12: Assigning of weights and threshold value-realization. Assign unity to all the incremental weights of any order in the final set of inequalities (1). Using (12), (13), and (14), obtain incremental weights of all orders and the original weights. Substitute all weights into inequalities (11). Set

$$T = \text{minimum } g_i(A), \quad \text{for } i = 1, \cdots, p'.$$

Then the realization of the given function F is

$$R[F] = (a_1, a_2, \cdots, a_n; T).$$

VI. Example

$$F(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$$
$$= x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_1x_6x_7 + x_1x_6x_8 + x_2x_3x_4$$
$$+ x_2x_3x_5 + x_2x_3x_6 + x_2x_3x_7 + x_2x_3x_8 + x_2x_4x_5 + x_2x_4x_6$$
$$+ x_2x_4x_7 + x_2x_4x_8 + x_2x_5x_6 + x_2x_5x_7 + x_2x_5x_8 + x_2x_6x_7x_8$$
$$+ x_3x_4x_5 + x_3x_4x_6 + x_3x_4x_7x_8 + x_3x_5x_6 + x_3x_5x_7x_8$$
$$+ x_4x_5x_6x_7 + x_4x_5x_6x_8 + x_4x_5x_7x_8.$$

F is symmetric in variables x_4 and x_5, and also symmetric in variables x_7 and x_8. We shall work out this example in detail, with the eight weights denoted by eight different symbols.

Solution

Step 1: F is a unate function.

Step 2:

$$\bar{F} = \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4 + \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_5 + \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_6\bar{x}_7 + \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_6\bar{x}_8$$
$$+ \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_7\bar{x}_8 + \bar{x}_1\bar{x}_2\bar{x}_4\bar{x}_5 + \bar{x}_1\bar{x}_2\bar{x}_4\bar{x}_6\bar{x}_7 + \bar{x}_1\bar{x}_2\bar{x}_4\bar{x}_6\bar{x}_8$$
$$+ \bar{x}_1\bar{x}_2\bar{x}_5\bar{x}_6\bar{x}_7 + \bar{x}_1\bar{x}_2\bar{x}_5\bar{x}_6\bar{x}_8 + \bar{x}_1\bar{x}_3\bar{x}_4\bar{x}_5\bar{x}_6 + \bar{x}_1\bar{x}_3\bar{x}_4\bar{x}_5\bar{x}_7$$
$$+ \bar{x}_1\bar{x}_3\bar{x}_4\bar{x}_5\bar{x}_8 + \bar{x}_1\bar{x}_3\bar{x}_4\bar{x}_6\bar{x}_7\bar{x}_8 + \bar{x}_1\bar{x}_3\bar{x}_5\bar{x}_6\bar{x}_7\bar{x}_8$$
$$+ \bar{x}_1\bar{x}_4\bar{x}_5\bar{x}_6\bar{x}_7\bar{x}_8 + \bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5\bar{x}_6 + \bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5\bar{x}_7\bar{x}_8.$$

Step 3: F and $\bar{F}$ are already in sum-of-product form.

Step 4:

$$a_1 > a_2 > a_3 > \begin{matrix} a_4 \\ a_5 \end{matrix} > a_6 > \begin{matrix} a_7 \\ a_8 \end{matrix}.$$

Steps 5 and 6: The irredundant inequalities $I_{p'}$ and $I_{q'}$ are given in Table I.

TABLE I

	i	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	
g_{Li}	1	1			1					$\geq T$
	2	1				1				
	3	1					1	1		
	4	1					1		1	
	5		1		1			1		
	6		1		1				1	
	7		1			1		1		
	8		1			1			1	
	9		1				1	1	1	
	10			1	1		1			
	11			1		1	1			
	12				1	1		1	1	

	j	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	
g_{Rj}	1	1					1			$< T$
	2	1						1	1	
	3		1	1						
	4		1				1	1		
	5		1				1		1	
	6			1	1			1		
	7			1	1				1	
	8			1		1		1		
	9			1		1			1	
	10			1			1	1	1	
	11				1	1	1			

Steps 7 and 8: Set

$$a_1 = \Delta a_1 + \Delta a_2 + \Delta a_3 + \Delta a_4 + \Delta a_6 + a_7$$
$$a_2 = \Delta a_2 + \Delta a_3 + \Delta a_4 + \Delta a_6 + a_7$$
$$a_3 = \Delta a_3 + \Delta a_4 + \Delta a_6 + a_7$$
$$a_4 = \Delta a_4 + \Delta a_6 + a_7$$
$$a_5 = \Delta a_5 + \Delta a_6 + a_7$$
$$a_6 = \Delta a_6 + a_7$$
$$a_7 = a_7$$
$$a_8 = a_8. \tag{15}$$

TABLE II

g_{Li}

i	Δa_1	Δa_2	Δa_3	Δa_4	Δa_5	Δa_6	a_7	a_8	S_{Li}
1	1	1	1	1		1			5
2	1	1	1		1	1			5
3	1	1	1			1	1		5
4	1	1	1			1		1	5
5		1	1	1		1	1		5
6		1	1	1		1		1	5
7		1	1		1	1	1		5
8		1	1		1	1		1	5
9		1	1			1	1	1	5
10			1	1		2	1		5
11			1		1	2	1		5
12					1	1	1	1	4

$> g_{Rj}$

j	Δa_1	Δa_2	Δa_3	Δa_4	Δa_5	Δa_6	a_7	a_8	S_{Rj}
1	1	1	1			1			4
2	1	1	1					1	4
3		1	2	1		1			5
4		1	1			1	1		4
5		1	1			1		1	4
6			1	1		1	1		4
7		1	1			1		1	4
8			1		1	1	1		4
9			1		1	1		1	4
10			1			1	1	1	4
11					1	2	1		4

TABLE III

g_{Li}

i	$\Delta^2 a_1$	Δa_2	Δa_3	Δa_4	$\Delta^2 a_5$	Δa_6	Δa_7	Δa_8	S_{Li}
1	1	1		1		1			4
2	1	1	1		1	1			5
3	1	1	1			1	1		5
4	1	1	1			1		1	5
5		1		1		1	1		4
6		1		1		1		1	4
7		1	1		1	1	1		5
8		1	1		1	1		1	5
9		1	1			1	1	1	5
10				1		2	1		4
11			1		1	2	1		5
12			1		1	1	1	1	5

$> g_{Rj}$

j	$\Delta^2 a_1$	Δa_2	Δa_3	Δa_4	$\Delta^2 a_5$	Δa_6	Δa_7	Δa_8	S_{Rj}
1	1	1				1			3
2	1	1	1					1	4
3		1		1		1			3
4		1				1	1		3
5		1				1		1	3
6				1		1	1		3
7				1		1		1	3
8			1		1	1	1		4
9			1		1	1		1	4
10			1			1	1	1	4
11					1	2	1		4

TABLE IV

g_{Li}

i	$\Delta^2 a_1$	Δa_2	Δa_3	$\Delta^2 a_4$	$\Delta^2 a_5$	Δa_6	Δa_7	Δa_8	S_{Li}
1	1	1		1	1	1			5
2	1	1	1		1	1			5
3	1	1	1			1	1		5
4	1	1	1			1		1	5
5		1		1	1	1	1		5
6		1		1	1	1		1	5
7		1	1		1	1	1		5
8		1	1		1	1		1	5
9		1	1			1	1	1	5
10				1	1	2	1		5
11			1		1	2	1		5
12			1		1	1	1	1	5

$> g_{Rj}$

j	$\Delta^2 a_1$	Δa_2	Δa_3	$\Delta^2 a_4$	$\Delta^2 a_5$	Δa_6	Δa_7	Δa_8	S_{Rj}
1	1	1				1			3
2	1	1	1					1	4
3		1		1	1	1			4
4		1				1	1		3
5		1				1		1	3
6				1	1	1	1		4
7				1	1	1		1	4
8			1		1	1	1		4
9			1		1	1		1	4
10			1			1	1	1	4
11					1	2	1		4

Substituting (15) into Table I, and subtracting the common terms $C=\Delta a_4+\Delta a_8+2a_7$ from each inequality, we obtain Table II.

Step 9: Condition b is satisfied.
Step 10: Condition b is satisfied.
Step 11: From Table II, we find

$$\begin{array}{ll} g_{L1} - g_{R3} & \Delta a_1 > \Delta a_3 \\ g_{L5} - g_{R3} & a_7 > \Delta a_3 \\ g_{L6} - g_{R3} & a_8 > \Delta a_3 \\ g_{L12} - g_{R8} & a_8 > \Delta a_3 \\ g_{L12} - g_{R9} & a_7 > \Delta a_3 \\ g_{L12} - g_{R10} & \Delta a_5 > \Delta a_3 \\ g_{L12} = g_{R11} & a_8 > \Delta a_6. \end{array}$$

Condition a is satisfied. Set

$$\begin{aligned} \Delta a_1 &= \Delta^2 a_1 + \Delta a_3 \\ \Delta a_5 &= \Delta^2 a_5 + \Delta a_3 \\ a_7 &= \Delta a_7 + \Delta a_3 \\ a_8 &= \Delta a_8 + \Delta a_3. \end{aligned} \quad (16)$$

Substituting (16) back into Table II, and subtracting the new common term $C=2\Delta a_3$, we obtain Table III.

Step 9: Condition b is satisfied.
Step 10: Condition b is satisfied.
Step 11: From Table III, we find

$$g_{L10} - g_{R11} \qquad \Delta a_4 > \Delta^2 a_5.$$

Condition a is satisfied. Set

$$\Delta a_4 = \Delta^2 a_4 + \Delta^2 a_5. \quad (17)$$

Substituting (17) back into Table III, we obtain Table IV.

Step 9: Condition a is satisfied. So we proceed to Step 12.

Step 12: Assign

$$\Delta^2 a_1 = \Delta a_2 = \Delta a_3 = \Delta^2 a_4 = \Delta^2 a_5 = \Delta a_6 = \Delta a_7 = \Delta a_8 = 1.$$

Substituting these values back into (15), (16), (17), and Table I, we obtain

$$a_1 = 9, \quad a_2 = 7, \quad a_3 = 6, \quad a_4 = 5, \quad a_5 = 5,$$
$$a_6 = 3, \quad a_7 = 2, \quad a_8 = 2, \quad \text{and} \quad T = 14.$$

Therefore,

$$R[F] = (9, 7, 6, 5, 5, 3, 2, 2; 14).$$

If in the very beginning symmetric variables are assumed to have equal weights, or $a_4=a_5$ and $a_7=a_8$, then the procedure will be greatly shortened. It is found that the solution can be obtained with only one substitution of incremental weights after secondary ordering, or with two substitutions of incremental weights altogether, as compared with three substitutions in the above solution. For this particular function, the solution obtained with symmetric variables having equal weights is the same as the above solution, or is a strictly minimal integral realization, although this is not necessarily true in general.

It is of interest to note that, in the solution of this example, no implicit relations are used. Therefore, there is no cut and trial at all. It should be emphasized that as long as only explicit relations and correct extracted explicit relations are used, if more relations than required are available, any relations chosen will work and will result in a minimal integral realization. For instance, in the above example, if we set $a_8=\Delta a_8+\Delta a_6$ instead of $\Delta a_8+\Delta a_3$, the same realization will be obtained eventually.

VII. Conclusions

We have presented a method for testing and realization of threshold functions. Although it is a modification of Sheng's secondary ordering method, it is not restricted to secondary ordering. It is rather based on the successive search for higher-order incremental weights. We propose that this method be called *successive higher ordering method.*

It may be concluded that the successive higher ordering method has the following advantages.

1) The procedure is relatively straightforward.

2) It needs no cut and trial at all when only explicit relations are used; it needs a little cut and trial only in extracting explicit relations when implicit relations are used.

3) It is mathematically rigorous, and the process is convergent.

4) It can be applied to functions of any number of variables.

5) For threshold functions symmetric in one or more subsets of the variables, this method results in a strictly minimal integral assignment if all the weights are denoted by different symbols, and results in a non-strictly minimal integral assignment if variables in a symmetric subset are assumed to have equal weights.

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Pattern Classification by Iteratively Determined Linear and Piecewise Linear Discriminant Functions

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Abstract—**This paper describes iterative procedures for determining linear and piecewise linear discriminant functions for multicategory pattern classifiers. While classifiers with the same structure have often been proposed, it is less well known that their parameters can be efficiently determined by simple adjustment procedures. For linear discriminant functions, convergence proofs are given for procedures that are guaranteed to yield error-free solutions on design samples, provided only that such solutions exist. While no similar results are known for piecewise linear discriminant functions, simple procedures are given that have been effective in various experiments. The results of experiments with artificially generated multimodal data and with hand-printed alphanumeric characters are given to show that this approach compares favorably with other classification methods.**

Introduction

In most pattern recognition systems, a set of measurements characterizing a pattern is used to classify the pattern into one of a finite number of categories. These systems are usually considered to be composed of two subsystems, a *receptor* or *preprocessor* that extracts the significant, characterizing measurements, and a *categorizer* or *classifier* that uses the output of the preprocessor to classify the patterns [1], [2]. As Chow has pointed out [3], the problem of designing the preprocessor " . . . not only has not been solved, but has not as yet been properly formulated with sufficient clarity and completeness." In this paper we shall assume that the preprocessor has been capably designed and shall concentrate on the design of the classifier, bearing in mind that even the best classifier cannot compensate for an inadequate choice of measurements.

The structure of the optimum (Bayes-sense) classifier has been derived by Chow [4]. For the case of a minimum-error-rate system, the classifier effectively computes the joint probability (or probability density) for the occurrence of the observed measurements and each category, and selects the category for which the joint probability (or probability density) is maximum. The structure of such a classifier is shown in Fig. 1. The measurements $x_1, x_2, \cdots, x_d$ can be viewed either as the coordinates of a point in *measurement space* or as the components of a *pattern vector* $\overline{X}$. The classifier assigns the "pattern" $\overline{X}$ to one of R categories $\omega_1, \omega_2, \cdots, \omega_R$ by computing R *discriminant functions* $g_1(\overline{X}), g_2(\overline{X}), \cdots, g_R(\overline{X})$ and by placing $\overline{X}$ in ω_i if $g_i(\overline{X}) > g_j(\overline{X})$ for all $j \neq i$ [2], [44]. Here the discriminant functions $g_i(\overline{X})$ are either the joint probabilities (or probability densities) $P(\overline{X}, \omega_i)$ or functions yielding

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