$$W_r = \sum_{i=0}^{K} \binom{L}{i} \cdot i \Big/ \sum_{i=0}^{K} \binom{L}{i} = \frac{N_1}{N_2}$$

then the average row a weight of the extended matrix is

$$W_r = \frac{N_1 + Z\cdot(K+1)}{N_2 + Z}.$$

Treating this as a function of the number of rows and differentiating, we get

$$W_r' = \frac{N_2(K+1) - N_1}{(N_2+Z)^2}.$$

From this we see that the average row a weight increases more rapidly when we add the first few rows of a weight $K+1$ and less rapidly as we approach matrix A_{K+1}. The result then follows since any $M\times L$ matrix is intermediate between A_K and A_{K+1} for some K.

Lemma 6:

$$\sum_{i=0}^{K+\sigma} \binom{L}{i} \leq 2^{LH(K+\sigma/L)}, \qquad \text{for } K \leq [L/3] \text{ and } L \geq 2.$$

Proof: It is well known that this bound is true for conventional summations, i.e., $\sigma=0$, so we need only show that it is true for $0<\sigma<1$ where the fractional summation is as defined above. To this end we consider the right and left sides of the inequality to be functions of σ. Between the points K and $K+1$, $\sum_{i=0}^{K+\sigma} \binom{L}{i}$ has the constant slope $\binom{L}{K+1}$ and $2^{LH(K+\sigma/L)}$ has slope

$$L\cdot 2^{LH(K+\sigma/L)}\cdot(\log_e 2)\cdot\left(\log_2 \frac{L-(K+\sigma)}{K+\sigma}\right).$$

$\binom{L}{K+1}$ can be rewritten as $\binom{L}{K}\cdot L-K/K+1$ and we observe that in the area of interest, $K\leq[L/3]$ and $L\geq 2$, the function $2^{LH(K+\sigma/L)}$ has the greater slope. Thus it is not possible for the value of $\sum_{i=0}^{K+\sigma} \binom{L}{i}$ to overtake the value of $2^{LH(K+\sigma/L)}$ between the points K and $K+1$.

We are now able to prove the theorem.

Proof: We first show that a set of M distinct output sequences, each of which has fractional a weight not greater than δ_a, must have some set of L consecutive positions in which the fractional a weight of the corresponding $M\times L$ matrix is less than δ_a. Let B be the $M\times T$ matrix whose rows are periods of the output sequences of fractional a weight δ_a or less. We consider $M\times L$ submatrices of consecutive columns, which we call frames, and we note that there are T distinct frames since B may be considered to wrap around in cylindrical fashion. We let $N_a(L_i)$ denote the a weight of the ith frame and $N_a(B)$ the a weight of B. Then, since each b_{ij} of B appears in exactly L frames, we have

$$L\cdot N_a(B) = \sum_{i=1}^{T} N_a(L_i).$$

Observing that $N_a(B)/M$ is the average row a weight of B and denoting the a weight of the frame with the least number of a's by $N_a(L_{\min})$, we obtain

$$L\cdot\delta_a T \geq \frac{L\cdot N_a(B)}{M} = \frac{\sum_{i=1}^{T} N_a(L_i)}{M} \geq \frac{T\cdot N_a(L_{\min})}{M}.$$

From this we conclude that

$$\frac{N_a(L_{\min})}{LM} \leq \delta_a.$$

Hence B contains at least one $M\times L$ submatrix with average fractional a weight less than or equal to δ_a, and the proof then follows immediately from Lemmas 1 and 6.

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An Approach for the Synthesis of Multithreshold Threshold Elements

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***Abstract*—A new approach for the realization of multithreshold threshold elements is presented. The procedure is based on the fact that the excitations at contradictory vertices of the switching function must be unequal. The weights of the multithreshold element, in general, satisfy simple relations of the form $U\cdot W=0$, where $U=(u_1, u_2, \cdots, u_n)$ and $W=(w_1, w_2, \cdots, w_n)$ such that $u_i \in \{1, 0, -1\}$, $i=1, 2, \cdots n$, and $W\in I^n$.**

Comparison of the excitations $E(X_i)=W\cdot X_i$ and $E(X_j)=W\cdot X_j$ at TRUE and FALSE vertices X_i and X_j, respectively, for all specified vertices result in some inequalities of the form $U\cdot W\neq 0$. Subsets of the remaining set of weight expressions $U\cdot W$ that are compatible are then determined, i.e., no linear combination of some or all of these expressions results in an expression $U_i\cdot W$ such that $U_i\cdot W\neq 0$ and independent of each other. Each expression of each of these subsets is then equated to zero, and simple relations between weights are established. These are then used to find the weights vectors W's. The threshold vector T for each W is next established. From the set of weight-threshold vectors (W, T) the desired solution is determined by some minimality criterion. An example has been worked out by hand and an algorithm is given for systematic synthesis procedure.

The method is applicable directly to the truth table or decimal number representation of the switching function, and a large number of permutation and negation of input variables [1], [2] is not necessary for finding the solution. Such processing of the switching function as decomposition and reconstruction [3] is also not necessary. The method, however, involves some amount of computation and efforts are in progress to program the algorithm.

***Index Terms*—Compatibility, multithreshold threshold elements, simple implication, weight expressions and inequalities.**

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I. Introduction

A multithreshold threshold element (MTTE) is defined by its weight-threshold vector $(\boldsymbol{W}, \boldsymbol{T})$, where the weight vector $\boldsymbol{W}=(w_1, w_2, \cdots, w_n)$ consists of an ordered set of n real numbers, called weights, corresponding to the n input variables $x_i \in \{0, 1\}$, $i=1, 2, \cdots, n$, constituting the input vector $\boldsymbol{X}=(x_1, x_2, \cdots, x_n)$ of the switching function $F(\boldsymbol{X})$ realized by the MTTE; and the threshold vector $\boldsymbol{T}=(T_1, T_2, \cdots, T_k)$ is an ordered set of k real numbers, called thresholds. The linear weighted sum

$$\boldsymbol{W}\cdot\boldsymbol{X} = \sum_{i=1}^{n} w_i x_i$$

is called the excitation and is represented as $E(\boldsymbol{X})$. The decision process of the MTTE can be expressed as follows:

$$\begin{aligned} F(\boldsymbol{X}) &= z, && \text{if } E(\boldsymbol{X}) \geq T_1 \\ & && \text{or if } T_{2j} \geq E(\boldsymbol{X}) \geq T_{2j+1} \\ F(\boldsymbol{X}) &= \bar{z}, && \text{otherwise, where } \bar{z} \text{ is the complement of } z \\ & && \text{and } z \in \{0, 1\} \end{aligned}$$

$$\begin{aligned} & j = 1, 2, 3, \cdots \\ & T_j \in \{T_1, T_2, \cdots, T_k\} \\ & T_j > T_{j+1}. \end{aligned}$$

A number of approaches have been made recently for the synthesis of multithreshold threshold elements. Haring [1] developed two algorithms, the first based on the "run-measure" minimization [4] by permuting and/or negating the weights assigned to the variables in the truth table of the given switching function, and the second based on single-threshold realizability conditions of the component subfunctions into which the given function is decomposed. The first algorithm requires $n!2^n$ permutations and/or negations of the weights assigned to variables to minimize the run measure of the given function. Even then a minimal solution with the smallest number of thresholds and the minimum sum of magnitudes of weights is not guaranteed. The second algorithm gives a minimal solution, but due to difficulties of testing and decomposition it is not useful for $k>2$ and $n>6$. The tabular method of Haring and Ohori [5] is based on the Rademacher–Walsh coefficients of the function and its classification into one of 221 equivalence classes (for a four-variable function). The method is applicable for $n \leq 4$ and minimality is not guaranteed. The algorithm developed by Necula [2] uses the run-measure minimization approach, but takes symmetry conditions into consideration and shows that only $n!2^{n-1}$ tests are sufficient. Mow and Fu [6] based their approach for MTTE synthesis on resolving contradictory pairs of vertices by using incremental weights. The synthesis procedure presented by Haring and Diephuis [3] uses a decomposition and reconstruction technique to realize the given function.

This note is a generalization and extension of the work done by Paldi and Sheng [7], and it presents a technique for determining sets of independent relations between the weights of the MTTE that do not contradict the separation of the excitation of TRUE and FALSE vertices. A set of weight vectors is then determined using minimal integral values, and the threshold vectors $\boldsymbol{T}$ for each of them are computed. The weight-threshold vector $(\boldsymbol{W}, \boldsymbol{T})$ that satisfies the minimality criteria is then selected.

The process involves some amount of computation, but is systematic and no preprocessing like decomposition and reconstruction is necessary.

II. Simple Weight Expressions and Weight Relations

Let

$$\boldsymbol{U} = (u_1, u_2, \cdots, u_n) \in \{0, 1, -1\}^n$$

and

$$G(w) \triangleq \boldsymbol{U}\cdot\boldsymbol{W} \tag{1}$$

be a *simple linear expression* of the weights of a multithreshold threshold element. Such equalities or inequalities as

$$G(w) = 0$$

or

$$G(w) \neq 0 \tag{2}$$

are defined as *simple linear relations* between the weights.

It is obvious that complimentary simple weight expressions, i.e., weight expressions $G_a(w)$ and $G_b(w)$ that satisfy

$$G_a(w) + G_b(w) = (\boldsymbol{U}_a + \boldsymbol{U}_b)\cdot\boldsymbol{W} = \boldsymbol{\theta}\cdot\boldsymbol{W} = 0$$

where $\boldsymbol{\theta}$ is the null vector, are not distinct in the sense that they do not generate different simple weight relations. Thus if $G_a(w)=0$, $G_b(w)=-G_a(w)=0$; and if $G_a(w)\neq 0$, $G_b(w)=-G_a(w)\neq 0$.

Two simple weight expressions are called *distinct* if they give rise to different simple weight relations. Weight expressions that have coefficients different from $\{1, 0, -1\}$ are called *complex weight expressions.*

For an n-variable problem, the total number of nontrivial excitations is (2^n-1). The set of vectors $\{\boldsymbol{U}\}$ is of order 3^n. Eliminating the null vector $\boldsymbol{\theta}$ and considering that for every vector $\boldsymbol{U}_a \in [\{\boldsymbol{U}\}-\boldsymbol{\theta}]$ there is another vector $\boldsymbol{U}_b \in [\{\boldsymbol{U}\}-\boldsymbol{\theta}]$ such that $\boldsymbol{U}_a \neq \boldsymbol{U}_b$ and $\boldsymbol{U}_a+\boldsymbol{U}_b=\boldsymbol{\theta}$; the total number of distinct simple expressions reduces to

$$N = \frac{(3^n-1)}{2}\,. \tag{3}$$

Example 1

All the distinct simple weight expressions for a three-variable problem can be expressed by the following column matrix:

$$[U]\cdot \boldsymbol{W} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}. \tag{4}$$

In (4), each dot product $\boldsymbol{U}\cdot\boldsymbol{W}$ corresponds to a simple distinct weight expression.

The distinction between an excitation $E(\boldsymbol{X})$ at a vertex $\boldsymbol{X}$ and a simple weight expression $G(w)$ is that while $E(\boldsymbol{X})$ is a linear combination of the weights with coefficients (1, 0), the simple weight expression $G(w)$ is a linear combination of the weights with coefficients 1, 0, and -1. Thus

$$\{E(\boldsymbol{X})\} \subset \{G(w)\}. \tag{5}$$

Now, consider a TRUE vertex $\boldsymbol{X}_i$ and a set of FALSE vertices $\boldsymbol{X}_{j1}, \boldsymbol{X}_{j2}, \cdots, \boldsymbol{X}_{js}$. Since the excitations of a TRUE and a FALSE vertex must be different, we have

$$E(\boldsymbol{X}_i) \neq E(\boldsymbol{X}_{jk}), \qquad k = 1, 2, \cdots, s \tag{6}$$

or

$$E(\boldsymbol{X}_i) - E(\boldsymbol{X}_{jk}) = G_k(w) \neq 0. \tag{7}$$

Thus corresponding to the TRUE vertex $\boldsymbol{X}_i$ and the FALSE vertices $\boldsymbol{X}_{j1}, \boldsymbol{X}_{j2}, \cdots, \boldsymbol{X}_{js}$ we have s distinct simple weight relations (inequalities) of the form (7). Let there be $\boldsymbol{X}_{i1}, \boldsymbol{X}_{i2}, \cdots, \boldsymbol{X}_{ir}$, a total of r TRUE vertices, then inequalities of the form (7) can be generated for each of r TRUE vertices and all the s FALSE vertices, but not all $r\cdot s$ inequalities are distinct. Thus the switching function being realized constrains a set of $q \leq r\cdot s$ simple weight expressions into inequalities. This leaves us with $(N-q)$ simple weight expressions, some or all of which may be equated to zero.

Example 2

Consider the three-variable example:

$$F(\boldsymbol{X})_3 = \sum 3, 5, 7.$$

From the comparison of excitations we obtain the following inequalities:

$$\begin{array}{lll} w_1 \neq 0 & w_1 + w_2 \neq 0 & w_1 - w_2 - w_3 \neq 0 \\ w_2 \neq 0 & w_1 + w_3 \neq 0 & w_2 - w_1 - w_3 \neq 0 \\ w_3 \neq 0 & w_2 + w_3 \neq 0 & w_1 + w_2 + w_3 \neq 0. \\ & w_1 - w_3 \neq 0 & \\ & w_2 - w_3 \neq 0 & \end{array} \tag{8}$$

This leaves us with only $(13-11)=2$ simple weight expressions that we may equate to zero giving us the following equality relations:

$$\begin{array}{lcl} w_1 - w_2 = 0 & & w_1 = w_2 \\ & \text{or} & \\ w_3 - w_1 - w_2 = 0 & & w_3 = w_1 + w_2. \end{array} \tag{9}$$

Setting $w_1 = w_2 = 1$ and $w_3 = 2$, a single-threshold realization is

$$(\boldsymbol{W}, \boldsymbol{T}) = (1, 1, 2; 2.5). \tag{10}$$

III. Determination of Compatible Weight Expressions

It has been observed in the previous section that the fact that excitations of TRUE and FALSE vertices are different can be utilized to partition the set of distinct simple linear weight expressions $\{G(w)\}$ into two subsets. The subset $\{G(w)\}_I$ contains all elements that are constrained to be inequalities.

Let us call the other subset the residual subset $\{G(w)\}_R$ of weight expressions.

In Example 2, the two weight expressions in (9),

$$G_1(w) = w_1 - w_2$$

and

$$G_2(w) = w_3 - w_1 - w_2$$

constitute the residual subset $\{G(w)\}_R$ and have been converted into equalities, as they are not contradictory and are independent of each other. However, this is a simple example and, in general, the number of elements in $\{G(w)\}_R = |\{G(w)\}_R| \gg n$.

A systematic method is, therefore, required to select only those weight expressions from $\{G(w)\}_R$ that may be equated to zero giving us the required weight relations useful for synthesis. An approach towards this end is given below.

Two weight expressions $G_a(w)$ and $G_b(w) \in \{G(w)\}_R$ are said to be *pairwise compatible* if both can be equated to zero without contradicting each other. They are *pairwise incompatible* otherwise. A set of simple weight expressions $\{G(w)\}' \subseteq \{G(w)\}_R$, each element of which has been equated to zero, is *completely compatible* if no linear combination results in a $G_i(w)$ such that $G_i(w) \in \{G(w)\}_I$, or if

$$\{G_1(w), G_2(w), \cdots, G_k(w)\} \subseteq \{G(w)\}_R$$

constitutes a completely compatible, or CC. Then

$$c_1G_1(w) + c_2G_2(w) + \cdots c_kG_k(w) = G_i(w) \notin \{G(w)\}_I$$

for all values of c_i, $i=1, 2, \cdots, k$, and $c_i \neq 0$.

A set of simple weight expressions $\{G_1(w), G_2(w), \cdots, G_k(w)\}$ is linearly independent if $c_1G_1(w)+c_2G_2(w)+\cdots+c_kG_k(w)=0$ implies $c_i=0$ for $i=1, 2, \cdots, k$. $G_1(w)$, $G_2(w) \in \{G(w)\}_R$ *imply* $G_3(w) \in \{G(w)\}_R$ if $c_1G_1(w)+c_2G_2(w)=G_3(w)$ for c_1, c_2 arbitrary integers. If $c_1, c_2=\pm 1$, $G_1(w)$ and $G_2(w)$ are said to *simply imply* $G_3(w)$.

Consider two distinct simple weight expressions $G_a(w)$, $G_b(w) \in \{G(w)\}_R$, and equate each of them to zero such that we have two independent weight relations

$$G_a(w) = 0 \qquad G_b(w) = 0. \tag{11}$$

Now, let us generate $H(w)$ such that

$$H(w) = pG_a(w) + qG_b(w) \tag{12}$$

where p and q are arbitrary integers.

Theorem 1: Any two arbitrary simple weight expressions $G_a(w)$, $G_b(w) \in \{G(w)\}_R$ equated to zero are pairwise incompatible if

$$1/r \cdot H(w) \in \{G(w)\}_I$$

where r is any integer and $r \neq 0$.

Proof: Let there be at least one $H_k(w)$ such that

$$\begin{aligned} H_k(w) &= p_kG_a(w) + q_kG_b(w) \\ &= rG_i(w), \qquad (p, q, r) \text{ are integers} \end{aligned}$$

and $G_i(w) \in \{G(w)\}_I$, i.e., $G_i(w) \neq 0$. Therefore, $H_k(w) \neq 0$. But this is a contradiction since $G_a(w)$ and $G_b(w)$ are both assumed to be zero. Thus $G_a(w)$ and $G_b(w)$ both cannot be equated to zero. Hence, they are pairwise incompatible. Q.E.D.

Example

Consider $\boldsymbol{U}_a=[1\ 0\ 0\ 1]$ and $\boldsymbol{U}_b=[1\ 0\ 0\ -1]$. Here, $\boldsymbol{U}_a+\boldsymbol{U}_b=[2\ 0\ 0\ 0]$ and $\boldsymbol{U}_a-\boldsymbol{U}_b=[0\ 0\ 0\ 2]$. Thus $\frac{1}{2}[\boldsymbol{U}_a+\boldsymbol{U}_b]\cdot \boldsymbol{W}$ and $\frac{1}{2}[\boldsymbol{U}_a-\boldsymbol{U}_b]\cdot \boldsymbol{W} \in \{G(w)\}$. The simple weight expressions $G_a(w)$, $G_b(w) \in \{G(w)\}_R$ are said to be *simply pairwise incompatible*, designated $G_a(w) \nsim G_b(w)$, if $p=q=r=\pm 1$ and $H(w) \in \{G(w)\}_I$.

Let us consider two arbitrary $\boldsymbol{U}$ vectors $\boldsymbol{U}_a$ and $\boldsymbol{U}_b$. The distribution of 1, 0, and -1 among these two vectors is as follows.

Distribution 1: A 0 in one corresponds to a 0 in the other.

Distribution 2: A 1 or -1 in one corresponds to a 1 or a -1 in the other, respectively.

Distribution 3: A 0 in one corresponds to a 1 or a -1 in the other.

Distribution 4: A 1 in one corresponds to a -1 in the other.

Distributions 1 and 2 constitute the identical part of the two vectors and Distribution 4 constitutes the complementary part. From the structure of the $\boldsymbol{U}$ vectors we have the following theorem.

Theorem 2: The weighted sum of two simple linear expressions

$$H(w) = [p \quad q]\cdot \begin{bmatrix} \boldsymbol{U}_a \\ \boldsymbol{U}_b \end{bmatrix} \cdot \boldsymbol{W}^T \notin \{G(w)\} \tag{13}$$

if the distribution of $\{1, 0, -1\}$ in $\boldsymbol{U}_a$ and $\boldsymbol{U}_b$ satisfy at least Distribution 1, 3, and 4 simultaneously.

Proof:

$$\begin{aligned} H(w) &= [p \quad q]\cdot \begin{bmatrix} \boldsymbol{U}_a \\ \boldsymbol{U}_b \end{bmatrix} \cdot \boldsymbol{W}^T, \qquad p \text{ and } q \text{ are arbitrary integers} \\ &= [p \quad q]\cdot (\text{Distribution 1, Distribution 3, Distribution 4}) \cdot \boldsymbol{W}^T \\ &= [(p+q) \text{ or } -(p+q),\ (\pm q \text{ or } \pm p),\ (p-q) \text{ or } (q-p)]\cdot \boldsymbol{W}^T. \end{aligned}$$

For $p, q \geq 1$,

$$H(w) = [\pm k, \pm p \text{ or } q, \pm l], \qquad k > 1, l = 0, 1. \tag{14}$$

Thus $H(w) \notin \{G(w)\}$. Distribution 1 contains all zeros and does not affect the status of $H(w)$.

Corollary 1: $\frac{1}{2}H(w)=\frac{1}{2}[\boldsymbol{U}_a \pm \boldsymbol{U}_b]\cdot \boldsymbol{W}^T \in \{G(w)\}$ if the distribution of 1, 0, -1 in $\boldsymbol{U}_a$ and $\boldsymbol{U}_b$ is Distribution 2 and 4 (or Distributions 2, 4, and 1).

Proof: Now, $\boldsymbol{U}_a \pm \boldsymbol{U}_b=[0, \cdots, \pm 2, \cdots, 0, \cdots \pm 2, \cdots, 0, \cdots]$.

Corollary 2: Distribution 3 only can occur singularly, and then, $H(w)=(U_a \pm U_b) \in \{G(w)\}$.

Proof: If Distribution 2 only exists, $\boldsymbol{U}_a=\boldsymbol{U}_b$. If Distribution 4 only exists, $\boldsymbol{U}_a+\boldsymbol{U}_b=\boldsymbol{\theta}$ and $\boldsymbol{U}_a$ and $\boldsymbol{U}_b$ are not distinct. Occurrence of Distribution 1 only implies $\boldsymbol{U}_a=\boldsymbol{U}_b=\boldsymbol{\theta}$.

IV. The Algorithm

The purpose of the proposed algorithm is to find a set or sets of simple weight expressions (SWE's) each of which can be equated to zero without contradictions or redundancy. This results in sets of relations between the weights of the MTTE that are then utilized for optimal realization. The fact that testing for complete compatibility involves generating $H(w)$'s with $|p|$, $|q| > 1$, and mutual incompatibility between three or more expressions, we confine our processing to simple pairwise compatibility only. This together with the use of the simple implication relations (SIR) results in sets of almost compatible weight expressions. The corresponding $\boldsymbol{U}$ vectors constituting the almost compatible weight expressions or maximal set of simple pairwise compatibles are now subjected to elementary row operations to remove any residual redundancy. Finally we get: 1) $h \leq n-1$ rows, each with two nonzero entries, one of which is unity, the rest of the entries are zeros; 2) some rows with all zeros (corresponding to any linear dependency left unremoved); 3) some rows with only entries in the ith column $i \leq n$ (this indicates incompatibility if the function is not independent of the ith variable).

If only 1) is present, the corresponding weight expressions are each equated to zero and they constitute a set of binary weight relations. If 1) and 2) are present, the all-zero rows of 2) are ignored and the solution is obtained from 1). If the set of rows of 3) is also present and the function is not independent of the ith variable, all possible lower order subsets of the $h<n-1$ binary weight relations are investigated for compatibility starting with the highest order.

The first step of the algorithm is to determine the set $\{G(w)\}_I$ of SWE's constrained to be inequalities by the specified switching function. The complementary subset $\{G(w)\}_R$ is next determined. Simple pairwise incompatibility and SIR's between SWE's using Theorem 2 and its corollaries are then established. These are then used [8] to determine the maximal subset of simple pairwise compatibles (MSPC) of SWE's. The SIR's of the form

$$A_i \pm A_j = \pm A_k \tag{15}$$

where A_i, A_j, and A_k are SWE's, are next utilized to remove any redundant weight relations from the MSPC's. Thus in an MSPC, if A_i, A_j, and A_k occur simultaneously and if they satisfy (15), any one of them, say A_k, is removed. All the MSPC's are thus tested with all the SIR's. If, however, an MSPC contains only two SWE's A_i, A_j, say of a relation as (15), they are replaced by the third, say A_k, and simple pairwise incompatibility between A_k and each of the remaining elements of the MSPC is tested using the simple pairwise incompatibility relations. Reduced MSPC's are thus generated whenever A_k is simple pairwise incompatible with some element of the original MSPC. In each of these reduced MSPC's, A_k is finally replaced by A_i and A_j wherever A_k occurs. The $[\boldsymbol{U}]$ submatrices corresponding to these MSPC's are then subjected to row operations as indicated earlier.

Once the set of binary weight relations is established, the smallest weight in each case is put equal to unity and the weight vector $\boldsymbol{W}$ is obtained. The threshold vector $\boldsymbol{T}$ for each $\boldsymbol{W}$ is realized by evaluating the excitation at each vertex and noting their boundaries at the TRUE and FALSE vertices.

The following two minimality criteria are now considered (any other minimality criteria can be used): 1) k, the number of thresholds is a minimum; 2) $S=\sum_{i=1}^{u}|w_i|$ is a minimum; and the $(\boldsymbol{W}, \boldsymbol{T})$ or sets of them that satisfy 1) and 2) are selected. The algorithm for the procedure mentioned above is given in Fig. 1.

V. A Numerical Representation of Simple Weight Expressions

As we have seen in the previous sections, a first step towards synthesis is to compare the excitations of each of the TRUE/FALSE vertices with each of the FALSE/TRUE vertices. This step consists of a possible maximum of $2^{2(n-1)}$ individual comparisons.[1] To simplify this process, the following numerical approach is found very useful.

Let us consider a set of weights

$$w_i = 3^{n-i}, \qquad i = 1, 2, \cdots, n. \tag{16}$$

The fact that in this system

$$w_i = 2(w_{i+1} + w_{i+2} + \cdots + w_n) + 1, \qquad i < n$$

or

$$w_i - (w_{i+1} + w_{i+2} + \cdots + w_n) = (w_{i+1} + w_{i+2} + \cdots + w_n) + 1$$

makes it possible to generate all possible simple expressions, no two of which are of identical value. Also, $\sum_{i=1}^{n} w_i$ is minimum.

Thus we first transform the specified vertices from the familiar decimal number representation into the number system mentioned above. The two sets of numbers corresponding to the TRUE and FALSE vertices are subtracted one from each of the rest. The magnitude of the numbers generated corresponds to the inequalities $\{G(w)\}_I$. From the set of successive numbers $1, 2, \cdots, N=(3^n-1)/2$, the numbers corresponding to $\{G(w)\}_I$ are removed, leaving behind the set $\{G(w)\}_R$.

VI. An Example of Synthesis

To clarify the ideas presented so far we introduce the following four-variable problem:

$$F(x_1, x_2, x_3, x_4) = \sum 1, 2, 3, 5, 7, 9, 15.$$

Using $w_1=27$, $w_2=9$, $w_3=3$, and $w_4=1$, we write the excitations at the TRUE vertices $E(\boldsymbol{X})_1$ and excitations at the FALSE vertices $E(\boldsymbol{X})_0$ as follows:

$$\{E(\boldsymbol{X})_1\} = \sum 1, 3, 4, 10, 13, 28, 40$$

and

$$\{E(\boldsymbol{X})_0\} = \sum 0, 9, 12, 27, 30, 31, 36, 37, 39.$$

Here, $N=(3^4-1)/2=40$.

The integers 1 to 40 are listed in Fig. 2, and those that correspond to the magnitude of difference of the excitations $E(\boldsymbol{X})_1$ and $E(\boldsymbol{X})_0$ are crossed. For example, consider $E(\boldsymbol{X})_1=4$ and all the $E(\boldsymbol{X})_0$. The magnitude of differences 0, 5, 8, 23, 26, 27, 32, 33, 35 are crossed. If a number is crossed once, it need not be crossed again. The result of this step is given in Fig. 2.

The $\boldsymbol{U}$ vectors corresponding to the numbers not crossed in the list given in Fig. 2 are shown in Table I; they are numbered as A, B, etc. The results of comparison of the $\boldsymbol{U}$ vectors are given in Table II.

Using the pairwise incompatibility relations of Table II, the following maximal simple pairwise compatibles are obtained: 1) $BDEF$; 2) DEG; 3) ABD; 4) ADG;

[1] Let p be the number of TRUE vertices. Then for a completely specified function, the number of FALSE vertices $q=2^n-p$. The number of comparisons $C=pq=p(2^n-p)$. This becomes maximum for $p=2^{n-1}$ and correspondingly $C_{p_{\max}}=2^{2(n-1)}$.

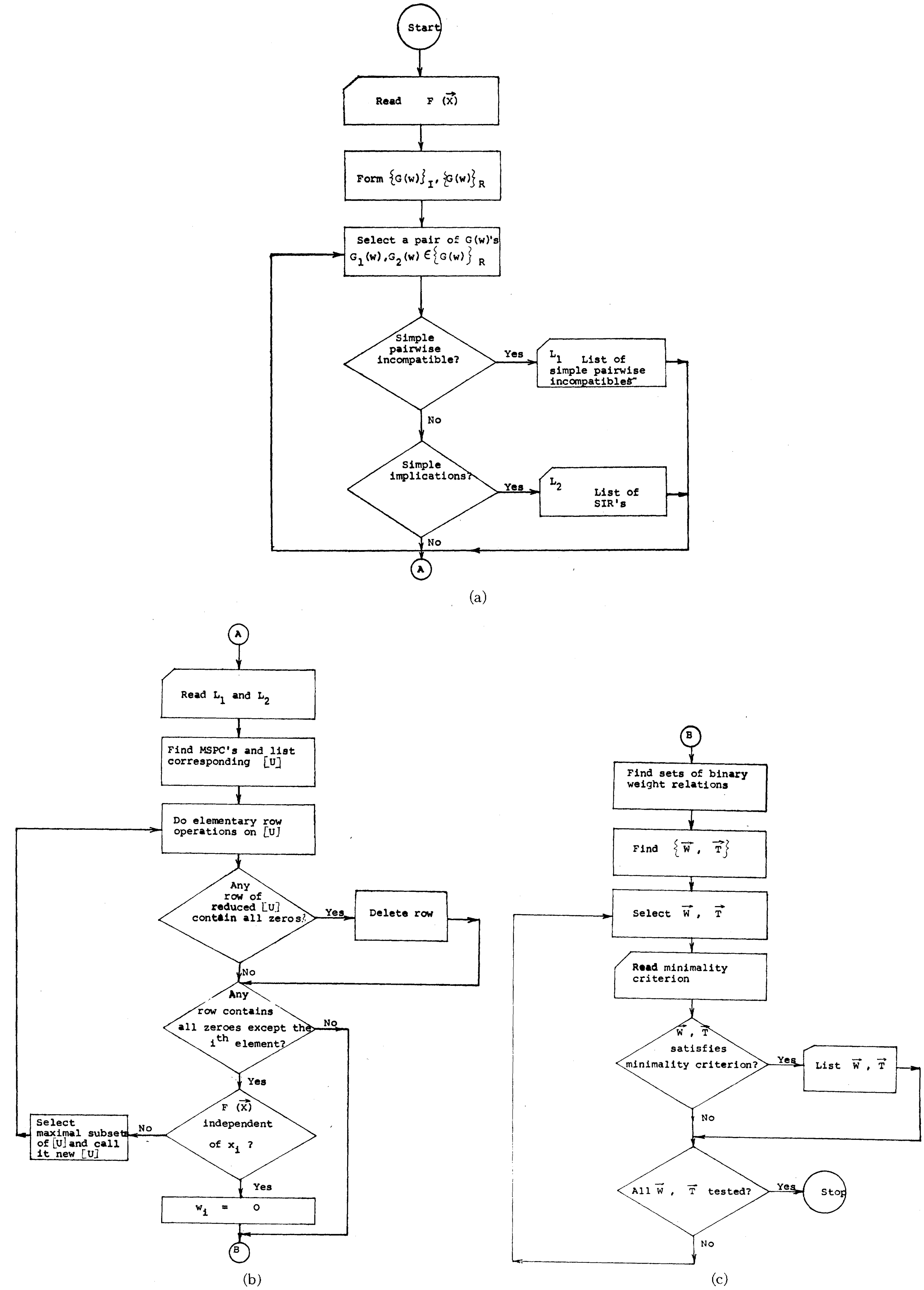

Fig. 1. MTTE synthesis algorithm. (Letters with arrows above them in figure appear as boldface letters in text.)

~~1~~	~~2~~	~~3~~	~~4~~	~~5~~	~~6~~	7	~~8~~	~~9~~	~~10~~
~~11~~	12	~~13~~	~~14~~	15	~~16~~	~~17~~	~~18~~	~~19~~	~~20~~
~~21~~	22	~~23~~	~~24~~	25	~~26~~	~~27~~	~~28~~	~~29~~	~~30~~
~~31~~	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	39	~~40~~

Fig. 2. Comparison of $\{E(\boldsymbol{X})_1\}$ and $\{E(\boldsymbol{X})_0\}$ of a four-variable function. (Letters with arrows above them in figure appear as boldface letters in text.)

TABLE I

$\boldsymbol{U}$ VECTORS OF EXAMPLE

	27	9	3	1	
(7)	0	1	−1	1	A
(12)	0	1	1	0	B
(15)	1	−1	−1	0	C
(22)	1	−1	1	1	D
(25)	1	0	−1	1	E
(37)	1	1	0	1	F
(39)	1	1	1	0	G

TABLE II

LIST OF SIMPLE PAIRWISE INCOMPATIBLES AND SIMPLE IMPLICATION RELATIONS

Incompatibles	Linear Dependencies
$A \nsim (E, F)$	
$B \nsim (C, G)$	$B+E=F$
$C \nsim (E, G)$	
$F \nsim (G)$	$G-C=2B$

TABLE III

LIST OF $(\boldsymbol{W}, \boldsymbol{T})$ AND THE CORRESPONDING k AND S VALUES

Subsets of $\{G(w)\}_R$	$\boldsymbol{W}=(w_1, w_2, w_3, w_4)$ $\boldsymbol{T}=(T_1, T_2, \cdots, T_k)$	k = Number of Thresholds	$S=\sum_{i=1}^{n}\lvert w_i\rvert$
DEG	$(-3, 2; 1, 4)$ $(3.5, 1.5, 0.5)$	3	10
ABD	$(-4, -1, 1, 2)$ $(0.5, -1.5, -2.5)$	3	8
ADG	$(-4, 1, 3, 2)$ $(4.5, 3.5, 1.5, -1.5, -2.5)$	5	10
CDF	$(-3, -1, -2, 4)$ $(0.5, -1.5, -2.5)$	3	10
ACD	$(-4, -3, -1, -2)$ $0.5, -0.5, -2.5, -5.5, -6.5$	5	10
BE[a]	a) $(1\ -3\ 3\ 2)$ $5.5, 4.5, 3.5, 1.5, -0.5, -1.5$	6	9
	b) $(2\ -3\ 3\ 1)$ $4.5, 2.5, 1.5, 0.5, -1.5, -2.5$	6	9

[a] Of the three possible 2's combination of the elements of BDE, i.e., BD, DE is included in sets ABD and DEG, respectively. Of the many choices for the weights in BE, only those with minimum S are included in this table.

5) CDF; and 6) ACD. Checking the MSPC's with the second column of Table II, it is observed that B, E, and F occur in MSPC1. Hence F is removed.

In the set BDE, B and E are now replaced by F, but as F is compatible with D (there is no pairwise incompatibility relation between D and F), the set BDE can now be used to find relations between weights.

The row operations on BDE are indicated below:

$$\begin{array}{c c} & \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \end{array} \\ \begin{array}{c} B \\ D \\ E \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}.$$

Remembering that all the weight expressions B, D, and E are equated to zero and observing that the last row of the transformed $\boldsymbol{U}$ matrix corresponds to a multiple of an inequality, it is concluded that B, D, and E are not compatible. Thus subsets of B, D, and E are to be investigated for compatibility and weight relations. For the present we consider the other pairwise compatibles and will return to this subset afterwards.

Results of row operations on the second MSPC, DEG, are given as follows:

$$\begin{array}{c c} & \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \end{array} \\ \begin{array}{c} D \\ E \\ G \end{array} & \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & -4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{3}{4} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{4} \end{bmatrix}.$$

Remembering each of D, E, and G are equalities, we have the following binary weight relations: 1) $w_1 = -\frac{3}{4}w_4$; 2) $w_2=\frac{1}{2}w_4$; and 3) $w_3=\frac{1}{4}w_4$. For integral weight values, w_4 is chosen equal to 4. Thus the weight vector $\boldsymbol{W}=(-3, 2, 1, 4)$. The corresponding threshold vector $\boldsymbol{T}$ can easily be obtained by constructing numbers with these weights and finding the boundaries of TRUE and FALSE regions. Here $\boldsymbol{T}=(3\cdot 5, 1\cdot 5, \cdot 5)$. For this solution $k=3$ and $S=10$. Here w_4 could have been chosen as -4, but the values of k and S would remain unchanged. The results of the other pairwise compatibles and the first set BDE are shown in Table III. From Table III, the desired solution that satisfies $k=\min$ and $S=\min$ corresponds to the set ABD and the solution is

$$\boldsymbol{W}, \boldsymbol{T} = (-4, -1, 1, 2) \qquad (0.5, -1.5, -2.5).$$

VII. CONCLUSION

A different approach towards multithreshold threshold element synthesis is presented. The method utilizes the simple fact that excitations at TRUE and FALSE vertices are not identical. Of course, the comparisons that have to be performed are large in number, but the fact that the technique is systematic makes it convenient for computer programming. A program for solving a six-variable problem is underway.

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Digital Multiplexing Analog Signals

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Abstract—**A method is presented that combines the multiplexing and A/D function to eliminate analog multiplexing switches.**

Index Terms—**Analog, A/D converter, comparator, digital, multiplexer.**

The conventional system for sampling a number of analog signals with a single A/D converter is shown in Fig. 1. One analog input at a time is switched through analog switches to the A/D converter under control of gating signals G_i. Often, FET switches are used as the switching elements.

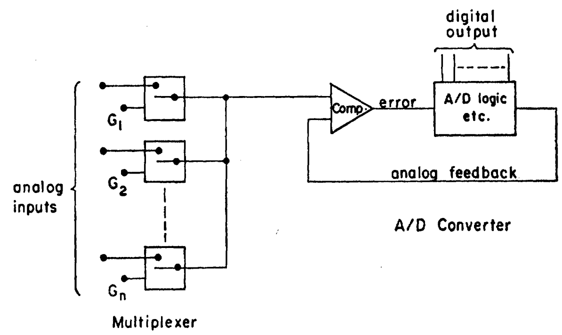

Fig. 1. Conventional multiplexor A/D system.

The proposed scheme is shown in Fig. 2. Rather than using a single comparator within the A/D converter, a comparator is used with each input. The digital error signal of the input to be sampled is routed to the A/D logic by applying the appropriate gating signal G_i.

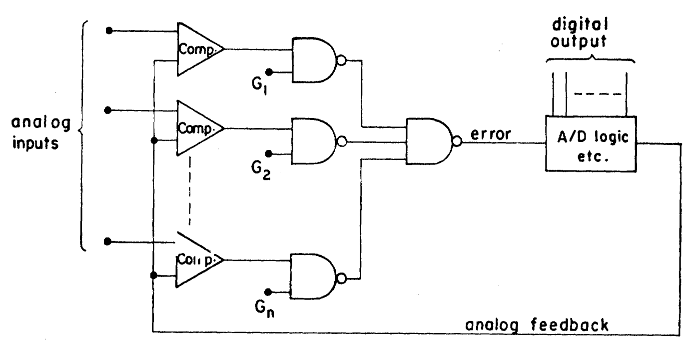

Fig. 2. Digital multiplexor for analog inputs.

For unbalanced inputs as shown in the figures, n analog switches plus one comparator are traded for n comparators. For balanced inputs, n 3-input comparators could replace $2n$ analog switches and a 2-input comparator. With the conventional method, for some applications it is possible to use a single amplifier between the multiplexer and the A/D converter if amplification is required. With the digital multiplexing method, however, an amplifier would be required for each channel. The proposed scheme, on the other hand, completely eliminates analog switches and their associated limitations such as settling time, offset, and drift. If the comparators used in the proposed scheme are the same as the comparator used in the conventional system, the proposed system will give improved performance because the switches have been eliminated. If economy rather than performance is of concern, the use of lower performance comparators than the comparator in the conventional system will result in equal performance of the two systems. Overall cost performance will, of course, depend upon switch comparator and A/D logic technology. The proposed scheme could be the preferred approach for some applications.

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