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Strong Connectivity in Symmetric Graphs and Generation of Maximal Minimally Strongly Connected Subgraphs

S. R. DAS

and

C. L. SHENG

Institute of Computer Science, National Chiao Tung University, Hsinchu, Taiwan, Republic of China

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ABSTRACT

In nonsymmetric graphs strong connectivity is an important concept. In this paper, extending the concept of strong connectivity of nonsymmetric graphs to the case of symmetric graphs, the idea of minimally strongly connected (MSC) and maximal minimally strongly connected (MMSC) subgraphs in a symmetric graph is introduced, and theoretical results are developed that pertain to certain useful properties of these subgraphs. A computer-oriented algorithm is also proposed for finding the MMSC subgraphs from a given symmetric graph, that seems efficient and simple, and tends to reduce computation in generating the subgraphs.

I. INTRODUCTION

Linear graph theory is finding increasing applications as a tool of analysis in widely differing areas of science and technology [1–10]. Linear symmetric graphs in particular have special applications in sequential switching theory, map coloring problems, transportation research, systems programming, etc. and have also been widely studied. Consider a symmetric graph G with n nodes v_i, $i=1,2,\ldots,n$. A *subgraph* G_s of G is a symmetric graph that has a subset of the set of nodes of G as its nodes and a subset of the set of edges of G as its edges. A *path* from node v_i to node v_j in G is a concatenation of r nondirected edges leading from v_i to v_j, r being the *length* of the path. A *circuit* in G is a path that starts from and terminates on the same node. Two *nodes* v_i and v_j in G are said to be *connected* if and only if there exists a path from v_i to v_j. A graph G is *connected* if every pair of its nodes is connected. The existence

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of a path between nodes in G is an equivalence relation, and as such it defines a partition of the nodes of G into disjoint subsets.

Similar to the concept of connectivity for symmetric graphs, for nonsymmetric or oriented graphs there is the concept of strong connectivity [6]. In the present paper in direct extension of the aforesaid concepts, we introduce a related concept, the concept of minimal strong connectivity, and define minimally strongly connected (MSC) subgraphs and maximal minimally strongly connected (MMSC) subgraphs in symmetric graphs as follows. Two nodes v_i and v_j in a symmetric graph G are said to be *minimally strongly connected* (MSC) if and only if v_i and v_j are connected by a path of length one. Otherwise, the nodes, if connected, are said to be *nonminimally strongly connected*. A subgraph G_s of G is said to be a *minimally strongly connected* (MSC) *subgraph* if and only if every possible pair of nodes in G_s is minimally strongly connected. Thus in a minimally strongly connected subgraph with k nodes, the total number of edges is $k(k-1)/2$. A minimally strongly connected subgraph G_s of G is a *maximal minimally strongly connected* (MMSC) *subgraph* if and only if there does not exist any node in G outside of the subgraph G_s which is minimally strongly connected with all the nodes of G_s. The existence of a path of length one between nodes of G, unlike that of paths of any length, is not an equivalence relation (since transitivity does not hold in this case), and accordingly it partitions the set of nodes of G into overlapping subsets. The MMSC subgraphs are thus not mutually exclusive and can have nonvoid intersections. Note that an MSC subgraph corresponds to a *complete subgraph* or *clique*, and an MMSC subgraph corresponds to a *maximal complete subgraph* or *maximal clique* of symmetric graphs. As illustration, refer to Fig. 1(a), which shows a symmetric graph G with six nodes $(v_1, v_2, v_3, v_4, v_5, v_6)$. In G, the node v_1 is minimally strongly connected with each of the nodes v_2, v_3, v_4, v_6, and the subgraph consisting of the nodes (v_1, v_2, v_3, v_4) is an MMSC subgraph.

In this paper theoretical results are developed that pertain to certain useful properties of symmetric graphs in relation to their MMSC subgraphs. A well-defined algorithm is also developed for finding all the MMSC subgraphs from a given symmetric graph, that follows a process of successive decomposition of the graph around some of the nonminimally strongly connected pairs of nodes of the graph, and tends to reduce computation in the generation of these subgraphs. The results established are significant, and the subgraph generation algorithm as given is explicitly simple and seems readily programmable. It is relevant to mention here that in the present discussion we consider symmetric graphs with no self-loops and multiple edges, that is, we do not consider *pseudographs* [10], and assume the graph to be a connected graph unless otherwise stated.

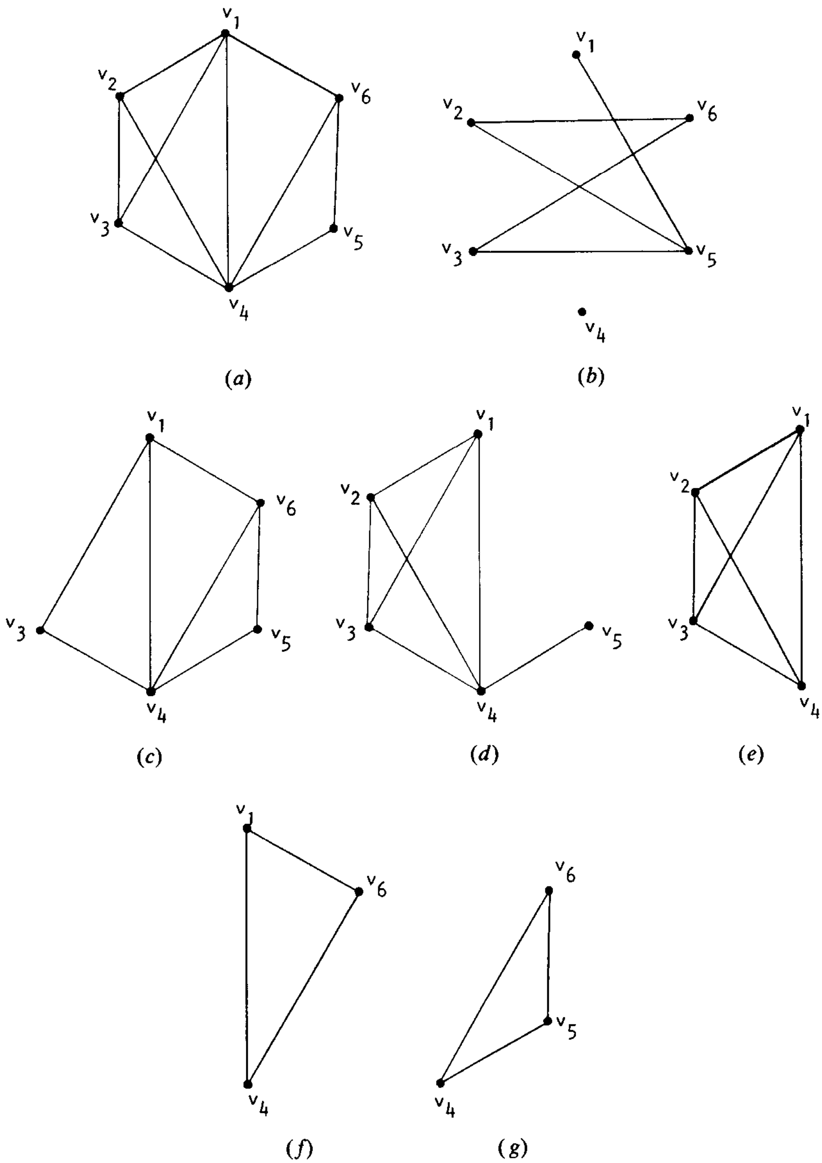
v_1 v_2 v_3 v_4 v_5 v_6
(a)
v_1 v_2 v_3 v_4 v_5 v_6
(b)
v_1 v_3 v_4 v_5 v_6
(c)
v_1 v_2 v_3 v_4 v_5
(d)
v_1 v_2 v_3 v_4
(e)
v_1 v_4 v_6
(f)
v_4 v_5 v_6
(g)

Fig. 1.

II. SYMMETRIC GRAPHS, MMSC SUBGRAPHS AND A SUBGRAPH GENERATION ALGORITHM

Consider a node v_i in a symmetric graph G. The *degree* of v_i, $d(v_i)$ is the number of edges of G incident in v_i. Two subgraphs G_a and G_b are said to be *complementary* to each other if and only if both G_a and G_b have the same set of nodes and one has edges connecting between those pairs of nodes that are not connected by edges in the other. Figure 1(b) shows the complementary graph $\bar{G}$ of the symmetric graph G of Fig. 1(a).

The theorems below, which follow rather obviously, develop theoretical results that establish certain important properties of symmetric graphs in relation to their MMSC subgraphs.

THEOREM 2. *Let G be a symmetric graph with n nodes, v_i, $i=1,2,\ldots,n$. Denote by $\bar{G}$ the complementary graph of G. If the degree of a node v_j, $d(v_j)$, in G is zero, then v_j must appear in all the MMSC subgraphs of $\bar{G}$.*

COROLLARY 1.1. *If the degree $d(v_i)$ of every node v_i, $i=1,2,\ldots,n$, in G is $n-1$, then G itself is an MMSC graph.*

THEOREM 2. *Let G be a symmetric graph with n nodes, v_i, $i=1,2,\ldots,n$. Denote by $\bar{G}$ the complementary graph of G. If the degree $d(v_j)$ of a node v_j in G is zero, then v_j must appear in all the MMSC subgraphs of $\bar{G}$.*

COROLLARY 2.1. *If the degree $d(v_i)$ of every node v_i, $i=1,2,\ldots,n$, in G is zero, then the complementary graph $\bar{G}$ of G is an MMSC graph.*

THEOREM 3. *Consider a symmetric graph G with n nodes v_i, $i=1,2,\ldots,n$. Let the nodes v_j and v_{j+1}, $j=1,2,\ldots,n-1$, be minimally strongly connected, and so also be the nodes v_1 and v_n in G; for all other pairs of nodes (v_k,v_m), let v_k and v_m be nonminimally strongly connected. In G, the path $r_{12}r_{23}\cdots r_{n1}$ is a circuit, r_{ij} being the edge connecting nodes v_i and v_j; the degree of every node v_j, $d(v_j)$, is two; and there are n MMSC subgraphs.*

THEOREM 4. *In a symmetric graph G, let the degree of a node v_i, $d(v_i)$, be k, and let $v_{i1},v_{i2},\ldots,v_{ik}$ be the k other nodes of G minimally strongly connected with v_i. If the degree of each of the nodes v_{im}, $m=1,2,\ldots,k$, is $d(v_{im})\geqslant k$, and each v_{im} is minimally strongly connected with at least the nodes $v_{i1},v_{i2},\ldots,v_{i(m-1)},v_{i(m+1)},\ldots,v_{ik}$ in addition to v_i, then the set of nodes $v_i,v_{i1},\ldots,v_{ik}$ forms an MMSC subgraph of G.*

In the symmetric graph G in Fig. 1(a), the nodes v_1,v_2,v_3,v_4 satisfy the requirements of Theorem 4, and thus the subgraph comprising the nodes (v_1,v_2,v_3,v_4) forms an MMSC subgraph of G.

The *degree complement* of a node v_i, $\bar{d}(v_i)$, in a symmetric graph G is the degree of v_i in the complementary graph $\bar{G}$. In an n-node graph G, if the degree of a node v_i, $d(v_i)$, is k, $k \leqslant n-1$, then the degree complement of v_i, $\bar{d}(v_i)$, is $n-k-1$. The *degree complement* of a nonminimally strongly connected pair of nodes (v_i, v_j) is $\bar{d}(v_i, v_j) = (k_1, k_2)$, where $\bar{d}(v_i) = k_1$, $\bar{d}(v_j) = k_2$. Considering Fig. 1(a), the degree complement of node v_5, $d(v_5)$, in G is three, while that of (v_2, v_6), which represents a nonminimally strongly connected pair of nodes in G, is given as $\bar{d}(v_2, v_6) = (2, 2)$.

For two subgraphs G_i and G_k of a symmetric graph G, let the set of nodes in G_i be a *proper* or an *improper* subset of the set of nodes in G_k, both G_i and G_k having all the existing edges of G connecting relevant pairs of nodes. Then $G_i \subseteq G_k$. Consider now a nonminimally strongly connected pair of nodes (v_i, v_j) in G. Then *decomposing* G into two subgraphs G_i and G_j around (v_i, v_j) means obtaining the subgraphs G_i and G_j from G such that G_i contains all the nodes of G except v_j and G_j contains all the nodes of G except v_i, both G_i and G_j having all the existing edges of G connecting relevant pairs of nodes. Obviously, $G_i, G_j \subseteq G$. Figure 1(c) and (d) shows two subgraphs G_i and G_j obtained by decomposing the graph G of (a). Evidently, $G_i, G_j \subseteq G$.

THEOREM 5. *Let G be a symmetric graph, and let (v_i, v_j) be a nonminimally strongly connected pair of nodes in G. Let G be decomposed around (v_i, v_j) into two subgraphs G_i and G_j, and let this process of decomposition around nonminimally strongly connected pairs of nodes be iteratively applied to G_i and G_j and to all their subgraphs until in the resulting subgraphs there exist no more nonminimally strongly connected pairs of nodes. The final set of these subgraphs then includes all the MMSC subgraphs of G.*

Proof. Since (v_i, v_j) is a nonminimally strongly connected pair of nodes in G, in no MMSC subgraph of G, both of v_i and v_j can occur. Thus decomposing G into two subgraphs G_i and G_j such that G_i contains all the nodes of G except v_j and G_j contains all the nodes of G except v_i eliminates the possibility of joint occurrence of v_i and v_j in any subgraph. By iterative application of this process to all the resulting subgraphs of G until in the subgraphs there does not exist any nonminimally strongly connected pair of nodes, we obtain a set of subgraphs that include all the MMSC subgraphs of G.

For any two distinct nonminimally strongly connected pairs of nodes (v_{i1}, v_{j1}) and (v_{i2}, v_{j2}) in a symmetric graph G, let $\bar{d}(v_{i1}, v_{j1}) = (k_1, k_2)$ and $\bar{d}(v_{i2}, v_{j2}) = (r_1, r_2)$. If $k_1 > r_1$, $k_2 \geqslant r_2$, or $k_1 \geqslant r_1$, $k_2 > r_2$, then an ordering of the degree complements of the pairs of nodes can be made as $\bar{d}(v_{i1}, v_{j1}) \geqslant \bar{d}(v_{i2}, v_{j2})$. If both $k_1 = r_1$, $k_2 = r_2$, the ordering can be made either as $\bar{d}(v_{i1}, v_{j1}) \geqslant \bar{d}(v_{i2}, v_{j2})$ or as $\bar{d}(v_{i2}, v_{j2}) \geqslant \bar{d}(v_{i1}, v_{j1})$. However, when $k_1 > r_1$, $k_2 < r_2$, the ordering of the degree complements will depend on whether $k_1 - r_1$ is greater than, equal to, or less than $r_2 - k_2$ as follows: (1) when $k_1 - r_1 > r_2 - k_2$, then $\bar{d}(v_{i1}, v_{j1}) \geqslant \bar{d}(v_{i2}, v_{j2})$; (2) when $k_1 - r_1 = r_2 - k_2$, then either $\bar{d}(v_{i1}, v_{j1}) \geqslant \bar{d}(v_{i2}, v_{j2})$ or

$\bar{d}(v_{i2}, v_{j2}) \geqslant \bar{d}(v_{i1}, v_{j1})$; and (3) when $k_1 - r_1 < r_2 - k_2$, then $\bar{d}(v_{i2}, v_{j2}) \geqslant \bar{d}(v_{i1}, v_{j1})$. The ordering of the degree complements can be similarly made when $k_1 < r_1$, $k_2 > r_2$. This kind of ordering ($\geqslant$) that can be established among degree complements of different nonminimally strongly connected pairs of nodes in a symmetric graph is called *magnitude ordering* of degree complements of the pairs of nodes. Referring to graph G in Fig. 1(a), the magnitude ordering of the degree complements of its five nonminimally strongly connected pairs of nodes can be made as $\bar{d}(v_2, v_5) \geqslant \bar{d}(v_3, v_5) \geqslant \bar{d}(v_1, v_5) \geqslant \bar{d}(v_2, v_6) \geqslant \bar{d}(v_3, v_6)$.

THEOREM 6. *Let G be a symmetric graph, and let (v_i, v_j) be a nonminimally strongly connected pair of nodes of G having the highest degree complement in the magnitude ordering. If now G is split around (v_i, v_j) into two subgraphs G_i and G_j, then in the resulting subgraphs the number of nonminimally strongly connected pairs of nodes will always be less than that when G is split into subgraphs around any other nonminimally strongly connected pair having nonhighest degree complement in the magnitude ordering.*

Proof. Let the degree complement of the nonminimally strongly connected pair of nodes (v_i, v_j) be $\bar{d}(v_i, v_j) = (k_1, k_2)$. Let (v_s, v_t) be another nonminimally strongly connected pair of nodes of G for which the degree complement is $\bar{d}(v_s, v_t) = (r_1, r_2)$. Assume that $k_1 > r_1$, $k_2 > r_2$. In the subgraph G_i, there are $m_1 + k_1 - 1$ nonminimally strongly connected pairs of nodes, whereas in the subgraph G_j, there are $m_1 + k_2 - 1$ nonminimally strongly connected pairs of nodes, m_1 being the number of nonminimally strongly connected pairs of nodes in G, excluding the pairs with v_i, v_j. Let now G be split around (v_s, v_t) into two subgraphs G_s and G_t. Then the numbers of nonminimally strongly connected pairs of nodes in G_s and G_t are, respectively, $m_2 + r_1 - 1$ and $m_2 + r_2 - 1$, where m_2 is the number of nonminimally strongly connected pairs of nodes in G, excluding the pairs with v_s, v_t. But $m_1 = h - (k_1 + k_2 - 1)$ and $m_2 = h - (r_1 + r_2 - 1)$, where h is the total number of nonminimally strongly connected pairs of nodes in G. Then the number of nonminimally strongly connected pairs of nodes in G_i and G_j is $(m_1 + k_1 - 1) + (m_1 + k_2 - 1) = 2m_1 + (k_1 + k_2 - 2) = 2h - 2(k_1 + k_2 - 1) + (k_1 + k_2 - 2) = 2h - k_1 - k_2$. Similarly, the number of nonminimally strongly connected pairs of nodes in G_s and G_t is $(m_2 + r_1 - 1) + (m_2 + r_2 - 1) = 2h - r_1 - r_2$. Now $(2h - r_1 - r_2) - (2h - k_1 - k_2) = (k_1 - r_1) + (k_2 - r_2)$, and this is always positive if $k_1 > r_1$, $k_2 > r_2$. Thus the theorem holds in this case. The theorem similarly holds in other cases for which (v_i, v_j) may have the highest degree complement, viz. $k_1 > r_1$, $k_2 = r_2$, or $k_1 = r_1$, $k_2 > r_2$, or $k_1 > r_1$, $k_2 < r_2$, but $k_1 - r_1 > r_2 - k_2$; or $k_1 < r_1$, $k_2 > r_2$, but $r_1 - k_1 < k_2 - r_2$.

Theorem 6 develops results that tend to minimize computation in the generation of MMSC subgraphs. To avoid generating non-MMSC subgraphs, we use the following obvious theorem.

THEOREM 7. *In the process of successively decomposing a symmetric graph G into subgraphs around nonminimally strongly connected pairs of nodes, let G_i and G_j be any two subgraphs obtained at different stages such that $G_i \subseteq G_j$, but G_i is not derived from G_j. Then in finding only MMSC subgraphs, G_i may be discarded.*

A formal algorithm to find the MMSC subgraphs from a given symmetric graph is presented next.

ALGORITHM. (1) *Given a symmetric graph G, find the magnitude ordering of the degree complements of the nonminimally strongly connected pairs of nodes in G.* (2) *Select a nonminimally strongly connected pair of nodes (v_i, v_j) in G, having the highest degree complement in the magnitude ordering. If more than one pair has the highest degree complement, select any one of these pairs, (v_i, v_j). Decompose G around (v_i, v_j) into two subgraphs G_i and G_j such that G_i contains all the nodes of G except v_j and G_j contains all the nodes of G except v_i. Consider $G_i(G_j)$. Check if there exists a subgraph $G_k(G_m)$ from which $G_i(G_j)$ is not derived, contains $G_i(G_j)$.* (*a*) *If so, discard $G_i(G_j)$;* (*b*) *if not, take $G_i(G_j)$ and go to* (1). (3) *Continue with* (1) *and* (2) *until in all the resulting subgraphs there is no nonminimally strongly connected pair of nodes. The final set of subgraphs include all the MMSC subgraphs of G.*

The application of the algorithm to graph G in Fig. 1(a) results in the generation of three MMSC subgraphs, as shown in (e), (f), and (g).

REFERENCES

1. K. E. Stoffers, Scheduling of traffic lights—a new approach, *Transportation Res.* **2**, 199–234 (1968).
2. K. E. Stoffers, Sequential algorithm for the determination of maximum compatibles, *IEEE Trans. Computers* **C-23**, 95–98 (Jan. 1974).
3. J. Nieminen, On finding maximum compatibles, *Proc. IEEE* **63**, 729–730 (Apr. 1975).
4. S. R. Das and C. L. Sheng, On finding maximum compatibles, *Proc. IEEE* **57**, 694–695 (Apr. 1969).
5. S. R. Das, On a new approach for finding all the modified cut-sets in an incompatibility graph, *IEEE Trans. Computers* **C-22**, 187–193 (Feb. 1973).
6. C. V. Ramamoorthy, Analysis of graphs by connectivity considerations, *J. Assoc. Comput. Mach.* **13**, 211–222 (Apr. 1966).
7. M. C. Paull and S. H. Unger, Minimizing the number of states in incompletely specified sequential switching functions, *IRE Trans. Electron. Comput.* **EC-8**, 356–367 (Sept. 1959).
8. J. Weissman, Boolean algebra, map coloring, and interconnections, *Amer. Math. Monthly* **69**, 608–613 (Sept. 1962).
9. S. Seshu and M. B. Reed, *Linear Graphs and Electrical Networks*, Addison-Wesley, Reading, Mass., 1961.
10. F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass., 1969.

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