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Note on the optimum span of control in a pure hierarchy

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Abstract: This paper deals with a continuous model of a pure hierarchy. The aim of this quantitative model is to minimize costs, which are the sum of wage costs and costs caused by delays in decision making. It offers a more concrete discussion on the comparative statics of the optimum structure, and provides a liberal interpretation of the organizational phenomenon.

Keywords: Organization design, span of control, pure hierarchy, comparative statics

Introduction

In the design of organization structure, among the important factors to be considered are the quality, the speed, and the cost of the staff of job planning. However, the quality of planning jobs does not have satisfactory measurement tools; and the other two factors all have a direct relation with the span of control in an organization. Therefore the discussion on the optimum span of control has always been a major topic for the study of organizations (Beckmann, 1960; Mackenzie, 1974; Williamson, 1967).

Keren en Levhari (1979) try to explain both the existence of hierarchies and their structure by posting that they serve the need to reduce the planning time of the general manager. The organization aims at executing a given set of iterative calculations whose outcome is a concerted plan of actions. They assume that the planning time of each level in the hierarchy is linear with the span of control, and they use the sum of the planning time of the level to measure the speed of the planning. The objective is to minimize costs, which are the sum of wage costs and costs caused by delays in decision making.

Although Keren and Levhari (1979) discussed the topic of the hierarchy structure in general, they did not convince us sufficiently that the planning time in the span of control of the level varies linearly with the span of control. In normal situations, the increase of the span of control will complicate the task resulting in a prolongation of the planning time, but this rate of increase does not necessary have to be a constant (Levis, 1984; Miller, 1956; Shannon and Warren, 1949). In fact, many authors (Caplow, 1957; Costs and Updegraff, 1973; Pugh and Hickson, 1976; Spyrons and Demitris, 1982; Zey-Ferrel, 1979) in the field of organization theory and bureaucracy have suggested that communication interactions and coordinates and control problems increase with a rate faster than size. This means that if $f(s)$ is the planning time required by a group leader, with the span of control s to complete his own task, then f has the following properties:

$$f(s)>0, \quad f'(s)>0 \quad \text{and } f''(s)\geqslant 0.$$

In this note we replace the linear planning time function of Keren and Levhari by a general convex function f, and find how this affects the

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structure of the hierarchy. Section 2 introduces the general model. Section 3 derives the characteristics of the optimum structure. Section 4 focuses on the comparative statics of the optimum structure. The effect of changing the wage rate is presented in Section 4.1. The effect of changing the number of productive units is presented in Section 4.1. These properties (Proposition 4.1 and Proposition 4.2) are general results of Keren and Levhari (1979). The effect of changing the indirect cost function is presented in Section 4.3. The effect of changing the planning time function is presented in Section 4.4. These properties (Proposition 4.3 and Proposition 4.4) were ignored in the paper of Keren and Levhari.

2. The model of a hierarchy

We use a model of a pure hierarchy which a given number of productive units. This hierarchy consists of identical employees, links head and productive units, and transmits information from the productive units to the head and instructions from the head to the productive units. The wage rate per employee is denoted by w.

The span of control of all group leaders of a given level will be the same since membership of the hierarchy is homogeneous, and the task at each level are identical.

The above assumptions are mainly adopted from Keren and Levhari (1979).

To indicate the speed of the planning and the cost of staff, the following notations are used:

N = the number of productive units;
H = the number of hierarchy levels;
s_h = the span of control of level h;
x_h = the number of group leaders of level h; $x_1 = 1$ and $x_{H+1} = N$,
M $= x_1 + x_2 + \cdots + x_H$, the number of group leaders in the whole hierarchy;
$f(s_h)$ = the planning time required by a group leader of level h, to complete his own task, where

$$f(s) > 0, \quad f'(s) > 0 \quad \text{and} \quad f''(s) \geqslant 0; \tag{2.1}$$

T $= f(s_1) + f(s_1) + \cdots + f(s_H)$, the total planning time for the whole hierarchy;
$c(T)$ = the cost of profits lost per unit time through slow planning, where

$$c(0) = 0, \quad c'(T) > 0 \quad \text{and} \quad c''(T) \geqslant 0. \tag{2.2}$$

The following relation x_h and s_h holds:

$$x_{h+1} = x_h s_h = s_1 s_2 \cdots s_h, \quad 1 \leqslant h \leqslant H,$$

or

$$\ln x_{h+1} = \ln s_1 + \ln s_2 + \cdots + \ln s_h, \quad 1 \leqslant h \leqslant H. \tag{2.3}$$

If the measurement of x_h is based on the sum of working times of level h during a day, then x_h and hence s_h do not necessary have to be integers. In this model, x_h and s_h are considered as continuous variables.

The objective is to minimize the total cost L subject to (2.3), i.e.

$$\begin{aligned} \text{Min} \quad & L = c\left(\sum_{h=1}^{H} f(s_h)\right) + w\left(\sum_{h=1}^{H} x_h\right), \\ \text{s.t.} \quad & \ln x_h = \ln s_1 + \ln s_2 + \cdots + \ln s_{h-1}, \\ & 1 \leqslant h \leqslant H+1, \quad x_1 = 1, \quad x_{H+1} = N \end{aligned} \tag{2.4}$$

where w and N are parameters.

A continuous analogue to the discrete case of (2.3) will be

$$\ln x_h = \int_0^h \ln s_t \, dt, \quad \text{i.e.} \quad \frac{d}{dh} x_h = x_h \ln s_h. \tag{2.5}$$

Therefore, a continuous model of the pure hierarchy is given by

$$\begin{aligned} \text{Min} \quad & L = c\left(\int_0^H f(s_h) \, dh\right) + w\left(\int_0^H x_h \, dh\right), \\ \text{s.t.} \quad & \frac{d}{dh} x_h = x_h \ln s_h, \quad x_0 = 1, \quad x_H = N. \end{aligned} \tag{2.6}$$

3. The solution of the model

We proceed by two steps. First, keep $M = \int_0^H x \, dhn$ fixed and consider the following problem:

$$\text{Min} \quad T=\int_0^H f(s)\,\mathrm{d}h,$$
$$\text{s.t.} \quad \int_0^H x\,\mathrm{d}h = M, \tag{3.1}$$
$$x' = x \ln s, \quad x_0 = 1, \quad x_H = N.$$

The Hamiltonian (Kamien and Schwartz, 1981) of (3.1) is

$$\mathbb{H} = f(s_h) + x_h x_h \ln s_h + u x_h$$

where u is a constant and the necessary conditions of optimality are

$$\frac{\partial \mathbb{H}}{\partial s} = f'(s) + \frac{\lambda x}{s} = 0$$
$$\text{(optimality conditions)}, \tag{3.2}$$
$$-\lambda' = \frac{\partial \mathbb{H}}{\partial x} = u + \lambda \ln s$$
$$\text{(multiplier conditions)}, \tag{3.3}$$
$$f(s_H) + u x_H + \lambda_H x_H \ln s_H = 0$$
$$\text{(transversality condition)}, \tag{3.4}$$
$$x' = x \ln s \quad \text{(equation of motion)}. \tag{3.5}$$

Differentiating (3.2) with respect to h and using (3.3) and (3.5) yields

$$\frac{\mathrm{d}}{\mathrm{d}h}(sf'(s)) - ux = 0, \tag{3.6}$$
$$\frac{\mathrm{d}}{\mathrm{d}s}(f'(s)s \ln s)$$
$$= \ln s \frac{\mathrm{d}}{\mathrm{d}s}(sf'(s)) + f'(s)s\frac{\mathrm{d}}{\mathrm{d}s}\ln s$$
$$\text{(by (3.5) and (3.6))}$$
$$= u\frac{\mathrm{d}x}{\mathrm{d}s} + f'(s). \tag{3.7}$$

Together with (3.2) and (3.4), this leads to

$$u x_H = f'(s_H) s_H \ln s_H - f(s_H). \tag{3.8}$$

Integrating (3.7) with respect to s and using (3.8) yields

$$ux = f'(s)s \ln s - f(s). \tag{3.9}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}s}(f'(s)s \ln s - f(s))$$
$$= f''(s)s \ln s + f'(s) \ln s > 0.$$

Now, consider the case where M is variable. In (3.9), if u increases then so does the value of s corresponding to x; this leads to an increase of the value of $x' = x \ln s$ for each x, and hence to decrease of the value of $M = \int_0^H x\,\mathrm{d}h$. This implies that

$$\frac{\mathrm{d}u}{\mathrm{d}M} < 0. \tag{3.10}$$

By the interpretation of the multiplier u (Kamien and Schwartz, 1981, p. 115),

$$u = -T'(M) > 0, \tag{3.11}$$
$$\frac{\mathrm{d}u}{\mathrm{d}M} = -T''(M) < 0. \tag{3.12}$$

Substituting $T = T(M)$ in (2.6) and differentiating L with respect to M yields

$$\frac{\mathrm{d}L}{\mathrm{d}M} = c'(T)T'(M) + w = 0. \tag{3.13}$$

Let M^* be the number of group leaders in the optimum structure and T^* be the total planning time of optimality; then (3.11) and (3.12) imply

$$u^* = -T'(M^*) = \frac{w}{c'(T^*)} > 0. \tag{3.14}$$

(3.12) yields that $u = -T'(M)$, the marginal time saving of an additional group leader, is decreasing in M. (3.11) yields that $y = w/(c'(T(M))$, the marginal costs in terms of time of an additional group leader, is increasing in M (c.f. Figure 1). (3.9) and (3.14) yield a relation between x and $s = s(x)$:

$$u^* x = f'(s(x))s(x) \ln s(x) - f(s(x)) \tag{3.15}$$

where $s(1) = s_0$ and $s(N) = s_H$. Differentiating (3.15) with respect to h and using (2.5) yields

$$u^* x = (f''(s)s + f'(s))s' > 0. \tag{3.16}$$

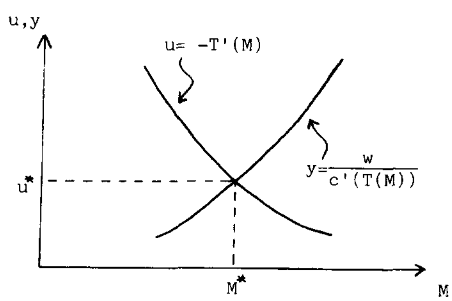

Figure 1. Relations between M^*, T^* and u^*

Proposition 3.1. *In the optimum structure, let s_h be the span of control of level h and let x_h be the number of group leaders of level h; then*

(1) $$\frac{\mathrm{d}s}{\mathrm{d}h}=\frac{f'(s)s\ \ln\ s-f(s)}{f''(s)s+f'(s)}>0,$$

(2) $$\frac{\mathrm{d}^2x}{\mathrm{d}h^2}=\frac{x}{s}\left(\frac{f'(s)s\ \ln\ s-f(s)}{f''(s)s+f'(s)}\right)+x(\ln\ s)^2>0.$$

4. Comparative statics

Given the parameters w, N and the functions c, f, the optimum structure of the hierarchy is determined by (3.15). Using this equation, the results of the comparative statics with respect to w, N, c and f are presented.

4.1. The effect of changing the wage rate w

A computation of $\partial u^*/\partial w$ in (3.15) yields the following properties. We defer its proof in Appendix A.

Proposition 4.1. *Keep N, c and f fixed, and consider w as a variable; then w and the key variables of the optimum structure have the following relations:*

(1) $\mathrm{d}u^*/\mathrm{d}w>0$;
(2) $\mathrm{d}T^*/\mathrm{d}w>0$;
(3) $\mathrm{d}M^*/\mathrm{d}w<0$;
(4) $\mathrm{d}H^*/\mathrm{d}w<0$;
(5) $\mathrm{d}s(1)/\mathrm{d}w>0$;
(6) $\mathrm{d}s(N)/\mathrm{d}w>0$;
(7) $\mathrm{d}L^*/\mathrm{d}w=M^*$;
(8) *For each h,* $\mathrm{d}s_h/\mathrm{d}w>0$ *and* $\mathrm{d}x_h/\mathrm{d}w>0$.

These results give many interesting properties. One of them is that if w increases then so do all values of u^*, T^*, L^*, $s(1)$ and $s(N)$, while the values of M^* and H^* decrease. Further results are obtained by comparing the magnitudes of these partial derivatives (c.f. Appendix A). For example, a simple computation yields that $(\mathrm{d}M^*/\mathrm{d}w)/(\mathrm{d}H^*/\mathrm{d}w)$ is greater than N if $M^*\ \ln\ s(1)>N-1$, and less than N if $M^*\ \ln\ s(1)<N-1$. The sign of $\mathrm{d}s(1)/\mathrm{d}w-\mathrm{d}s(N)/\mathrm{d}w$ is indefinite. However, if f is linear then its sign is negative. Another characteristic is that the marginal total cost is equal to the number of group leaders in the whole hierarchy.

4.2. The effect of changing the number of productive units N

A computation of $\partial u^*/\partial N$ in (3.15) yields the following properties. We defer its proof in Appendix B.

Proposition 4.2. *Keep w, c and f fixed, and consider N as a variable, then N and the key variables of the optimum structure have the following relations:*

(1) $\mathrm{d}u^*/\mathrm{d}N<0$;
(2) $\mathrm{d}T^*/\mathrm{d}N>0$;
(3) $\mathrm{d}M^*/\mathrm{d}N>0$;
(4) $\mathrm{d}H^*/\mathrm{d}N>0$;
(5) $\mathrm{d}s(1)/\mathrm{d}N<0$;
(6) $\mathrm{d}L^*/\mathrm{d}N>0$;
(7) *For each h,* $\mathrm{d}s_h/\mathrm{d}N<0$ *and* $\mathrm{d}x_h/\mathrm{d}N<0$.

This implies that if N increases then so do all values of T^*, M^*, H^* and L^*, while the values of u^* and $s(1)$ decrease. Further results could be obtained by comparing the magnitudes of these partial derivatives (c.f. Appendix B). For example, if c is a linear function, then

(i) u^*, s_h are independent of N, and $\mathrm{d}s(N)/\mathrm{d}N>0$;

(ii) The ratio between the increments of M^* and H^*, is equal to N (i.e. $\mathrm{d}M^*/\mathrm{d}N=N\,\mathrm{d}H^*/\mathrm{d}N$).

4.3. The effect of changing the indirect cost function c

Suppose that the indirect cost function c is changed into another function, say c_1, and suppose this

$$c'(T)\leqslant c_1'(T)\quad \text{for all } T. \tag{4.1}$$

The curve $T=T(M)$, determined by problem (3.1), is independent of the indirect cost function. Property (4.1) states that the curve $y=w/c_1'(T(M))$ is below the curve $y=w/c'(T(M))$. See Figure 2.

Proposition 4.3. *When the marginal indirect cost increase (shifting the marginal indirect cost curve*

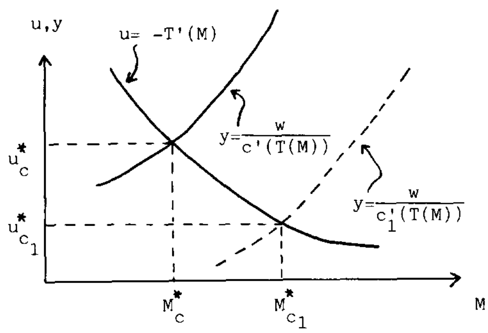

Figure 2. The effect of increasing the marginal indirect cost

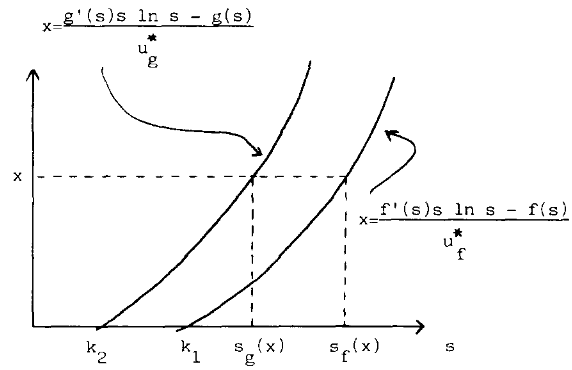

Figure 3. Relation between two graphs of x with different planning time functions

upward), the optimum structure will change as follows:

(1) *For each h, s_h and x_h decrease;*

(2) *u^*, T^*, H^*, $s(1)$, $s(N)$ decrease, while M^* and H^* increase.*

4.4. The effect of changing the planning time function f

The exact form of the planning time function f is difficult to identify because it usually depends on the complexity of organization tasks. Let g be the planning time function induced by increasing the amount of complexity. The assumption we made here is

$$f(s) < g(s), \qquad \frac{f'(s)}{f(s)} < \frac{g'(s)}{g(s)} \tag{4.2}$$

(i.e. the elasticity of $f \leqslant$ the elasticity of g).

Property (4.2) yields that

$$f'(s)s \ln s - f(s) < g'(s)s \ln s - g(s) \quad \text{for all } s. \tag{4.3}$$

Therefore, the value of s corresponding to x in (3.9) is greater than the value of s corresponding to x in the equation: $ux = g'(s)s \ln s - g(s)$. By the same argument as in (3.10) the curve $u = -T_f'(M)$ lies to the left of the curve $u = -T_g'(M)$. It is valid that $T_f(M) < T_g(M)$ for all M (c.f. problem (3.1), and hence the curve $y = w/c'(T_g(M))$ is below the curve $y = w/c'(T_f(M))$. This implies that $M_f^* < M_g^*$ (c.f. Figure 1). Property (4.3) yields that the number k_1, determined by $f(k_1)k_1 \ln k_1 - f(k_1) = 0$, is greater than k_2, determined by $g(k_2)k_2 \ln k_2 - g(k_2) = 0$. By the same argument as in (3.10), a relation between the graph of

$$x = \frac{f'(s)s \ln s - f(s)}{u_f^*}$$

and the graph of

$$x = \frac{g'(s)s \ln s - g(s)}{u_g^*}$$

could be obtained as shown in Figure 3.

Proposition. *When the complexity of organization tasks increases (increasing the elasticity of planning time), the optimum structure will change as follows:*

(1) *For each h, s_h and x_h decrease;*

(2) *M^*, H^* increase, while $s(1)$ and $s(N)$ decrease.*

5. Conclusions

The general results show that the (optimum) spans of control increase as one goes down the levels of the hierarchy. However when time saving become more and wage costs less important, the increment in the span of control from one level to the next will diminish; in the limit, when wage costs are not considered at all (i.e. $u^* = 0$), and they become equal (c.f. (3.15)). These results support the findings of some empirical studies (Starbuck, 1979, p. 88).

In the optimum structure of the Keren and Levhari model, the ratio between the variation of the span of control of the level and the variation of level is increasing in level; but this is not always true in the general model (c.f. Proposition 3.1).

We provided a more concrete discussion on the comparative statics. The partial derivatives of key variables with respect to the wage rate, and with respect to the number of productive units were formulated. Further results in the comparative statics could be obtained by comparing the magnitude of these partial derivatives. In the optimum structure of the Keren and Levhari model, the variation of the top level span of control, with respect to wage rate, is greater than that of the bottom level span of control; but this is not always true in the general model (c.f. Appendix A).

Regarding the question whether the span of control of the top level increases or decreases when the organization size increases, Starbuck (1979, p. 94) and Keren and Levhari (1979) obtained different results. This question can be explained from our model as follows: if the increase in organization size (the number of productive units) does not cause change in variables such as wage rate, planning time function, then the top level span of control decreases (c.f. Proposition 4.2). If the increase in organization size results in the increase in complexity of the planning job, and thus causes change in the planning time function, then the span of control of the top level have an uncertain direction of change (c.f. Proposition 4.2 and Proposition 4.4). If the increase in organization size leads to the demand of labor and thus causes change in wage rate, then the direction of change in the top level span of control is still uncertain (c.f. Proposition 4.1 and Proposition 4.2).

Appendix A. Proof of Proposition 4.1

Let $s=s(w)$ and $s_\delta=s(w+\delta)$ be the optimum span of control corresponding to w and $w+\delta$ respectively. Let $u=u^*(w)$ and $u_\delta=u^*(w+\delta)$ be the value of u^* corresponding to w and $w+\delta$ respectively. Then, by (3.15),

$$ux=f'(s(x))s(x)\ln(s(x))-f(s(x)),$$
$$u_\delta x=f'(s_\delta(x))s_\delta(x))\ln(s_\delta(x))-f(s_\delta(x)).$$

Together with these equations, this leads to

$$s\left(u^{-1}u_\delta x\right)=s_\delta(x)\quad \text{for all } x. \tag{A.1}$$

(1). Using $x'=x\ln s$ and changing a variable yields

$$T^*=\int_0^{H^*}f(s)\,\mathrm{d}h=\int_1^N f(s(x))\left(x\ln(s(x))\right)^{-1}\mathrm{d}x.$$

This implies that

$$\begin{aligned}
\frac{\mathrm{d}T^*}{\mathrm{d}w}&=\lim_{\delta\to0}\frac{1}{\delta}\left[\int_1^N(x\ln(s_\delta(x)))^{-1}f(s_\delta(x))\,\mathrm{d}x-\int_1^N(x\ln(s(x)))^{-1}f(s(x))\,\mathrm{d}x\right]\\
&=\lim_{\delta\to0}\frac{1}{\delta}\left[\int_{u^{-1}u_\delta}^{u^{-1}u_\delta N}(x\ln(s(x)))^{-1}f(s(x))\,\mathrm{d}x-\int_1^N(x\ln(s(x)))^{-1}f(s(x))\,\mathrm{d}x\right]\\
&=\frac{1}{u}\left[\frac{f(s(N))}{\ln(s(N))}-\frac{f(s(1))}{\ln(s(1))}\right]\frac{\mathrm{d}u^*}{\mathrm{d}w}\quad\text{(by (3.15))}\\
&=\left[\left(\frac{f'(s(N))s(N)}{u}-\frac{N}{\ln(s(N))}\right)-\left(\frac{f'(s(1))s(1)}{u}-\frac{1}{\ln(s(1))}\right)\right]\frac{\mathrm{d}u^*}{\mathrm{d}w}\quad\text{(by (3.6))}\\
&=\left(M^*-\frac{N}{\ln(s(N))}+\frac{1}{\ln(s(1))}\right)\frac{\mathrm{d}u^*}{\mathrm{d}w}.
\end{aligned}\tag{A.2}$$

Differentiate $u^* = (c'(T^*))^{-1}w$ with respect to w:

$$\frac{\mathrm{d}T^*}{\mathrm{d}w} = (wc''(T^*))^{-1}\left[c'(T^*) - (c'(T^*))^2 \frac{\mathrm{d}u^*}{\mathrm{d}w}\right]. \tag{A.3}$$

Together with (A.3) and (A.2), this leads to

$$\frac{\mathrm{d}u^*}{\mathrm{d}w} = \left[\left(M^* - \frac{N}{\ln(s(N))} + \frac{1}{\ln(s(1))}\right)\frac{wc''(T^*)}{c'(T^*)} + c'(T^*)\right]^{-1}.$$

(2) This is a result of (A.2).

(3)
$$\frac{\mathrm{d}M^*}{\mathrm{d}w} = \lim_{\delta\to 0}\frac{1}{\delta}\left[\int_0^{H_\delta^*} x_{\delta h}\,\mathrm{d}h - \int_0^{H^*} x_h\,\mathrm{d}h\right] \quad (\text{using } x' = x \ln s)$$
$$= \lim_{\delta\to 0}\frac{1}{\delta}\left[\int_1^N (\ln(s_\delta(x)))^{-1}\,\mathrm{d}x - \int_1^N ((\ln s(x)))^{-1}\,\mathrm{d}x\right] \quad (\text{using (A.1)})$$
$$= \lim_{\delta\to 0}\frac{1}{\delta}\left[u_\delta^{-1}u\int_{u^{-1}u_\delta}^{u^{-1}u_\delta N} (\ln(s(x)))^{-1}\,\mathrm{d}x - \int_1^N (\ln(s(x)))^{-1}\,\mathrm{d}x\right]$$
$$= \frac{-c'(T^*)}{w}\left(M^* - \frac{N}{\ln(s(N))} + \frac{1}{\ln(s(1))}\right)\frac{\mathrm{d}u^*}{\mathrm{d}w}$$

(4) Let $h = h(x)$, $h_\delta = h_\delta(x)$ be the inverse function of $x = x_h$ and $x = x_{\delta h}$ respectively; then

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}h}\right)^{-1} = (x\ln(s(x)))^{-1}, \qquad \frac{\mathrm{d}h_\delta}{\mathrm{d}x} = \left(\frac{\mathrm{d}x_{\delta h}}{\mathrm{d}h}\right)^{-1} = (x\ln(s_\delta(x)))^{-1},$$
$$H^* = h(N) = \int_1^N (x\ln(s(x)))^{-1}\,\mathrm{d}x, \qquad H_\delta^* = h_\delta(N) = \int_1^N (x\ln(s_\delta(x)))^{-1}\,\mathrm{d}x.$$

This implies that

$$\frac{\mathrm{d}H^*}{\mathrm{d}w} = \lim_{\delta\to 0}\frac{1}{\delta}\left[\int_1^N \left(x\ln\left(s\left(u^{-1}u_\delta x\right)\right)\right)^{-1}\mathrm{d}x - \int_1^N (x\ln(s(x)))^{-1}\,\mathrm{d}x\right]$$
$$= \lim_{\delta\to 0}\frac{1}{\delta}\left[\int_{u^{-1}u_\delta}^{u^{-1}u_\delta N} (x\ln(s(x)))^{-1}\,\mathrm{d}x - \int_1^N (x\ln(s(x)))^{-1}\,\mathrm{d}x\right]$$
$$= \frac{c'(T^*)}{w}\left[\frac{1}{\ln(s(N))} - \frac{1}{\ln(s(1))}\right]\frac{\mathrm{d}u^*}{\mathrm{d}w}.$$

(5) By differentiating $u^* = f'(s(1))\ln(s(1)) - f(s(1))$ with respect to w.
(6) By differentiating $u^*N = f'(s(N))\ln(s(N)) - f(s(N))$ with respect to w.
(7) By differentiating $L^* = c(T) = wM^*$ with respect to w and using properties (2) and (3).
(8) By the same argument as (3.10).

Appendix B. Proof of Proposition 4.2

Let $s = s(N)$ and $s_\delta = s(N+\delta)$ be the optimum span of control corresponding to N and $N+\delta$ respectively. Let $u = u^*(N)$ and $u_\delta = u^*(N+\delta)$ be the value of u^* corresponding to N and $N+\delta$ respectively. (3.15) yields that

$$s\left(u^{-1}u_\delta x\right) = s_\delta(x) \quad \text{for all } x. \tag{B.1}$$

(1) Using $x' = x\ \ln(s(x))$ and changing a variable yields

$$\frac{\mathrm{d}T^*}{\mathrm{d}N} = \lim_{\delta \to 0} \frac{1}{\delta}\left[\int_1^{N+\delta} \frac{f(s_\delta(x))}{x\ \ln(s_\delta(x))}\mathrm{d}x - \int_1^{N} \frac{f(s(x))}{x\ \ln(s(x))}\mathrm{d}x\right] \quad \text{(using } (B.1))$$

$$= \frac{1}{u}\left[\frac{f(s(N))}{\ln(s(N))} - \frac{f(s(1))}{\ln(s(1))}\right]\frac{\mathrm{d}u^*}{\mathrm{d}N} + \frac{f(s(N))}{N\ \ln(s(N))} \quad \text{(by (3.6) and (3.15))}$$

$$= \left(M^* - \frac{N}{\ln(s(N))} + \frac{1}{\ln(s(1))}\right)\frac{\mathrm{d}u^*}{\mathrm{d}N} + \frac{f(s(N))}{N\ \ln(s(N))}. \tag{B.2}$$

Differentiate $u^* = (c'(T^*))^{-1}w$ with respect to N:

$$\frac{\mathrm{d}T^*}{\mathrm{d}N} = -(wc'(T^*))^{-1}(c'(T^*))^2\frac{\mathrm{d}u^*}{\mathrm{d}N}. \tag{B.3}$$

By (B.2) and (B.3), we have the desired result.

(2) This is a result of (B.3).

(3) By the same argument as for $\mathrm{d}u^*/\mathrm{d}w$.

(4) By the same argument as for $\mathrm{d}H^*/\mathrm{d}w$.

(5) By differentiating $u^* = f'(s(1))s(1)\ \ln(s(1)) - f(s(1))$ with respect to N.

(6) By differentiating $L^* = c(T^*) + wM^*$ with respect to N.

(7) By the same arguments as in (3.10).

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