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NOTE ON THE PRODUCTION FUNCTION FOR ORGANIZATIONS DOING CASE WORK

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This paper examines the decision-makers' contribution to an organization's output by constructing a production function. In this production function the employees' work time enters as inputs and the average delay of a case work in the organization hierarchy is considered as the output. The model is pondered as a multi-factor production function for the multi-level organizations. Such a production function has some properties which are like the common properties of the microeconomic production function, but it is non-homothetic.

Key words: Organization design; production function.

1. Introduction

In recent years, some authors (Kochen and Deutsch, 1974; Keren and Levhari, 1979; Beckmann, 1977, 1982; Tarng and Chen, 1988) have been using mathematical models to discuss the relationships among variables in an organizational structure. They observe the functions of an organization based on the concept of input–output, and use mathematical equations to discuss more concretely some problems in an organization.

Beckmann (1982) followed a similar production function structure as used in economics to describe the characteristics of the production function of an organization doing case work. He viewed employees in a hierarchical structure as factors of production (input); the expected time a case work spent in the hierarchy as a parameter; and the finished case work as a product (output). Normally, the output of a firm's production function is a controllable variable. This means that the control of the production level can be achieved through manipulation of the quantities of production factors used.

The production function proposed by Beckmann used 'cases handled' as output. The kind of organization he analyzed was the rate of arrival of case work, which can be controlled. However, some organizations, such as judicial courts, tax offices, post offices, banks, hospitals, etc., have to receive and handle the case whenever they come. They can only accept the imposed rate of arrival, and never have the

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right to select them. This indicates that the production function proposed by Beckmann is not completely appropriate for the organizations mentioned above, whose primary concern is 'output efficiency'. Hence, how to construct a production function concerned primarily with 'work efficiency', and the characteristics of the cost functions of such production functions have become widely discussed topics.

This paper develops an organization doing case work as a production system, and uses the relationship between the input and output of this production system to establish an organization production function. For inputs, employees at different levels of the hierarchy will be treated as different production factors, which means that an organization having H levels of employees will have H different production factors. At the same time, X_i, the total working hours rendered by all employees at the ith level within a unit time, is treated as the used quantity of the ith production factor. For outputs, we assume that there are H types of work to be handled by the organization. Of these, the first type of work can be viewed as products of a single-level production process, which means that the work will be considered finished and will leave the organization once it is served by a first-level employee. The second type of work can be viewed as products of a dual-level production process. This means that the work is first handled by a first-level employee and then passed on to a second-level employee to finish. Normally, the ith type of work can be viewed as a product of an i-level production process, and is finished once it is served by an ith-level employee. If θ_i, $\sum_{i=1}^{H} \theta_i = 1$, indicates the percentage of the ith type of work and $T_i = T_i(X_1, X_2, \ldots, X_H)$ is regarded as the expected time an ith type work spends in the hierarchy, then $T = \sum_{i=1}^{H} \theta_i T_i$ indicates the expected time a work spends in the hierarchy. In order to compare the production efficiencies among different organizations and to have a common basis for comparison, we will adopt variable S,

$$S = T(\infty, \infty, \ldots, \infty)/T(X_1, X_2, \ldots, X_H),$$

to indicate the production quantity of an organization. Then we know $0 \le S \le 1$.

The mathematical model of the production function S for an organization doing case work can help decision-makers to understand the relationship between the inputs $(X_1, X_2, \ldots, X_H)$ and the output S. Hence, we can control the effect and the efficiency of this organization at the same time.

What are the characteristics of an organizational production function? Does it have similar characteristics as the production functions discussed in economics? These are the two primary topics to be discussed in this paper.

The production function mentioned above can only be used to describe the relationship between the input and output of a single-attribute work. If an organization has to handle m numbers of work with different attributes, and assuming that $S_k = S_k(X_1, X_2, \ldots, X_{H_k})$ indicates the production function of the kth-attribute work, then we can use m different production functions, $S_1, S_2, \ldots, S_m$, to describe the function of an organization. Using production functions $S_1, S_2, \ldots, S_m$, and their corresponding cost functions $C_1, C_2, \ldots, C_m$, we can establish the optimal structure of the hierarchy. These results can be used as references for an organization designer.

2. Model

This is a model of the production function for organizations doing case work. Its aim is to identify the characteristics of such a production function, and compare them with the well-known properties of the firm's production function.

The organization populates identical employees in a given level of the hierarchy; and each employee has a well-defined area of competence, determined by technical criteria which are easy to verify. We consider the first level of the hierarchy as a queueing system (regarding the case works and the first-level employees as customers and servers, respectively); and assume that this queueing system has a Poisson input process with a rate λ.

All arriving ith-level cases, including the ith type to the Hth type, should be assigned to the ith-level employees to complete the ith-level work without any delay. Cases from the $(i+1)$th type to the Hth type are moved to the next level. The expected number of cases received by any ith-level's employee per hour is $P_i\lambda/X_i$, where $P_i=\sum_{k=i}^{H}\theta_k$ and $1=P_1>P_2>\ldots>P_H$.

Assume that the completion rate at which an ith-level employee handles cases is μ_i and the completion time is exponentially distributed with mean $1/\mu_i$. An elementary result of queueing theory (Gross and Harris, 1974, pp. 40–64) yields that the expected time of a case work spent at the ith level is:

$$\frac{1}{\mu_i-P_i\lambda/X_i},$$

if $\mu_i>P_i\lambda/X_i$.

And hence T_k, the expected time a kth-type work spends in the hierarchy, is:

$$T_k=\sum_{i=1}^{k}\frac{1}{\mu_i-P_i\lambda/X_i}, \tag{1}$$

if $\mu_i>P_i\lambda/X_i$, for all $i=1,2,\ldots,k$.

This implies T, the expected time a case work spends in the hierarchy, is:

$$\begin{aligned}
T&=\sum_{k=1}^{H}\theta_k T_k, \quad \text{by using (1)}\\
&=\sum_{k=1}^{H}\theta_k\left(\sum_{i=1}^{k}\frac{1}{\mu_i-P_i\lambda/X_i}\right)\\
&=\sum_{i=1}^{H}\left(\sum_{j=1}^{H}\theta_j\right)\left(\frac{1}{\mu_i-P_i\lambda/X_i}\right)\\
&=\sum_{i=1}^{H}\frac{1}{\mu_i/P_i-\lambda/X_i};
\end{aligned}\tag{2}$$

if $\mu_i > P_i\lambda/X_i$, for all $i = 1, 2, \ldots, H$.

It is valid that T has a lower bound, $\bar{T}$, where

$$\bar{T} = \lim_{(X_1,\ldots,X_H)\to(\infty,\ldots,\infty)} T(X_1, X_2, \ldots, X_H)$$
$$= \sum_{i=1}^{H} P_i/\mu_i. \tag{3}$$

Equation (2) yields that T not only depends on the controllable variables, $X_1, X_2, \ldots, X_H$, but also depends on the uncontrollable variables, $\lambda, \mu_1, \ldots, \mu_H$, and $P_1, \ldots, P_H$. Since the measurement of the efficiency of an organization should be limited to its controllable variables, we therefore adopt the variable S, $S = \bar{T}/T$, to represent the service quality of an organization doing case work.

Now equations (2) and (3) define a production function for output S as a function of inputs X_i, the working hours at various levels $i = 1, 2, \ldots, H$:

$$S = \sum_{i=1}^{H} \frac{P_i}{\mu_i} \Big/ \sum_{i=1}^{H} \frac{1}{\mu_i/P_i - \lambda/X_i}. \tag{4}$$

An illustration of the case presented here is the examination of a company's expenditure. The executive department should apply funds to the examination department and have their approval. In general, the larger the amount of money it examines, the more data and complex procedure it needs. Hence, the cases of expenditure investigation must be divided into many types. When the budget case work proceeds, we find that different levels of employees have different kinds of emphases and limitations to examine and consider.

Assume that the budget cases can be divided into four types: (1) under 2000 dollars; (2) between 2000 to 9999 dollars; (3) between 10 000 to 99 999 dollars; and (4) more than 100 000 dollars.

The model parameters could be estimated as follows: $\lambda = 8$ cases/hour, $\mu_1 = 6$ cases/hour, $\mu_2 = 5$ cases/hour, $\mu_3 = 4$ cases/hour, $\mu_4 = 2$ cases/hour, and $\theta_1 = 2/8$, $\theta_2 = 3/8$, $\theta_3 = 2/8$, $\theta_4 = 1/8$; then $P_1 = \theta_1 + \theta_2 + \theta_3 + \theta_4 = 1$, $P_2 = \theta_2 + \theta_3 + \theta_4 = 6/8$, $P_3 = \theta_3 + \theta_4 = 3/8$, and $P_4 = \theta_4 = 1/8$.

Because the model must satisfy the rule $\mu_i > P_i\lambda/X_i$, for all $1 \le i \le 4$, therefore $X_1 > 8/6$ hours, $X_2 > 6/5$ hours, $X_3 > 3/4$ hour, and $X_4 > 1/2$ hour. The results are presented in Table 1.

Table 1

Input				Output
X_1	X_2	X_3	X_4	S
1.4	1.3	0.8	0.6	0.0646
2	2	1	1	0.3439
20	15	10	5	0.9229
200	150	100	50	0.9923

3. Properties of the production function

A firm's production $f(X_1, X_2, \ldots, X_n)$ is said to be homogeneous of degree k if $f(tX_1, tX_2, \ldots, tX_n) = t^k f(X_1, X_2, \ldots, X_n)$, for all $t \geq 0$. A production function that can be expressed as a monotonic increasing function of a homogeneous function is called homothetic (Henderson and Quandt, 1980, p. 106). We examine the production function $S = S(X_1, \ldots, X_H;\ \lambda, \mu_1, \ldots, \mu_H, P_1, \ldots, P_H)$ deriving from (4) to the firm's production function with the following properties:

(a) $S = 0$, if $X_i = 0$ for any i.
(b) S increases with every X_i.
(c) S is bounded with respect to any X_i.
(d) S is homogeneous of degree 0 in variables $(X_1, X_2, \ldots, X_H, \lambda)$, that is:

$$S(rX_1, \ldots, rX_H;\ r\lambda, \mu_1, \ldots, \mu_H, P_1, \ldots, P_H)$$
$$= S(X_1, \ldots, X_H;\ \lambda, \mu_1, \ldots, \mu_H, P_1, \ldots, P_H),$$

for all $r > 0$.

(e) S is strictly concave in variables $(X_1, X_2, \ldots, X_H)$.
(f) S is not homothetic in variables $(X_1, X_2, \ldots, X_H)$.

All except the last two properties are easy to prove by a simple computation from (4); the last two properties could be shown as below.

3.1. The proof of property (e)

Let $(X_1^{(0)}, X_2^{(0)}, \ldots, X_H^{(0)})$, $(X_1^{(1)}, X_2^{(1)}, \ldots, X_H^{(1)})$ be two distinct points of $R^{(H)}$, and let $g(X) = 1/X$, where X, $X_i^{(0)}$, $X_i^{(1)}$ are all positive numbers. Since $g''(X) = 2/X^3 > 0$, $g(X)$ is strictly convex on X, so for any $\alpha \in (0, 1)$ we have:

$$\begin{aligned}
&\alpha \frac{1}{\mu_i/P_i - \lambda/X_i^{(0)}} + (1-\alpha)\frac{1}{\mu_i/P_i - \lambda/X_i^{(1)}} \\
&= \alpha g\left(\frac{\mu_i}{P_i} - \frac{\lambda}{X_i^{(0)}}\right) + (1-\alpha) g\left(\frac{\mu_i}{P_i} - \frac{\lambda}{X_i^{(1)}}\right) \\
&> g\left(\alpha\left(\frac{\mu_i}{P_i} - \frac{\lambda}{X_i^{(0)}}\right) + (1-\alpha)\left(\frac{\mu_i}{P_i} - \frac{\lambda}{X_i^{(1)}}\right)\right) \\
&= 1\Big/\left\{\frac{\mu_i}{P_i} - \lambda\left[\alpha \frac{1}{X_i^{(0)}} + (1-\alpha)\frac{1}{X_i^{(1)}}\right]\right\} \\
&= 1\Big/\left\{\frac{\mu_i}{P_i} - \lambda[\alpha g(X_i^{(0)}) + (1-\alpha) g(X_i^{(1)})]\right\} \\
&> 1\Big/\left\{\frac{\mu_i}{P_i} - \lambda g(\alpha X_i^{(0)} + (1-\alpha) X_i^{(1)})\right\} \\
&= 1\Big/\left\{\frac{\mu_i}{P_i} - \lambda/[\alpha X_i^{(0)} + (1-\alpha) X_i^{(1)}]\right\}.
\end{aligned} \tag{5}$$

Note that $X_i^{(0)} \neq X_i^{(1)}$ for some $i \in \{1, 2, \ldots, H\}$.

Summing up (5) from $i=1$ to $i=H$ leads to:

$$\alpha \sum_{i=1}^{H} \frac{1}{\mu_i/P_i - \lambda/X_i^{(0)}} + (1-\alpha) \sum_{i=1}^{H} \frac{1}{\mu_i/P_i - \lambda/X_i^{(1)}}$$
$$> \sum_{i=1}^{H} \frac{1}{\mu_i/P_i - \lambda/(\alpha X_i^{(0)} + (1-\alpha) X_i^{(1)})},$$

and hence by (4) we have:

$$\alpha \frac{1}{S(X_1^{(0)}, \ldots, X_H^{(0)})} + (1-\alpha) \frac{1}{S(X_1^{(1)}, \ldots, X_H^{(1)})}$$
$$> \frac{1}{S(\alpha X_1^{(0)} + (1-\alpha) X_1^{(1)}, \ldots, \alpha X_H^{(0)} + (1-\alpha) X_H^{(1)})}.$$

This implies that $1/S(X_1, \ldots, X_H)$ is strictly convex in the variables $(X_1, \ldots, X_H)$ and hence, by the results of convex analysis (Rockafellar, 1970, p. 32), we know $S = S(X_1, \ldots, X_H)$ is strictly concave in the variables $(X_1, \ldots, X_H)$.

3.2. The proof of property (f)

If $S = S(X_1, \ldots, X_H)$ is homothetic on variables $(X_1, \ldots, X_H)$, then there exist monotonic increasing function $f(X)$ and a constant k such that the composite function $f(S(X_1, \ldots, X_H))$ is a homogeneous function of degree k, that is

$$f(S(tX_1, \ldots, tX_H)) = t^k f(S(X_1, \ldots, X_H)), \quad \text{for all } t > 0. \tag{6}$$

Computing the ratio between two partial derivatives of (6) yields:

$$\frac{f'(S(tX_1, \ldots, tX_H)) \cdot (\partial S(tX_1, \ldots, tX_H)/\partial X_i) \cdot t}{f'(S(tX_1, \ldots, tX_H)) \cdot (\partial S(tX_1, \ldots, tX_H)/\partial X_j) \cdot t}$$
$$= \frac{t^k f'(S(X_1, \ldots, X_H)) \cdot (\partial S(X_1, \ldots, X_H)/\partial X_i)}{t^k f'(S(X_1, \ldots, X_H)) \cdot (\partial S(X_1, \ldots, X_H)/\partial X_j)},$$

and hence:

$$\frac{(\partial S(tX_1, \ldots, tX_H)/\partial X_i)}{(\partial S(tX_1, \ldots, tX_H)/\partial X_j)} = \frac{(\partial S(X_1, \ldots, X_H)/\partial X_i)}{(\partial S(X_1, \ldots, X_H)/\partial X_j)}. \tag{7}$$

On the other hand, equation (4) yields:

$$\frac{(\partial S(tX_1, \ldots, tX_H)/\partial X_i)}{(\partial S(tX_1, \ldots, tX_H)/\partial X_j)} = \frac{X_j^2(\mu_j/P_j - \lambda/tX_j)^2}{X_i^2(\mu_i/P_i - \lambda/tX_i)^2}$$
$$\neq \frac{X_j^2(\mu_j/P_j - \lambda/X_j)^2}{X_i^2(\mu_i/P_i - \lambda/X_i)^2}$$
$$= \frac{(\partial S(X_1, \ldots, X_H)/\partial X_i)}{(\partial S(X_1, \ldots, X_H)/\partial X_j)}. \tag{8}$$

From (7) and (8) we get a contradiction, so the production function $S = S(X_1, \ldots, X_H)$ is not homothetic in variables $(X_1, \ldots, X_H)$, as asserted.

The above-mentioned non-homothetic production function S indicates that the output of S cannot respond to the proportionate increase of all its inputs, i.e. $S(tX_1, \ldots, tX_H) \neq t^k S(X_1, \ldots, X_H)$, for all $k \geq 0$. In other words, the production function does not show its increasing, decreasing and constant returns to scale.

4. Extension

Given an output level, the minimization of wage cost in a hierarchically structured organization can be studied as a cost function. By using the production function (4), the cost function of an organization doing case work could be obtained as follows.

Let W_i be the wage cost of an ith-level employee working per hour. Given an output level $s = S(X_1, \ldots, X_H)$, minimization of the total wage cost forms a cost function $C = C(s)$. That is:

$$C(s) = \min \sum_{i=1}^{H} W_i X_i$$

$$\text{subject to} \quad \frac{\sum_{i=1}^{H} P_i/\mu_i}{\sum_{i=1}^{H} 1/(\mu_i/P_i - \lambda/X_i)} = s.$$

Such a cost function $C(s)$ can be used to determine the most effective cost hierarchy structure. What are the characteristics of such a cost function? Does it have similar characteristics as the firm's cost functions discussed in economics? These are valuable problems in the design of a hierarchy structure.

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