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# On a modified binomial distribution of order $k$

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*Abstract*: We propose a modified model of the binomial distribution of order $k$ and obtain its probability generating function (pgf). We also extend the modified model and compute its pgf. The Poisson limit is also shown under some conditions.

*Keywords*: Binomial of order $k$, binomial of order $(k_1, k_2)$, binomial of order $(k_1, k_2; k_3)$, Poisson limit.

## 1. Introduction

Discrete distributions of order $k$ have recently attracted special attention. Since Philippou and Muwafi (1982) and Philippou et al. (1983) initiated some study in this direction, a series of papers on this subject have been published. Among them, Aki, Kuboki and Hirano (1984) calculated the probabilities and moments of various discrete distributions of order $k$ such as geometric, negative binomial, Poisson and logarithmic series distribution. In Hirano (1984) a binomial distribution of order $k$ was defined and a closed form of the exact distribution has been derived. This result was also independently given in Philippou and Makri (1986). Later, Aki (1985) considered some extensions of geometric, negative binomial, Poisson, logarithmic series and binomial of order $k$ on a binary sequence. More recently, Xekalaki, Panaretos and Philippou (1987) considered some mixtures of distributions of order $k$ and obtained some distributions of order $k$ such as the beta-geometric distribution and the beta-negative binomial distribution. The limiting distributions lead to the Poisson and gamma-compound Poisson distributions of order $k$. Philippou (1988) also considered some extensions of some discrete distributions of order $k$ to be multiparameter distributions of order $k$. Panaretos and Xekalaki (1986a) studied the cluster binomial distribution and its relation to generalized Poisson distributions.

On the other hand, recently, Panaretos and Xekalaki (1986b) and Xekalaki and Panaretos (1989) proposed an alternative approach to generalize the classic discrete distributions and provide interpretations of some important generalized discrete distributions via considering various urn sampling schemes. They generalized most of the classic discrete distributions and found relationships among them. When a class of generalized discrete distributions is set up via properly defining a certain urn sampling scheme, a wider class of distributions is obtained by extending the parameter space.

Consider an urn that contains $\alpha$ balls marked 0 and $\beta_i$ balls marked $i$ ($i = 1, 2, \ldots, l$). Each time a ball is randomly drawn from the urn and its number is recorded. Then, $\gamma$ balls (including the one just drawn) are added to the urn before the next ball is drawn. In a total time of $n$ (prefixed) samplings, let $X$ denote the number of runs of 0's with length $k$. For a prefixed event $E$, let $N$ denote the total number of drawings so that for the first time $E$ occurs (an inverse sampling). Panaretos and Xekalaki (1986) made a detailed study of the distributions of $X$ and $N$ under various conditions on $l$, $\alpha$, $\beta_i$, $\gamma$ and $E$. For instance, when $l = 1$, $\gamma \geqslant 1$, $X$ becomes a generalized Pólya variable. When $\gamma = 1$ and $E$ denotes the event that there are $m$ 0's, then $N$ becomes a generalized negative binomial variable. If again, the values of the involved rational parameters $\theta = (\alpha/\delta)^m$, $\theta_i = (\beta_i/\delta)^{x_i}$ with $\delta = \alpha + \Sigma\beta_i$ ($i = 1, 2, \ldots, l$) are extended to be any real values in (0, 1) with $\theta + \Sigma\theta_i = 1$, then, a wider class of generalized distributions is obtained which is defined as the cluster negative binomial (see Xekalaki and Panaretos (1989)).

According to this point of view, if we permit $\gamma$ to take negative integer values with some restrictions or allow $\gamma$ to be dependent on values of $\alpha_j$ and $\beta_{ij}$, where $\alpha_j$ and $\beta_{ij}$ denote, respectively, the values of $\alpha$ and $\beta_i$ at the $j$th sampling, we may obtain a wider class of generalized distributions.

In this article, we focus on the binomial model of order $k$ and consider a modified model of order, ($k_1$, $k_2$) which is defined in Section 2. It can be seen that this modified model includes the binomial model of order $k$ in Hirano (1984) as a special case of order $(0, k)$ or $(k, 0)$. This modification can be extended along the lines studied in Aki (1985) for a binary sequence associated with a sequence $\{p_1, p_2, \ldots\}$ with values in (0, 1). However, we extend the model in another direction in Section 3.

## 2. Binomial of order $(k_1, k_2)$

Consider a box with security installation. Usually there is a wheel attached to the box. Each time, the wheel either turns one step left (counter clockwise), or one step right (clockwise). If we consider turning left as a failure (F) and turning right as a success (S), then the random walk on a circle results in a binary sequence. The box can be opened if for the $m$th time $k_1$ consecutive F's followed by $k_2$ consecutive S's have been produced. Here $m$, $k_1$ and $k_2$ may be some unknown parameters needed to be guessed or determined. If $n$, $m$, $k_1 + k_2$, and $p$ are given, it may be interesting to ask what values of $k_1$ and $k_2$ make $\mathrm{B}_{k_1,k_2}(m;\ n,\ p)$ minimum.

For another example, in a biological context, consider a process of random mating. As a result, in chromosomes of each cell of an offspring, there exists a certain arrangement of genes. Suppose two categories of genotypes (A and B) are considered. A certain arrangement of genotypes, say $k_1$ A-genotype followed by $k_2$ B-genotype, reflects a certain characteristic, for example, a defect or symptom etc. Then, it may be interesting to evaluate the probability of such a characteristic.

Consider now the urn sampling scheme described in the previous section with $l = 1$, $\gamma = 0$ or $\gamma = 1$. Let $Y$ denote the total number of events which consists of exactly $k_1$ 0's followed by exactly $k_2$ 1's. Then, the limiting distribution of $Y$ becomes $N(n;\ k_1,\ k_2)$, which is defined in the following, when $\alpha \to \infty$ and $\alpha/(\alpha + \beta_1) \to p$ $(0 < p < 1)$. See Panaretos and Xekalaki (1986) for the case of a binomial of order $k$.

**Definition 1.** In a sequence of Bernoulli trials with probability $p$ (a success denoted by S and a failure by F), a $(k_1,\ k_2)$-event is said to have occurred if for the first time the event that $k_1$ consecutive F followed by $k_2$ consecutive S (i.e. ...FF...FSS...S...) has occurred in the trial, where $(k_1,\ k_2)$ is any pair of non-negative integers (including 0) excluding (0, 0).

For convenience, let $N(n;\ k_1,\ k_2)$ denote the number of occurrences of a $(k_1,\ k_2)$-event in $n$ trials. The distribution of $N(n;\ k_1,\ k_2)$ will be referred to as the binomial of order $(k_1,\ k_2)$ and will be denoted by $\mathrm{B}_{k_1,k_2}(\cdot;\ n,\ p)$. Clearly, it is seen that $\mathrm{B}_{0,k}(\cdot;\ n,\ p)$ or $\mathrm{B}_{k,0}(\cdot;\ n,\ 1\text{-}p)$ are identical to $\mathrm{B}_k(\cdot;\ n,\ p)$ in Hirano (1984) or Philippou and Makri (1986).

In the following, we derive the probability generating function of $N(n;\ k_1,\ k_2)$. Unfortunately, the method we use can not be applied for cases of $(k_1,\ k_2)$ with $k_1 = 0$ and $k_2 \geqslant 2$. The techniques used for deriving $\mathrm{B}_k(\cdot;\ n,\ p)$ given in Hirano (1984) or Philippou and Makri (1986) fail for our model.

**Lemma 1.**

(i)
$$\begin{aligned} \mathrm{B}_{k_1,k_2}(x;\ n,\ p) &= 0 && \textit{if } n < k_1 + k_2,\ x > 0, \\ &= 1 && \textit{if } n < k_1 + k_2,\ x = 0, \\ &= q^{k_1}p^{k_2} && \textit{if } n = k_1 + k_2,\ x = 1, \\ &= 1 - q^{k_1}p^{k_2} && \textit{if } n = k_1 + k_2,\ x = 0. \end{aligned}$$

(ii)
$$\mathrm{B}_{k_1,k_2}(0;\ n,\ p) = \mathrm{B}_{k_1,k_2}(0;\ n-1,\ p) - q^{k_1}p^{k_2}\mathrm{B}_{k_1,k_2}(0;\ n-k_1-k_2,\ p) \quad \textit{for } n \geqslant k_1 + k_2.$$

(iii)
$$\mathrm{B}_{k_1,k_2}(x;\ n,\ p) = \sum_{j=0}^{n-k_1-k_2} q^{k_1}p^{k_2}\mathrm{B}_{k_1,k_2}(x-1;\ j,\ p)\mathrm{B}_{k_1,k_2}(0;\ n-k_1-k_2-j) \quad \textit{for } n \geqslant k_1 + k_2,\ x \geqslant 1,\ q \equiv 1-p.$$

(iv)
$$\begin{aligned} \mathrm{B}_{k_1,k_2}(x;\ n+1,\ p) = \mathrm{B}_{k_1,k_2}(x;\ n,\ p) + q^{k_1}p^{k_2}\big[&\mathrm{B}_{k_1,k_2}(x-1;\ n-k_1-k_2+1,\ p) \\ &- \mathrm{B}_{k_1,k_2}(x;\ n-k_1-k_2+1,\ p)\big] \end{aligned}$$
$$\textit{for } n \geqslant k_1 + k_2,\ 1 \leqslant x \leqslant \left[\frac{n}{k_1+k_2}\right],$$

*where* $[a]$ *denotes the largest integer not exceeding* $a$.

**Proof.** (i) and (ii) are straightforward. To see (iii), consider a $(k_1,\ k_2)$-event that occurs between the $j$th and the $(j + k_1 + k_2 - 1)$th Bernoulli trials, and suppose that there are $x - 1$, $(k_1,\ k_2)$-events which have occurred before the $j$th trial and that no $(k_1,\ k_2)$-events occur after the $(j + k_1 + k_2 - 1)$th trial, where $j = 1, 2, \ldots, n - k_1 - k_2 + 1$. To see (iv), suppose there are $x$ $(k_1,\ k_2)$-events that occur in $n+1$ trials and consider the following situations.

*Case 1.* Suppose there are $x$ $(k_1,\ k_2)$-events in $n$ trials. Then, there are either $x$ $(k_1,\ k_2)$-events in $n$ trials or $x+1$ $(k_1,\ k_2)$-events in $n+1$ trials. The probability is then given by $\mathrm{B}_{k_1,k_2}(x;\ n,\ p) - q^{k_1}p^{k_2}\mathrm{B}_{k_1,k_2}(x;\ n-k_1-k_2+1,\ p)$.

*Case 2.* There are $x-1$ $(k_1,\ k_2)$-events in the first $n - k_1 - k_2 + 1$ trials and there are $x$ $(k_1,\ k_2)$-events in $n+1$ trials. This results in a probability $q^{k_1}p^{k_2}\mathrm{B}_{k_1,k_2}(x-1;\ n-k_1-k_2+1,\ p)$. □

Let $\Phi_n(t;\ k_1,\ k_2)$ denote the probability generating function (pgf) of $N(n;\ k_1,\ k_2)$. Then, we have the following main result.

**Theorem 1.**

$$\Phi_n(t;\ k_1,\ k_2) = \sum_{i=i_0}^{n} C_{\alpha_i}^{i}\left[p^{k_2}q^{k_1}(t-1)\right]^{i-\alpha_i}$$

*where* $i_0 = -[-n/(k_1+k_2)]$, $\alpha_i = ((k_1+k_2)i - n)/(k_1+k_2-1)$ *if the right hand side is an integer, and* $\alpha_i = -1$ *otherwise. Define* $C_{-1}^{a} \equiv 0$, *for any* $a$.

**Proof.** By Lemma 1, we have

$$\begin{aligned}\Phi_n(t;\ k_1,\ k_2) &= 1 && \text{if } 0 \leqslant n < k_1 + k_2,\\ &= 1 + q^{k_1}p^{k_2}(t-1) && \text{if } n = k_1 + k_2, \\ &= \Phi_{n-1}(t;\ k_1,\ k_2) + q^{k_1}p^{k_2}(t-1)\Phi_{n-k_1-k_2}(t;\ k_1,\ k_2) && \text{if } n > k_1 + k_2.\end{aligned} \tag{2.1}$$

Define

$$\Phi(z;\ t,\ k_1,\ k_2) = \sum_{n=0}^{\infty} \Phi_n(t;\ k_1,\ k_2) z^n. \tag{2.2}$$

It follows from (2.1) that

$$\Phi(z;\ t,\ k_1,\ k_2) = 1 + z\Phi(z;\ t,\ k_1,\ k_2) + (t-1)q^{k_1}p^{k_2}z^{k_1+k_2}\Phi(z;\ t,\ k_1,\ k_2).$$

Solving for $\Phi(z;\ t,\ k_1,\ k_2)$, we have

$$\Phi(z;\ t,\ k_1,\ k_2) = \frac{1}{1 - z - (t-1)q^{k_1}p^{k_2}z^{k_1+k_2}} \tag{2.3}$$

where $t$ and $z$ lie in the domain

$$D \equiv \left\{ |t| \leqslant 1,\ |z| < 1 \text{ and } |z + (t-1)q^{k_1}p^{k_2}z^{k_1+k_2}| < 1 \right\}.$$

Again, note that the right hand side of (2.3) is equal to

$$\sum_{n=0}^{\infty} \left\{ z + (t-1)q^{k_1}p^{k_2}z^{k_1+k_2} \right\}^n \quad \text{for } (z,\ t) \in D.$$

Collect all coefficients of $z^n$ in the polynomial expansion to obtain $\Phi_n(t;\ k_1,\ k_2)$. This completes the proof. □

Let $M_n(t;\ k_1,\ k_2)$ denote the factorial moment generating function of $N(t;\ k_1,\ k_2)$. Then we have, by Theorem 1,

$$M_n(t;\ k_1,\ k_2) = \Phi_n(t+1;\ k_1,\ k_2) = \sum_{i=i_0}^{n} C_{\alpha_i}^{i}\left(p^{k_2}q^{k_1}t\right)^{i-\alpha_i} \tag{2.4}$$

where $i_0$ and $\alpha_i$ are defined in Theorem 1. For given $i$, let $i - \alpha_i = m$, a positive integer. Then, the $m$th factorial moment of $N(n;\ k_1,\ k_2)$ is given by

$$\mu_{(m)} = C_{\alpha_i}^{i}\left(p^{k_2}q^{k_1}\right)^m m! = \left(n - m(k_1 + k_2 - 1)\right)^{(m)}\left(p^{k_2}q^{k_1}\right)^m,$$

noting that $i - \alpha_i = m$ implies $i = n - m(k_1 + k_2 - 1)$, where $a^{(m)} = a(a-1)\cdots(a-m+1)$. Note that $(n - m(k_1 + k_2 - 1))^{(m)}/n^m \to 1$ as $n \to \infty$. On the other hand, the $m$th factorial moment of the Poisson distribution with parameter $\lambda$ is given by $\lambda^m$. Also note that the Poisson distribution is uniquely determined by its moments. Instead of the moment of order $m$, we consider the $m$th factorial moment of $N(n;\ k_1,\ k_2)$. It follows then from Kendall (1967):

**Corollary 1.** *The distribution of* $\mathrm{B}_{k_1,k_2}(\cdot;\ n,\ p)$ *converges to the Poisson distribution with parameter* $\lambda$ *if* $n(1-p)^{k_1}p^{k_2} \to \lambda$ *as* $n \to \infty$ *and* $p \to 0$. □

Finally, we consider the waiting time of $N(n;\ k_1,\ k_2)$. Let $S_n$ denote the waiting time of the $n$th occurence of a $(k_1,\ k_2)$-event. Then

$$S_n = T_1 + T_2 + \cdots + T_n \tag{2.5}$$

where $T_i$ is $k_1 + k_2$ plus the number of trials between the $(i-1)$th and $i$th occurrence of a $(k_1,\ k_2)$-event. It can be seen that $T_1, T_2, \ldots, T_n$ are i.i.d. Let $T$ denote an i.i.d. copy and let $\phi(t,\ k_1,\ k_2)$ denote the pgf of $T$. Then, we have:

**Corollary 2.**

(i) $\phi(t;\ k_1,\ k_2) = \dfrac{q^{k_1}p^{k_2}t^{k_1+k_2}}{1 - t + q^{k_1}p^{k_2}t^{k_1+k_2}}.$

(ii) $ET = \left(q^{k_1}p^{k_2}\right)^{-1}.$

(iii) $\operatorname{Var} T = \left\{q^{k_1}p^{k_2}(1 - 2k_1 - 2k_2) + 1\right\}/\left(q^{k_1}p^{k_2}\right)^2.$

**Proof.** (ii) and (iii) follow directly from (i). To show (i), note that

$$\begin{aligned} P(T = n) &= 0 && \text{if } n < k_1 + k_2, \\ &= q^{k_1}p^{k_2} && \text{if } n = k_1 + k_2, \\ &= q^{k_1}p^{k_2}\mathrm{B}_{k_1,k_2}(0; n - k_1 - k_2,\ p) && \text{if } n > k_1 + k_2. \end{aligned}$$

It follows then

$$\begin{aligned} \phi(t;\ k_1,\ k_2) &= \sum_{n=k_1+k_2}^{\infty} q^{k_1}p^{k_2}\Phi_{n-k_1-k_2}(0;\ k_1,\ k_2)t^n \\ &= t^{k_1+k_2}q^{k_1}p^{k_2}\sum_{n=0}^{\infty} \Phi_n(0;\ k_1,\ k_2)t^n \\ &= t^{k_1+k_2}q^{k_1}p^{k_2}\Phi(t;\ 0,\ k_1,\ k_2) \quad (\text{by } (2.2)) \\ &= q^{k_1}p^{k_2}t^{k_1+k_2}/\left(1 - t + q^{k_1}p^{k_2}t^{k_1+k_2}\right) \quad (\text{by } (2.3)). \qquad \square \end{aligned}$$

## 3. An extension

If we consider a $(k_1,\ k_2)$-event as a unit and generalize the concept of a run to be an event which consists of exactly $k_3$ such units, then we can define the following event which is more general.

**Definition 2.** A $(k_1,\ k_2;\ k_3)$-event is said to have occurred if there are exactly $k_3$ consecutive $(k_1,\ k_2)$-events that occur in $n$ Bernoulli trials.

Obviously, a $(k_1,\ k_2;\ 1)$-event is actually a $(k_1,\ k_2)$-event as defined in Definition 1. Any event can be considered as a composition of a $(k_{i1},\ k_{i2};\ k_{i3})$-event followed by a $(k_{i+1,1},\ k_{i+1,2};\ k_{i+1,3})$-event corresponding to some sequence $\{(k_{i1},\ k_{i2},\ k_{i3}),\ i = 1, 2, \ldots, l\}$.

Let $N_{k3}(n;\ k_1,\ k_2)$ denote the number of occurrences of a $(k_1,\ k_2;\ k_3)$-event in $n$ Bernoulli trials. We note that

$$\begin{aligned} \Pr\left\{N_{k_3}(n;\ k_1,\ k_2) = r\right\} &= \Pr\left\{rk_3 \leqslant N(n;\ k_1,\ k_2) < (r+1)k_3\right\} \\ &= \Pr\left\{S_{rk_3} \leqslant n\right\} - \Pr\left\{S_{(r+1)k_3} \leqslant n\right\} \end{aligned}$$

where $S_n$ is the waiting time defined by (2.5). Let $p_{n,r} = \Pr\{N_{k_3}(n;\ k_1,\ k_2) = r\}$ for fixed $k_1$, $k_2$ and $k_3$. Then, it can be obtained that (see, for example, Feller (1968))

$$\sum_{n=0}^{\infty} p_{n,r} z^n = \phi^{rk_3}(z;\ k_1,\ k_2)\left[1 - \phi^{k_3}(z;\ k_1,\ k_2)\right]/(1-z) \tag{3.1}$$

where $\phi(\cdot)$ is the pgf of $T$ given by Corollary 2. Let $\xi_n(t)$ $(\equiv \Sigma_{r=0}^{\infty} p_{n,r} t^r)$ denote the pgf of $N_{k_3}(n;\ k_1,\ k_2)$ for fixed $k_1$, $k_2$ and $k_3$. Then,

$$\begin{aligned}
\sum_{n=0}^{\infty} \xi_n(t) z^n &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} p_{n,r} t^r z^n \\
&= \sum_{r=0}^{\infty} t^r \sum_{n=0}^{\infty} p_{n,r} z^n \\
&= \sum_{r=0}^{\infty} t^r \phi^{rk_3}(z;\ k_1,\ k_2)\left[1 - \phi^{k_3}(z;\ k_1,\ k_2)\right]/(1-z) \quad \text{(by (3.1))} \\
&= \left\{1 - \phi^{k_3}(z;\ k_1,\ k_2)\right\}/\left\{(1-z)\left[1 - t\phi^{k_3}(z;\ k_1,\ k_2)\right]\right\}.
\end{aligned}$$

We therefore conclude:

**Theorem 2.** *The pgf of $N_{k_3}(z;\ k_1,\ k_2)$ is the coefficient of $z^n$ in the polynomial expansion of the ratiional function*

$$\left\{1\text{-}\phi^{k_3}(z:\ k_1,\ k_2)\right\}/\left\{(1-z)\left[1 - t\phi^{k_3}(z;\ k_1,\ k_2)\right]\right\},$$

*where $\phi(\cdot)$ is given by Corollary 2.* □

**Corollary 3.** *The first and the second moment of $N_{k_3}(n;\ k_1,\ k_2)$ are, respectively, the coefficient of $z^n$ in the polynomial expansions of the following rational functions*:

$$\phi^{k_3}(z;\ k_1,\ k_2)/(1-z)\left[1 - \phi^{k_3}(z;\ k_1,\ k_2)\right]$$

*and*

$$\left[\phi^{k_3}(z;\ k_1,\ k_2) + \phi^{2k_3}(z;\ k_1,\ k_2)\right]/(1-z)\left[1 - \phi^{k_3}(z;\ k_1,\ k_2)\right]^2. \qquad \square$$

**Proof.** Applying analogous arguments as above, we can conclude the results.

Usually such a coefficient of $z^n$ can be obtained by differentiating the associated rational function with respect to $z$ and then taking $z = 0$.

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