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Estimation of the exponential mean under type I censorded sampling

Wen-Tao Huang

Institute of Statistical Science, Academia Sinica, Taipei, Taiwan, ROC

Huann-Sheng Chen

Institute of Statistical Science, Academia Sinica, Taipei, Taiwan, ROC

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Abstract: The main purpose of the present article is to propose a new estimator for the exponential mean whose bias and mean square error are both less than that of the MLE under type I censored sampling. Some other porperties of the proposed estimators are also studied. Comparisons of the proposed and the known estimators are also made for small sample size.

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1. Introduction

The exponential distribution has been widely used as a model in areas ranging from studies on industrial life testing to studies involving survival times in clinical trials. Extensive bibliography can be found in Johnson and Kotz (1970, Chapter 18), Mann et al. (1974), Lawless (1982), among others. For Bayes formulation, readers are referred to Martz and Waller (1982).

In many situations, the experiment is conducted over a fixed period of time, and hence the observed failure times may be truncated by the time limit of the experiment. Such data are said to be type I censored.

In this paper we consider the estimation of the exponential mean based on type I censored data. We propose an estimator of the mean which has less bias and mean square error (MSE) than that of the MLE which was studied by Bartholomew (1957), Epstein and Sobel (1954) and others.

Correspondence to: Dr. W.T. Huang, Institute of Statistical Science, Academia Sinica, Taipei 11529, Taiwan, ROC.

Let n denote the number of items being put on test at time zero. It is assumed that life length of each item on test has an exponential distribution with density

$$f(x) = (1/\theta)\exp(-x/\theta) \quad \text{for } 0<x<\infty,\ \theta>0. \tag{1.1}$$

For a fixed positive constant T, let $[0, T]$ denote the time interval of the experiment and N the number of failures observed in $[0, T]$. Let $X_{[1]} \le X_{[2]} \le \cdots \le X_{[N]}$ denote the ordered observed failure times. The likelihood for θ under type I censoring is

$$L(\theta) = (1/\theta)^N \exp\left\{(-1/\theta)\left[\sum_{j=1}^{N} X_{[j]} + (n-N)T\right]\right\}, \quad N \ge 0.$$

The MLE for θ is given by

$$\hat{\theta} = \left[\sum_{i=1}^{N} X_{[i]} + (n-N)T\right]/N, \quad \text{for } N>0. \tag{1.2}$$

Bartholomew (1957, 1963) has studied $\hat{\theta}$ and derived its sampling distribution. Later, Yang and Sirvanci (1977) conducted an important study on the behaviors of N, $\hat{\theta}$ and related quantities. Some large sample properties of these statistics were also studied. Sirvanci and Levent (1982) studied its numerical results.

We propose a new estimator $\tilde{\theta}$ (defined in Section 2). If not all items are censored, it has been shown that the bias of $\tilde{\theta}$ is of order $O(1/n^4)$ which is less than that of $\hat{\theta}$, the MLE, whose bias is of order $O(1/n)$. Hence, bias is computed under the condition $N>0$. Furthermore, we have shown that under condition $N \ge 1$, $n^2(\mathrm{MSE}(\hat{\theta}) - \mathrm{MSE}(\tilde{\theta})) = aT^2 + O(1/n^3)$ for some $a>0$. Properties of consistency, asymptotic unbiasedness and normality of $\tilde{\theta}$ are proven in Section 3. Comparisons of $\tilde{\theta}$ with $\hat{\theta}$ and another estimator θ^* (defined by (2.8)) in terms of bias and mean square error for respective sample sizes $n=10$ and $n=30$ are plotted in Figures 1 and 2. Some of these values are also tabulated in Tables 1 and 2.

2. A new alternative estimator

2.1. Bias and mean square error

Let E_c and P_c denote the conditional expectation and the probability calculated under the condition $N \ge 1$. For convenience, set $p = 1 - q = 1 - \exp(-T/\theta)$. The bias of $\hat{\theta}$ has been computed by Yang and Sirvanci (1977)

$$\mathrm{Bias}(\hat{\theta}) = E_c(\hat{\theta}) - \theta = -T/p + nTE_c(1/N). \tag{2.1}$$

From Lemma 1 below we have

$$E_c(1/N) = (1/np) + (q/n^2p^2) + O(1/n^3). \tag{2.2}$$

Hence,

$$\mathrm{Bias}(\hat{\theta}) = Tq/np^2 + O(1/n^2). \tag{2.3}$$

Note that

$$E_c\left[\sum_{i=1}^{N} X_{[i]}/N+T(p^{-1}-1)\right]=\theta. \tag{2.4}$$

This suggests using

$$\sum_{i=1}^{N} X_{[i]}/N+S(T,N) \tag{2.5}$$

to estimate θ, where $S(T,N)$ is an estimator for $(p^{-1}-1)$. The MLE $\hat{\theta}$ in (1.2) is to let

$$S(T,N)=T(nN^{-1}-1), \tag{2.6}$$

where N/n is the MLE of p. Alternatively, Huang and Wang (1989) considered

$$S(T,N)=-T+(n+1)T/N-nT/N^2 \tag{2.7}$$

for the estimation of $1/p$ against that of n/N. For this case, they obtained an alternative estimator for θ defined by

$$\theta^*=\left(\sum_{i=1}^{N} X_{[i]}/N\right)+(1-1/N)(n-N)T/N. \tag{2.8}$$

It has been shown that θ^* has less bias and MSE than that of $\hat{\theta}$ when n is large. Even when n is 10, its absolute bias and MSE of θ^* are still smaller when T is away from 0.5 (see Tables 1 and 2). In order to further reduce its bias, we replace the factor $1-1/N$ by $1+\alpha/N+\beta/N^2+\gamma/N^3$ in (2.8) and consider the following class of estimators which include both θ^* and $\hat{\theta}$

$$\theta(\alpha,\beta,\gamma)=\left(\sum_{i=1}^{N} X_{[i]}/N\right)+[(n-N)T/N](1+\alpha/N+\beta/N^2+\gamma/N^3). \tag{2.9}$$

It is then noted that $\theta(0,0,0)\equiv\hat{\theta}$ and $\theta(-1,0,0)\equiv\theta^*$ which are defined, respectively, by (1.2) and (2.8). Since the MSE of $\hat{\theta}$ includes the only term of θ^2/np of the smallest order of n, we cannot reduce the order of MSE of $\theta(\alpha,\beta,\gamma)$ by varying values of α,β and γ. Hence, we focus on the reduction of its bias. Through computation, it is found that the bias of $\theta(\alpha,\beta,\gamma)$ attains its minimum order of n when $\alpha=\gamma=-1$ and $\beta=1$. Accordingly, we propose the following estimator

$$\tilde{\theta}=\left(\sum_{i=1}^{N} X_{[i]}/N\right)+[(n-N)T/N](1-1/N+1/N^2-1/N^3). \tag{2.10}$$

Before we compute its bias and MSE of $\tilde{\theta}$, we need the following lemma. Because of its own usefulness, we expand each term $E(1/N^i)$ up to $O(1/n^7)$ which is more than we need.

Lemma 1.

$$E_c(1/N) = 1/np + q/n^2p^2 + (q+q^2)/n^3p^3 + (q+4q^2+q^3)/n^4p^4 + (q+11q^2+11q^3+q^4)/n^5p^5 + (q+26q^2+66q^3+26q^4+q^5)/n^6p^6 + O(1/n^7),$$

$$E_c(1/N^2) = 1/n^2p^2 + 3q/n^3p^3 + (4q+7q^2)/n^4p^4 + (5q+30q^2+15q^3)/n^5p^5 + (6q+91q^2+146q^3+31q^4)/n^6p^6 + O(1/n^7),$$

$$E_c(1/N^3) = 1/n^3p^3 + 6q/n^4p^4 + (10q+25q^2)/n^5p^5 + (15q+120q^2+90q^3)/n^6p^6 + O(1/n^7),$$

$$E_c(1/N^4) = 1/n^4p^4 + 10q/n^5p^5 + (20q+65q^2)/n^6p^6 + O(1/n^7),$$

$$E_c(1/N^5) = 1/n^5p^5 + 15q/n^6p^6 + O(1/n^7),$$

$$E_c(1/N^6) = 1/n^6p^6 + O(1/n^7),$$

$$E_c(1/N^i) = O(1/n^7), \quad i = 7, 8.$$

Proof. A continuous analogue of $1/N$ is the function $f(x) = 1/x$, for $x > 0$. We apply Taylor's expansion to $f(x)$ about $x = np$. We then compute the conditional expection E_c of the Taylor expansion with x replaced by N. In the computation, we use the fact that if Y has a binomial distribution $B(n, p)$, the moments of N and Y are related by

$$E_c(N-np)^r = \left(\frac{1}{1-q^n}\right)[E(Y-np)^r - (-1)^r(np)^r q^n], \quad \text{for } r = 1, 2, \ldots.$$

The lemma then follows from straight forward calculation.

Theorem 1.
(i) $\mathrm{Bias}(\hat{\theta}) + (np)^3\mathrm{Bias}(\tilde{\theta}) = O(1/n^2)$,
(ii) $n^2(\mathrm{MSE}(\hat{\theta}) - \mathrm{MSE}(\tilde{\theta})) = aT^2 + O(1/n^3)$,
where

$$a = (2q+3q^2)/p^4 + (3q+20q^2+9q^3)/np^5 + (4q+69q^2+110q^3+21q^4)/n^2p^6 > 0.$$

Proof. Applying Lemma 1, we have

$$E_c(\tilde{\theta}) = \theta - Tq/n^4p^5 + T\,O(1/n^5).$$

Since

$$\mathrm{Bias}(\hat{\theta}) = Tq/np^2 + Tq(1+q)/n^2p^3 + O(1/n^4),$$

this proves (i). (ii) can be proven easily using Lemma 1.

2.2. Comparisons of $\hat{\theta}, \theta^*$ and $\tilde{\theta}$

Let δ denote either one of the estimators, $\hat{\theta}, \theta^*, \tilde{\theta}$. We denote the bias and MSE of the estimator δ for given values of θ and T by $B(\delta;\theta,T)$ and $\mathrm{MSE}(\delta;\theta,T)$ respectively. Then, for any $c>0$, we have

$$B(c\delta;c\theta_0,ct_0) = cB(\delta;\theta_0,t_0), \tag{2.11}$$

$$\mathrm{MSE}(c\delta;c\theta_0,ct_0) = c^2\mathrm{MSE}(\delta;\theta_0,t_0). \tag{2.12}$$

For numerical computation, it is necessary to specify values for n, T, and θ. In view of (2.11),

$$B(\delta;\theta,T) = \theta B(\delta;1,T/\theta). \tag{2.13}$$

Thus we only have to compute $B(\delta;1,\alpha)$, for $\theta=1$ and $T=\alpha$. Bias for other values of θ and T can be easily obtained from (2.13). Similar relation holds for MSE. Hence, comparing $\hat{\theta}, \theta^*$ and $\tilde{\theta}$ in terms of exact bias and MSE are possible for various T with fixed $\theta=1$. We plot these quantities for small sample sizes of $n=10$ and $n=30$ in Figures 1 and 2. Biases and MSE of $\hat{\theta}, \theta^*$ and $\tilde{\theta}$ for some values of T for $n=10$, and 30 are tabulated respectively in Tables 1 and 2.

3. Sample distribution and asymptotic properties of $\tilde{\theta}$

We note that, by (2.10),

$$\tilde{\theta} = \hat{\theta} + A(N) \tag{3.1}$$

where

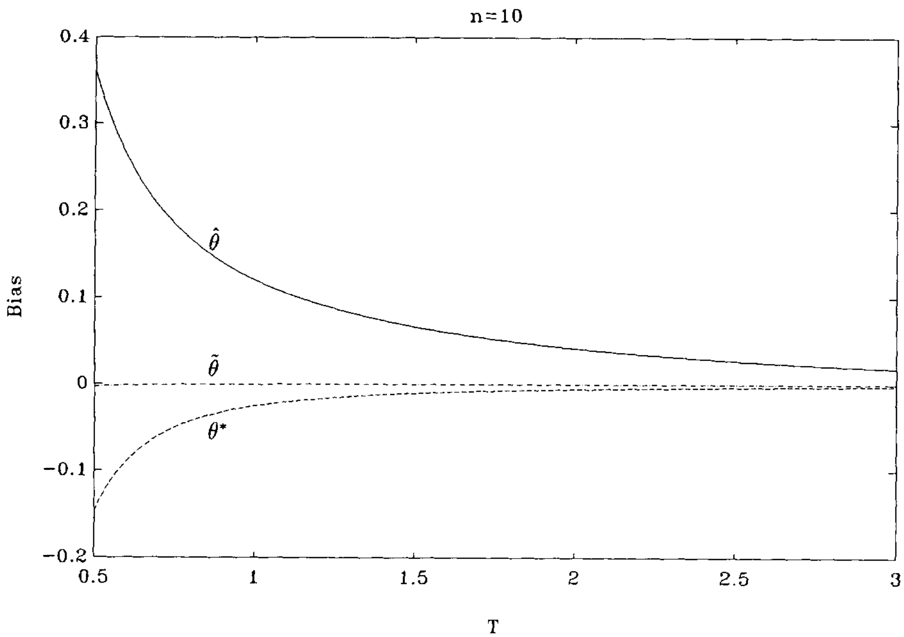
$$A(N) = T\{1/N - (n+1)/N^2 + (n+1)/N^3 - n/N^4\}. \tag{3.2}$$

Let $g(x)$ be the density of $\hat{\theta}$ and $f_c(\theta_0)$ the conditional density of $\tilde{\theta}$ under condition $N \geq 1$. Then,

$$\begin{aligned} f_c(\theta_0) &= \sum_{m=1}^{n} f_c(\theta_0 \mid N=m)P[N=m] \\ &= \sum_{m=1}^{n} g(\theta_0 - A(m))P[N=m] \\ &= \sum_{m=1}^{n} \left\{ \sum_{k=1}^{n} \left[\binom{n}{k} (k/\theta)^k \exp(-k(\theta_0 - A(m))/\theta)/\Gamma(k) \right] \right. \\ &\qquad \left. \times \sum_{i=0}^{k} \left[\binom{k}{i} (-1)^i D^{k-1}(m,k,i) 1_{(D(m,k,i)>0)} \right] \right\} \\ &\qquad \times \binom{n}{m} p^m q^{n-m}/(1-q^n)^2 \end{aligned} \tag{3.3}$$

where

$$D(m,k,i) = \theta_0 - A(m) - T(n-k+i)/k, \tag{3.4}$$



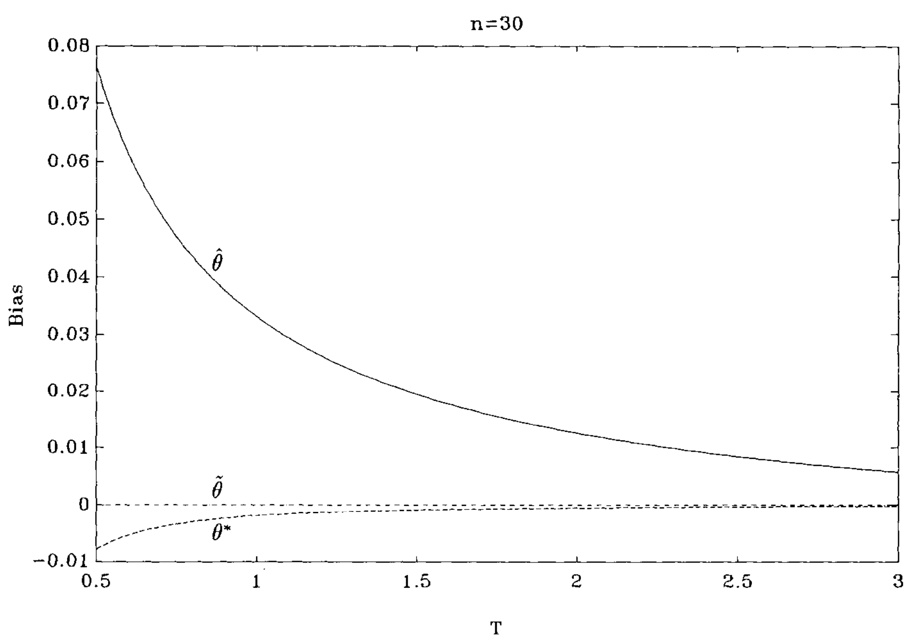


Fig. 1. Plot of biases with respect to T.

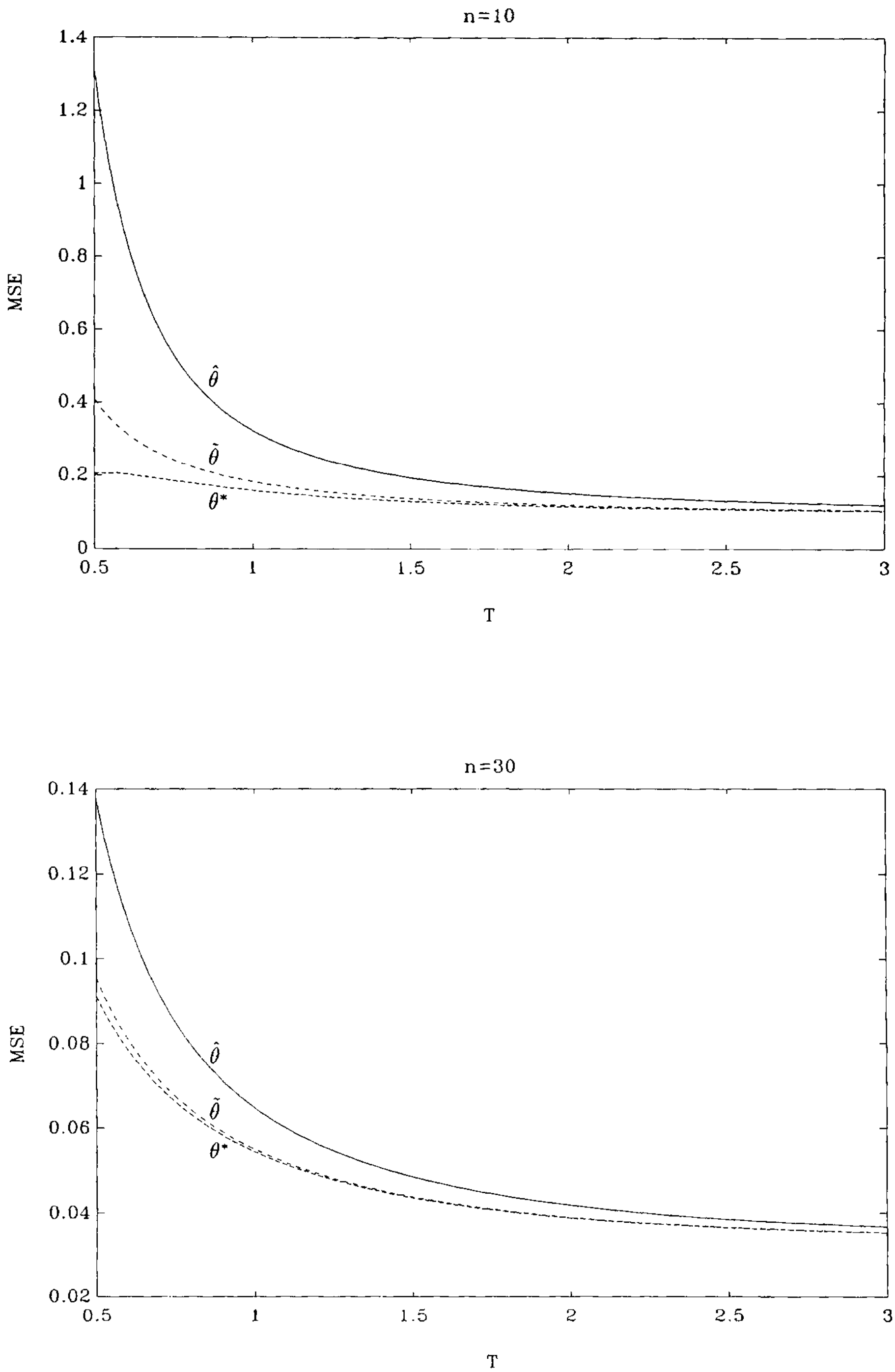


Fig. 2. Plot of MSE with respect to T.

1_S denotes the indicator function of set S and

$$g(x) = \sum_{k=1}^{n} \left[\binom{n}{k} (k/\theta)^k \exp(-kx/\theta)/\Gamma(k) \right] \times \sum_{i=0}^{k} \left[\binom{n}{i} (-1)^i S^{k-1}(k,i) 1_{(S(k,i)>0)} \right]$$

with $S(k,i) = x - T(n-k+i)/k$ (see Bartholomew (1963)).

Theorem 2.

(i) *$\tilde{\theta}$ is strongly consistent and asymptotically unbiased,*

(iia) $\sqrt{n}(\tilde{\theta}-\theta) \xrightarrow{\mathscr{L}} \mathrm{N}(0,\sigma^2(\theta))$,

(iib) $\sqrt{n}\sigma^{-1}(\tilde{\theta})(\tilde{\theta}-\theta) \xrightarrow{\mathscr{L}} \mathrm{N}(0,1)$,

(iiia) $n^{-1/2}\mathrm{N}(\tilde{\theta}-E_c(\tilde{\theta})) \xrightarrow{\mathscr{L}} \mathrm{N}(0,\sigma^2(\theta))$,

(iiib) $n^{-1/2}\sigma^{-1}(\tilde{\theta})\mathrm{N}(\tilde{\theta}-E_c(\tilde{\theta})) \xrightarrow{\mathscr{L}} \mathrm{N}(0,1)$,

Table 1
Biases of θ^*, $\hat{\theta}$ and $\tilde{\theta}$ for some values of T

	T	θ^*	$\hat{\theta}$	$\tilde{\theta}(\times 10^{-2})$
$n=10$	0.2	−2.69033	3.24833	−8.36663
	0.4	−0.28201	0.55918	−0.68846
	0.6	−0.08928	0.26358	−0.17611
	0.8	−0.04285	0.17324	−0.07099
	1.0	−0.02530	0.12039	−0.03645
	1.5	−0.01041	0.06645	−0.01183
	2.0	−0.00564	0.04186	−0.00560
	2.5	−0.00343	0.02767	−0.00315
	3.0	−0.00220	0.01862	−0.00193
	3.5	−0.00145	0.01258	−0.00123
	4.0	−0.00096	0.00848	−0.00080
	4.5	−0.00064	0.00569	−0.00053
	5.0	−0.00042	0.00380	−0.00035
$n=30$	0.2	−0.07705	0.26796	−0.10329
	0.4	−0.01312	0.10075	−0.00850
	0.6	−0.00536	0.06135	−0.00217
	0.8	−0.00298	0.04349	−0.00088
	1.0	−0.00193	0.03315	0.00045
	1.5	−0.00089	0.01953	0.00015
	2.0	−0.00051	0.01263	0.00007
	2.5	−0.00032	0.00845	0.00004
	3.0	−0.00021	0.00573	0.00002
	3.5	−0.00014	0.00388	0.00002
	4.0	−0.00009	0.00262	0.00001
	4.5	−0.00006	0.00176	0.00001
	5.0	−0.00004	0.00118	0.00000

Table 2
MSE of θ^*, $\hat{\theta}$ and $\tilde{\theta}$ for some values of T

	T	θ^*	$\hat{\theta}$	$\tilde{\theta}(\times 10^{-2})$
$n=10$	0.2	2.03441	22.25450	3.19667
	0.4	0.15968	2.40964	0.59686
	0.6	0.20200	0.84899	0.31504
	0.8	0.17904	0.46909	0.22502
	1.0	0.15865	0.32232	0.18249
	1.5	0.12904	0.19360	0.13704
	2.0	0.11530	0.15118	0.11930
	2.5	0.10831	0.13156	0.11073
	3.0	0.10455	0.12078	0.10617
	3.5	0.10248	0.11419	0.10362
	4.0	0.10132	0.10988	0.10214
	4.5	0.10069	0.10694	0.10128
	5.0	0.10034	0.10489	0.10077
$n=30$	0.2	0.19717	0.75730	0.28354
	0.4	0.11040	0.19123	0.11898
	0.6	0.07840	0.10900	0.08089
	0.8	0.06306	0.07945	0.06416
	1.0	0.05429	0.06473	0.05490
	1.5	0.04348	0.04839	0.04371
	2.0	0.03880	0.04177	0.03892
	2.5	0.03642	0.03845	0.03650
	3.0	0.03512	0.03658	0.03518
	3.5	0.03439	0.03546	0.03442
	4.0	0.03396	0.03475	0.03398
	4.5	0.03370	0.03429	0.03372
	5.0	0.03355	0.03398	0.03357

where

$$\sigma^2(\theta) = \theta^2/(1-\exp(-T/\theta)).$$

Proof. Note that $n/N \to 1/p$ a.s. under the conditional probability (P_c). Hence, $\sqrt{n}A(N)=o(1)$ a.s. under P_c. By Yang and Sirvanci (1977) we have that $E_c(n/N)\to 1/p$ and $E_c(n/N)^2 \to (1/p)^2$. Hence, $E_cA(N)=o(1)$. Since $\hat{\theta}$ is consistent and asymptotically unbiased, so is $\tilde{\theta}$. Hence (i) holds.

To show (iia), note that $\sqrt{n}(\tilde{\theta}-\theta)=\sqrt{n}(\hat{\theta}-\theta)+\sqrt{n}A(N)$. Again, by Yang and Sirvanci (1977), (iia) holds for $\hat{\theta}$ and $\sqrt{n}A(N)=o(1)$ under P_c. By the Slutsky theorem, (iia) holds. Here $\sigma^2(\theta)$ has been corrected in Yang and Sirvanci (1979).

To show (iib), note that $\sigma^2(\theta)$ is continuous in $\theta>0$. Hence, $\sigma^2(\tilde{\theta})/\sigma^2(\hat{\theta})\to 1$ under P_c (noting that $\tilde{\theta}=\hat{\theta}+o(1/\sqrt{n})$ under P_c). Again, by Yang and Sirvanci (1977) (iib) holds for $\hat{\theta}$ and we may conclude (iib) by noting that $\sqrt{n}\sigma^{-1}(\tilde{\theta})(\tilde{\theta}-\theta)=(\sigma^{-1}(\tilde{\theta})/\sigma^{-1}(\hat{\theta}))\sqrt{n}\sigma^{-1}(\hat{\theta})(\hat{\theta}-\theta+o(1/\sqrt{n}))$, where $\sigma(\hat{\theta})/\sigma(\tilde{\theta})\to 1$ under P_c.

To show (iiia), note that it holds for $\hat{\theta}$. On the other hand, we can show that $(N/n)\sqrt{n}(A(N)-E_c(A(N)))=o(1)$ under P_c. Again, by the Slutsky theorem, we can conclude (iiia). For (iiib), follow analogous arguments for (iib). The proof is thus complete.

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