The optimal division of N areas under an arbitrary distribution of communication demand

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THE OPTIMAL DIVISION OF N AREAS UNDER AN ARBITRARY DISTRIBUTION OF COMMUNICATION DEMAND

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The problem of telecommunication trunk congestion caused by a heavily loaded circuit-switched network in a large area can be solved by dividing a large area into several small areas. Instead of building a transmitting center for a large area, we suggest that each small area should have its own transmitting center. All the transmitting centers are then linked together to form a single trunk line. Using the quantitative model presented in this paper, the problem of deciding how to divide the large area into small ones and where to build the transmitting centers can be worked out.

Keywords: transmitting center, communication demand

1. Introduction

The problem of telecommunication trunk congestion in circuit-switched networks under a heavy-loaded communication condition was discussed by Wang and Saadawi (1992). The efficiency of the network depends on whether the trunk communication condition is effective or not. Yum and Schwartz (1987) applied the method of alternate routing to avoid the congestion and to maintain the efficient performance of the network system. The problems of routing and communication flowing control are also discussed in Filipiak (1988), using the method of dynamic flow theory.

Most perspectives of the network location including those of Cavalier (1985), Cavalier and Sherali (1986) and Chiu (1987) are limited to the allocation of proper places where communication demand is needed. Boffey and Karazis (1989) proposed a new method to solve the problem of locating bridges in local area computer networks (LANs) in order to overcome this drawback. They considered the communication demand in an area to be different than the communication demand on nodes.

The synthesis and analysis of communication flowing management and control in communication networks are a complicated problem. Nevertheless, the problems of how to compare with the methods of alternate routing

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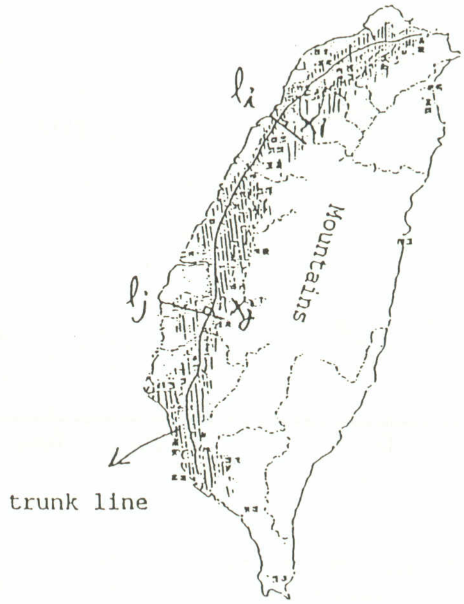
and dynamic flow theory or where the proper place should be located are not our main concern in this paper. This paper, using the concept of Boffey and Karkazis's (1989), mainly deals with how to avoid the telecommunication trunk congestion in heavily loaded circuit-switched networks and to figure out communication demand of an area (such as a state or a small country). The large area is divided into several small areas, each small area having its own transmitting center, and all transmitting centers are linked together to form a single trunk line. Extending Boffey and Karkazis's (1989) concept to communication network and using the communication demand to present a formula, we provide an optimal solution and explore its properties.

2. Preliminaries

Virtually all telecommunication networks are arranged in a hierarchical structure to achieve efficient loading of transmission and switching facilities (Elbert (1990)). For example, if user A wants to communicate with user B by phone, both living in the same local area, the hierarchical communication network system will send the message to the local transmitting center C_A. C_A may again use the hierarchical communication network system to send back the message to user B. If user A and user B are not in the same area, then when user A calls, the hierarchical communication network system will send its message to C_A, then using the telecommunication trunk line, the message will be sent to the transmitting center C_B where user B calls. Then C_B will send the message to user B through the hierarchical communication network system.

Fundamentally, the quantity of communication demand depands on the information demand of two areas. In a general model, since the quantity of communication demand varies from area to area, this paper assumes that the potential demand for communication in two areas are identical. The farthest two points in an appropriate area are linked by what will be called the trunk line, which is as shown below:



$f(x_1)$ is the communication demand at ℓ_1

Figure 1. Allocate the trunk line and form a communication demand function $f(x)$

If there are N ordered areas: $R_1, R_2, \ldots$ and R_N in a communication system with N_i parties in need of communication in areas R_i, then the amount of communication from area R_i to area R_j and from area R_j to area R_i are both equal to $N_i N_j$. Meanwhile, the amount in area R_i itself is equal to $(N_i)^2$. Because all transmitting centers C_i use the same trunk line for connection (see Figure 1) communicate between area R_i and area R_j $R_j (i < j)$ must be reached through the transmitting center of areas $R_{i+1}, R_{i+2}, \ldots$ and R_{j-1}. Therefore, the amount of communication incurred at the center of area R_i is equal to

$$F_i = \begin{cases} N_i^2 + 2N_i \sum\limits_{k=i+1}^{N} N_k, & i = 1 \\ N_i^2 + 2N_i \sum\limits_{k=i+1}^{N} N_k + 2N_i & 1 < i < n \\ \quad \sum\limits_{k=i+1}^{N} N_k + 2\sum\limits_{k=1}^{i-1} N_k \sum\limits_{k=i+1}^{N} N_k, & \\ N_i^2 + 2N_i \sum\limits_{k=1}^{i-1} N_k, & i = n \end{cases} \tag{2.1}$$

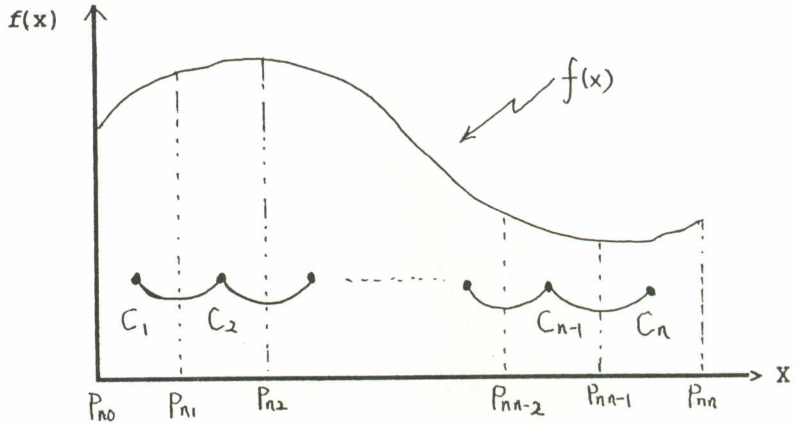


Figure 2. The division of n areas under communication demand function by $P_{n1}, P_{n2}, ..., P_{nn-1}$
(where C_i is the center of zone R_i)

We can extend the concept of the discrete distribution of communication demand to that of the continuous communication demand. The trunk line is treated as the X-axis and a communication demand function $f(x)$ formed (c.f. Figure 1), as shown below:

We may assume that $\int_0^1 f(x)dx = 1$, for if $f(x)$ is replaced by the function $\frac{f(x)}{\int_0^1 f(x)dx}$, we have the result: $\int_0^1 f(x)dx = 1$. As in Figure 2, if P_{ni} is the location of the i-th communication bridge, $P_{n0} = 0$ and $P_{nn} = 1$, then by the same argument as for (2.1)

$$
\begin{aligned}
F_i = & \left[\int_{P_{ni-1}}^{P_{ni}} f(x)dx\right]^2 + 2\int_{P_{ni-1}}^{P_{ni}} f(x)dx \int_0^{P_{ni-1}} f(x)dx + \\
& 2\int_{P_{ni-1}}^{P_{ni}} f(x)dx \int_{P_{ni}}^{1} f(x)dx + 2\int_0^{P_{ni-1}} f(x)dx \int_{P_{ni}}^{1} f(x)dx \\
= & \left[\int_{P_{ni-1}}^{P_{ni}} f(x)dx\right]\left[2\int_0^1 f(x)dx - \int_{P_{ni-1}}^{P_{ni}} f(x)dx\right] + \\
& 2\int_0^{P_{ni-1}} f(x)dx \int_{P_{ni}}^{1} f(x)dx \\
= & \left[\int_0^{P_{ni}} dx - \int_0^{P_{ni-1}} f(x)dx\right]\left[2 - \left(\int_0^{P_{ni}} f(x)dx - \right.\right.
\end{aligned}
$$

$$\int_0^{P_{ni-1}} f(x)dx\Big)\Big] + 2\Big[\int_0^{P_{ni-1}} f(x)dx\Big]\Big[1 - \int_0^{P_{ni}} f(x)dx\Big]$$
$$= 1 - \Big[1 - \int_0^{P_{ni}} f(x)dx\Big]^2 - \Big[\int_0^{P_{ni-1}} f(x)dx\Big]^2$$
$$\text{for all} \quad i = 1, 2, ..., n. \tag{2.2}$$

The quantity Max $\{F_1(P_{n1}, P_{n2}, ..., P_{nn}),\ F_2(P_{n1}, P_{n2}, ..., P_{nn})\ ,...,\ F_n(P_{n1}, P_{n2}, ..., P_{nn})\}$ is used to represent the index of waiting time or the index of service level in a given communication system. Therefore the problem of determining the optimal location of the n bridges can be written as follows:

$$\min_{0 \le P_{n1} < P_{n2} < ... < P_{nn-1} \le 1} \text{Max}[F_1(P_{n1}, ..., P_{nn}), ..., F_n(P_{n1}, ..., P_{nn})]. \tag{2.3}$$

3. The Optimal Setting of n Communication Bridges

If $P^*_{n1}, P^*_{n2}, ..., P^*_{nn-1}$ is the optimal solution of problem (2.3), then the necessary condition for optimality is

$$F_1(P^*_{n1}, P^*_{n2}, ..., P^*_{nn}) \\ = F_2(P^*_{n1}, P^*_{n2}, ..., P^*_{nn}) = ... = F_n(P^*_{n1}, P^*_{n2}, ..., P^*_{nn}) \tag{3.1}$$

and hence by (2.2), we have

$$\Big[1 - \int_0^{P^*_{n1}} f(x)dx\Big]^2 = \Big[1 - \int_0^{P^*_{n1}} f(x)dx\Big]^2 + \Big[\int_0^{P^*_{ni-1}} f(x)dx\Big]^2$$
$$= \Big[\int_0^{P^*_{nn-1}} f(x)dx\Big]^2 \quad \text{for all} \quad i = 1, 2, ..., n. \tag{3.2}$$

This implies that

$$\int_0^{P^*_{ni}} f(x)dx = 1 - \sqrt{K_n^2 - \Big(\int_0^{P^*_{ni-1}} f(x)dx\Big)^2} \quad \text{for } i = 1, 2, ..., n,$$
$$\text{where} \quad K_n = 1 - \int_0^{P^*_{n1}} f(x)dx. \tag{3.3}$$

Given a constant $K \in (0,1)$, if $\mathcal{G}_K(x)$ is the function defined by

$$\mathcal{G}_K(x) = 1 - \sqrt{K^2 - x^2} \quad 0 \le x \le K \tag{3.4}$$

then equation (3.3) can be written as

$$\int_0^{P^*_{ni}} f(x)dx = \mathcal{G}_{K_n}\left(\int_0^{P^*_{ni-1}} f(x)dx\right), \quad i = 1,2,...,n \tag{3.5}$$

and

$$1 = \int_0^{P^*_{nn}} f(x)dx = \mathcal{G}^{(n)}_{K_n}\left(\int_0^{P^*_{n0}} f(x)dx\right) = \mathcal{G}^{(n)}_{K_n}(0) \tag{3.6}$$

where $\mathcal{G}^{(n)}_{K_n}(x) = \mathcal{G}_{K_n} \circ \mathcal{G}_{K_n} \circ ... \circ \mathcal{G}_{K_n}(x)$ is a composite function of $\mathcal{G}_{K_n}(x)$ with itself n times.

4. Properties of the Optimal Solution

Theorem 1. *For any $i = 1,2,...,n$, $\int_{P^*_{ni-1}}^{P^*_{ni}} f(x)dx = \int_{P^*_{nn-i}}^{P^*_{nn-i+1}} f(x)dx$. If $f(x)$ is a constant function, then the order values $0 = P_{n0} < P_{n1} < P_{n2} < ... < P_{nn} = 1$ are symmetric with respect to the value $1/2$.*

Proof. The equality of the first term and the last term of (3.2) shows that this theorem holds for the case of $i = 1$. Using (3.2), this theorem can be proved by a similar process.

The functions $\mathcal{G}_K(x)$, defined by (3.4), have the following properties:

Lemma 1. *For any constant $K \in (0,1)$, $\mathcal{G}^{(i+1)}_K(0) > \mathcal{G}^{(i)}_K(0)$, $i = 1,2,...$ where $\mathcal{G}^{(i)}_K(x)$ is the composite function $\mathcal{G}_K$ with itself i times.*

Proof. Since $\mathcal{G}^{(1)}_K(0) > 0$, and $\mathcal{G}_K(x)$ is an increasing function of x, so $\mathcal{G}^{(2)}_K(0) = \mathcal{G}_K \circ (\mathcal{G}_K(0)) > \mathcal{G}_K(0)$. Similarly $\mathcal{G}^{(3)}_K(0) = \mathcal{G}_K \circ (\mathcal{G}^{(2)}_K(0)) > \mathcal{G}_K \circ (\mathcal{G}_K(0)) = \mathcal{G}^{(2)}_K(0)$. Using mathematical induction, we have the desired results.

Lemma 2. *For any two constants K, $K' \in [0,1]$, if $K \geq K'$, then $\mathcal{G}^{(r)}_K(0) \leq \mathcal{G}^{(r)}_{K'}(0)$, $r = 1,2,...$.*

Proof. Since $K \geq K'$, $\mathcal{G}_K(x) = 1 - \sqrt{K^2 - x^2} \leq \mathcal{G}_{K'}(x) = 1 - \sqrt{K'^2 - x^2}$ for all $x \in (0, K')$ and hence $\mathcal{G}^{(2)}_K(0) = \mathcal{G}_K(\mathcal{G}_K(0)) \leq \mathcal{G}_{K'}(\mathcal{G}_K(0)) \leq \mathcal{G}_{K'}(\mathcal{G}_{K'}(0)) = \mathcal{G}^{(2)}_{K'}(0)$. By the same argument as above, we have $\mathcal{G}^{(r)}_K(0) \leq \mathcal{G}^{(r)}_{K'}(0)$ for all $r = 1,2,...$.

Theorem 2. *Given a positive integer n, let K_n, $K_n \in (0,1)$, be a real number determined by equations (3.5) and (3.6). If $i < j$, then $K_i < K_j$.*

Proof. Suppose that $K_i \geq K_j$. Lemma 2 and (3.6) yield that $\mathcal{G}^{(j)}_{ki}(0) \leq \mathcal{G}^{(j)}_{kj}(0) = 1 = \mathcal{G}^{(i)}_{ki}(0)$. This contradicts the result of Lemma 1.

Lemma 3. *If n is an odd integer, then* $\int_0^{P^*_{n(n-1)/2}} f(x)dx = \frac{K_n}{\sqrt{2}}$.

Proof. Using (3.2), (3.3) we have

$$K_n^2 = \left[1 - \int_0^{P^*_{n(n+1)/2}} f(x)dx\right]^2 + \left[\int_0^{P^*_{n(n-1)/2}} f(x)dx\right]^2. \quad (4.1)$$

Theorem 1 yields that

$$1 - \int_0^{P^*_{n(n+1)/2}} f(x)dx = \int_0^{P^*_{n(n-1)/2}} f(x)dx. \quad (4.2)$$

Substituting (4.2) into (4.1) leads to the desired result.

Theorem 3.

(1) If n is an even integer, then

$$\int_{P^*_{ni-1}}^{P^*_{ni}} f(x)dx < \int_{P^*_{ni-2}}^{P^*_{ni-1}} f(x)dx, \quad \text{for} \quad i = 2, ..., n/2.$$

(2) If n is an odd integer, then

$$\int_{P^*_{ni-1}}^{P^*_{ni}} f(x)dx < \int_{P^*_{ni-2}}^{P^*_{ni-1}} f(x)dx, \quad \text{for} \quad i = 2, ..., (n+1)/2.$$

Proof. See appendix.

Corollary 1. $1/2 \le K_n \le 1/\sqrt{2}$, *for* $n = 2, 3, ...$.

Proof. Using the result of Theorem 3, we have

$$\lim_{n\to\infty} \int_{P^*_{n(n/2-1)}}^{P^*_{n(n/2)}} f(x)dx = 0. \quad (4.3)$$

$$\lim_{n\to\infty} \int_{P^*_{n(n-1)/2}}^{P^*_{n(n+1)/2}} f(x)dx = 0. \quad (4.4)$$

Together with Theorem 1, Lemma 3, (4.3) and (4.4), it leads to

$$\lim_{n\to\infty} K_n = 1/\sqrt{2}.$$

5. Experimental Results

Table 1 displays K_n for various values of n.

Table 1
Computational Results of K_n

No. of Centers	K_n	No. of Centers	K_n
2	0.500	14	0.696
3	0.586	15	0.697
4	0.625	16	0.698
5	0.648	17	0.699
6	0.662	19	0.700
7	0.671	20	0.701
8	0.678	23	0.702
9	0.683	26	0.703
10	0.687	30	0.704
11	0.690	37	0.705
12	0.692	52	0.706
13	0.694		

6. Conclusion

In our findings, we assume that the distribution of communication demand is an arbitrary and continuous function. The communication variables are divided into several transmitting centers and they can be linked to a single trunk line, in order to enforce the performance efficiency in communication.

As an extension of the present work, we can consider the following issues: what is the best choice of n to minimize communication waiting time? Also, how does one divide the communication variables into several transmitting centers when the facility cost of building transmitting centers is considered.

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Appendix

Proof of Theorem 3.

Property (3.2) yields that

$$\left[1-\int_0^{P^*_{ni}} f(x)dx\right]^2+\left[\int_0^{P^*_{ni-1}} f(x)dx\right]^2=\left[1-\int_0^{P^*_{ni+1}} f(x)dx\right]^2+\left[\int_0^{P^*_{ni}} f(x)dx\right]^2 \qquad (A.1)$$

for all $i=1,2,...,n-1$.

The left side of (A.1) can be written as

$$\left[\left(1-\int_0^{P^*_{ni+1}} f(x)dx\right)+\int_{P^*_{ni}}^{P^*_{ni+1}} f(x)dx\right]^2+\left[\int_0^{P^*_{ni-1}} f(x)dx\right]^2 \qquad (A.2)$$

and the right side of (A.1) can be written as

$$\left[1-\int_0^{P^*_{ni+1}} f(x)dx\right]^2+\left[\int_0^{P^*_{ni-1}} f(x)dx+\int_{P^*_{ni-1}}^{P^*_{ni}} f(x)dx\right]^2. \qquad (A.3)$$

Substituting (A.2) and (A.3) into (A.1) leads to

$$2\left[\int_{P^*_{ni+1}}^{1} f(x)dx\right]\left[\int_{P^*_{ni}}^{P^*_{ni+1}} f(x)dx\right]+\left[\int_{P^*_{ni}}^{P^*_{ni+1}} f(x)dx\right]^2+\left[\int_0^{P^*_{ni-1}} f(x)dx\right]^2=\left[\int_0^{P^*_{ni-1}} f(x)dx+\int_{P^*_{ni-1}}^{P^*_{ni}} f(x)dx\right]^2. \qquad (A.4)$$

If $i \le n/2$ then $P^*_{ni+1} \le P^*_{nn-i+1}$, and hence by Theorem 1 we have

$$\int_0^{P^*_{ni-1}} f(x)dx=\sum_{k=1}^{i-1}\int_{P^*_{nk-1}}^{P^*_{nk}} f(x)dx=\sum_{k=1}^{i-1}\int_{P^*_{nn-k}}^{P^*_{nn-k+1}} f(x)dx$$
$$=\int_{P^*_{nn-i+1}}^{1} f(x)dx \le \int_{P^*_{ni+1}}^{1} f(x)dx. \qquad (A.5)$$

Together (A.4) and (A.5), lead to

$$\left[\int_0^{P^*_{ni-1}} f(x)dx+\int_{P^*_{ni}}^{P^*_{ni+1}} f(x)dx\right]^2<\left[\int_0^{P^*_{ni-1}} f(x)dx+\int_{P^*_{ni-1}}^{P^*_{ni}} f(x)dx\right]^2$$

and therefore

$$\int_{P^*_{ni}}^{P^*_{ni+1}} f(x)dx < \int_{P^*_{ni-1}}^{P^*_{ni}} f(x)dx \qquad \text{for all } i \leq n/2.$$

To complete the proof of this theorem, we show that if n is an odd integer, then

$$\int_{P^*_{n(n-1)/2}}^{P^*_{n(n+1)/2}} f(x)dx \leq \int_{P^*_{n(n-3)/2}}^{P^*_{n(n-1)/2}} f(x)dx. \tag{A.6}$$

By (3.5) and the mean-value theorem, we have

$$\int_0^{P^*_{n(n+1)/2}} f(x)dx - \int_0^{P^*_{n(n-1)/2}} f(x)dx$$

$$= \mathcal{G}_{k_n}\left(\int_0^{P^*_{n(n-1)/2}} f(x)dx\right) - \mathcal{G}_{k_n}\left(\int_0^{P^*_{n(n-3)/2}} f(x)dx\right)$$

$$= \mathcal{G}'_{k_n}(Z)\left(\int_0^{P^*_{n(n-1)/2}} f(x)dx - \int_0^{P^*_{n(n-3)/2}} f(x)dx\right)$$

$$= \frac{Z}{\sqrt{K_n^2 - Z^2}} \int_{P^*_{n(n-3)/2}}^{P^*_{n(n-1)/2}} f(x)dx \quad \text{where } Z \text{ satisfies} \tag{A.7}$$

$$\int_0^{P^*_{n(n-3)/2}} f(x)dx < Z < \int_0^{P^*_{n(n-1)/2}} f(x)dx \tag{A.8}$$

(A.8) and Lemma 3 yield that

$$Z < \int_0^{P^*_{n(n-1)/2}} f(x)dx = \frac{K_n}{\sqrt{2}}$$

and therefore

$$\mathcal{G}'_{k_n}(Z) = \frac{Z}{\sqrt{K_n^2 - Z^2}} < 1 \tag{A.9}$$

(A.7) and (A.9), together give the inequality (A.6).

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