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Fuzzy multi-level minimum cost flow problems

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Abstract

Both multiple objectives and multiple hierarchies minimum cost flow problems with fuzzy costs and fuzzy capacities in the arcs are investigated. To reduce the complexity, a possibility programming is used to handle the vagueness in the parameters. Fuzzy approach can considerably simplify the problem and thus a fairly general multi-level problem can be solved reasonably easily in spite of the fact that the multi-level problem is NP-hard and very difficult to solve. Several numerical examples are considered to illustrate the approach.

Keywords: Decision making; Multicriteria analysis; Minimum cost flow problem; Multi-level programming; Network flow

1. Introduction

Minimum cost flow (MCF) problem is a general form of the network flow problem whose aim is to find the least cost of the shipment of a commodity through a capacitated network in order to satisfy demands at certain nodes from available supplies at other nodes. Because it represents a general form of the network flow, the results from the study of the MCF problem can be applied to many other network problems such as transportation, maximum flow, assignment, shortest path, and trans-shipment problems. Furthermore, the results from the study of the MCF problems frequently offer a clue or a lower bound to the more complicated network flow problems. Therefore, the MCF problem plays a crucial role in the understanding of network flows. The MCF problem is also very practical, it has been used to solve several real-world applicational problems such as multi-stage production inventory planning, mold allocation, nurse scheduling, project assignment, faculty-course assignment, and automobile routing [1].

In actual practice, the costs and the capacities of the network are generally vague or uncertain. Fuzzy set theory appears to be ideally suited to solve such vague aspects. In addition to apply the fuzzy approach to the basic MCF problem, we also extended the approach to more practical problems where the system has a multiple objectives and multiple hierarchy levels. To reduce the complexity of the problem, we used the possibility of linear programming to handle the vagueness of the parameters and the traditional max–min

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optimization approach to handle the multi-level aspects. The former uses the results of Negi and Lee [16, 17] and the latter uses the development of multi-level programming due to Shih et al. [19]. Another problem considered is the use of compensatory operations in obtaining the optimum. Each approach is illustrated by a numerical example.

Many investigations have been carried out to solve the multi-level programming problem. Ruefli [18] approached this problem by the use of goal decomposition. More systematic approaches were carried out by Bard [2], Ben-Ayed et al. [3], and Bialas and coworkers [4, 5]. The general multi-level programming problem has been shown to be non-convex and NP-hard. The only effective numerical approach to solve large practical problems appears to be the fuzzy approach proposed by Shih et al. [19].

2. Fuzzy minimum cost flow problem

Let $G(N,A)$ be a directed network with a cost c_{ij} and a capacity of upper bound u_{ij} and lower bound l_{ij} associated with every arc $(i,j)\in A$. We also let each node $i\in N$ possess a number of resources $b(i)$, which indicates its supply, demand, or transient node depending on whether $b(i)>0$, $b(i)<0$ or $b(i)=0$, respectively. The minimum cost flow (MCF) problem can be formulated as follows:

$$\text{Min} \quad f(x)=\sum_{(i,j)\in A} c_{ij}^{\mathrm{T}}\, x_{ij}$$

$$\text{s.t.} \quad \sum_{\{j:\,(i,j)\in A\}} x_{ij} - \sum_{\{j:\,(j,i)\in A\}} x_{ji} = b(i), \quad \forall i\in N, \tag{1a}$$

$$l_{ij}\leqslant x_{ij}\leqslant u_{ij}, \quad \forall (i,j)\in A, \tag{1b}$$

$$x_{ij}\geqslant 0 \text{ and integer}, \quad \forall (i,j)\in A. \tag{1c}$$

The objective is to minimize the total cost. Constraint (1a) represents the conservation of flows and constraint (1b) is the capacity constraint for each arch. In general, the MCF problem also has some additional assumptions such as: (a) supplies, demands, and capacities must be integers; (b) the network is directed; (c) the supply/demand at each node satisfies the condition $\sum b(i)=0$; and (d) the MCF problem has a feasible solution. Obviously, the MCF problem reduces to the maximum flow problem if the objective is maximization, constraint (1b) is removed, and all the flow cost $c_{ij}=1$, $\forall i$ and j. Transportation problem results if the network is bipartite and all the arcs are directed from source to sink without any capacity restriction.

In actual practice, both the capacity constraints and the cost parameter are vague and can be considered fuzzy and thus the fuzzy MCF problem can be represented by

$$\begin{aligned}
\text{Min} \quad & f(x)=\sum_{(i,j)\in A} \tilde{c}_{ij}^{\mathrm{T}}\, x_{ij}\\
\text{s.t.} \quad & \sum_{\{j:\,(i,j)\in A\}} x_{ij} - \sum_{\{j:\,(j,i)\in A\}} x_{ji} = b(i), \quad \forall i\in N,\\
& \tilde{l}_{ij}\leqslant x_{ij}\leqslant \tilde{u}_{ij}, \quad \forall (i,j)\in A,\\
& x_{ij}\geqslant 0 \text{ and integer}, \quad \forall (i,j)\in A,
\end{aligned} \tag{2}$$

where $\tilde{c}_{ij}$, $\tilde{l}_{ij}$, and $\tilde{u}_{ij}$ represent the fuzzy cost, fuzzy lower bound, and fuzzy upper bound of each arc, respectively. Trapezoidal fuzzy numbers will be used in this paper.

2.1. The possibility approach

Using the concept of possibility proposed by Zadeh, Luhandjula [15] and Buckley [6, 7] proposed the possibility programming. The formulation of Beckley results in nonlinear programming problems. Negi [16, 17] reformulated the approach of Buckley by the use of trapezoidal fuzzy numbers and reduced the problem into a fuzzy linear programming problem. We shall follow the formulation of Negi with some modifications to reduce the number of constraints. Possibility linear programming can be represented as

$$\begin{aligned} \text{Max/Min} \quad & Z = \tilde{\boldsymbol{c}}^{\mathrm{T}} \boldsymbol{x} \\ \text{s.t.} \quad & \tilde{\boldsymbol{A}}_i \boldsymbol{x} \leqslant (\text{or } \geqslant) \tilde{b}_i, \quad \forall i, \\ & \boldsymbol{x} \geqslant 0, \end{aligned} \tag{3}$$

where $\tilde{\boldsymbol{c}} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n)$, $\tilde{\boldsymbol{A}}_i = (\tilde{a}_{i1}, \tilde{a}_{i2}, \ldots, \tilde{a}_{in})$ and $\tilde{c}_j, \tilde{a}_{ij}$ and $\tilde{b}_i$ are the possibilistic variables, $\forall i$ and j.

By assuming exceedance possibility for comparison purposes [9] and by assuming that the decision maker has decided on a cut-off value for α, the above possibility programming problem, equation (3), can be reduced to the following crisp linear programming problem [17]:

$$\begin{aligned} \text{Max} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & \delta_1, \delta_2, \ldots, \delta_m, \quad \theta_2 \geqslant \alpha, \qquad b_{i3} \leqslant \sum_j a_{ij2}^{\mathrm{T}} x_j, \quad b_{i4} \geqslant \sum_j a_{ij1}^{\mathrm{T}} x_j, \quad \forall i, \\ & \sum_j c_{ij3}^{\mathrm{T}} x_j \leqslant f \leqslant \sum_j c_{ij4}^{\mathrm{T}} x_j, \quad \forall i, \qquad \delta_1, \delta_2, \ldots, \delta_m, \theta_2, \text{ and } \alpha \in [0,1], \\ & x_j \geqslant 0, \quad \forall j, \end{aligned} \tag{4}$$

or

$$\begin{aligned} \text{Min} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & \delta_1, \delta_2, \ldots, \delta_m, \quad \theta_1 \geqslant \alpha, \\ & b_{i1} \leqslant \sum_j a_{ij4}^{\mathrm{T}} x_j, \quad b_{i2} \geqslant \sum_j a_{ij3}^{\mathrm{T}} x_j, \quad \forall i, \\ & \sum_j c_{ij1}^{\mathrm{T}} x_j \leqslant f \leqslant \sum_j c_{ij2}^{\mathrm{T}} x_j, \quad \forall i, \\ & \delta_1, \delta_2, \ldots, \delta_m, \theta_1, \text{ and } \alpha \in [0,1], \\ & x_j \geqslant 0, \quad \forall j, \end{aligned} \tag{5}$$

where the trapezoidal fuzzy number has been assumed for $\tilde{c}_j, \tilde{a}_{ij}$ and $\tilde{b}_i$; and

$$\theta_1 = \left[f - \sum_j c_{j1}^{\mathrm{T}} x_j \right] \Big/ \left[\sum_j c_{j2}^{\mathrm{T}} x_j - \sum_j c_{j1}^{\mathrm{T}} x_j \right],$$

$$\theta_2 = \left[\sum_j c_{j4}^{\mathrm{T}} x_j - f \right] \Big/ \left[\sum_j c_{j4}^{\mathrm{T}} x_j - \sum_j c_{j3}^{\mathrm{T}} x_j \right],$$

$$\delta_i = (b_{i4} - r_{i1}) / [(b_{i4} - b_{i3}) + (r_{i2} - r_{i1})]$$

with $r_{i1} = \sum_j a_{ij1} x_j$, and $r_{i2} = \sum_j a_{ij2} x_j$

According to the above expressions, the possibilistic linear programming problem can be reduced to a crisp linear programming problem with $3m+3$ constraints. Three constraints due to the original objective function and $3m$ constraints due to the original m constraints. In addition, if the α value is assumed to be unknown, the above possibilistic programming problem will form a nonlinear programming problem. However, a linear programming problem results if we assume a fixed cut-off value $\alpha \in [0,1]$.

In the MCF problem, Eq. (2), there exist both a lower bound l_{ij} and an upper bound u_{ij} for each arc. We can use one trapezoidal fuzzy number to represent each arc constraint, i.e. $\tilde{b}_i = (l_{ij1}, l_{ij2}, u_{ij1}, u_{ij2})$, where the intervals $[l_{ij1}, l_{ij2}]$ and $[u_{ij1}, u_{ij2}]$ represent the tolerances in the lower and upper bounds, respectively. Since we seek the maximum possibility $\delta_i \geqslant \alpha$, $i = 1$ or 2, based on the cut-off value, the left-hand side of the fuzzy number can be represented by

$$\delta_1 = (x_{ij} - l_{ij1})/(l_{ij2} - l_{ij1}) \geqslant \alpha,$$

or

$$x_{ij} \geqslant \alpha(l_{ij2} - l_{ij1}) + l_{ij1}$$

and the right-hand side of the fuzzy number:

$$\delta_2 = (u_{ij2} - x_{ij})/(u_{ij2} - u_{ij1}) \geqslant \alpha,$$

or

$$x_{ij} \leqslant u_{ij2} - \alpha(u_{ij2} - u_{ij1}).$$

Thus, we can form the fuzzy interval as

$$\alpha(l_{ij2} - l_{ij1}) + l_{ij1} \leqslant x_{ij} \leqslant u_{ij2} - \alpha(u_{ij2} - u_{ij1}). \tag{6}$$

By using Eqs. (5) and (6), the fuzzy MCF problem, Eq. (2), can be reduced to the following crisp linear programming problem:

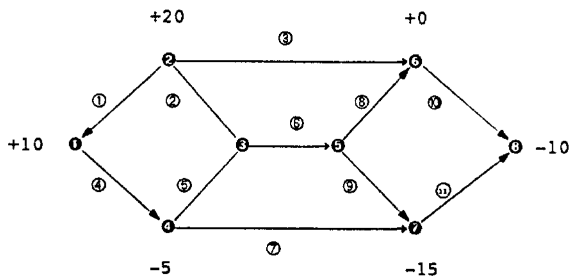
$$\begin{aligned}
\text{Min} \quad & f(\boldsymbol{x}) \\
\text{s.t.} \quad & \theta_1 \geqslant \alpha, \\
& \alpha(l_{ij2} - l_{ij1}) + l_{ij1} \leqslant x_{ij} \leqslant u_{ij4} - \alpha(u_{ij2} - u_{ij1}), \quad \forall i \text{ and } j, \\
& \sum_i \sum_j c_{ij1}^{\mathrm{T}} x_{ij} \leqslant f \leqslant \sum_i \sum_j c_{ij2}^{\mathrm{T}} x_{ij}, \forall i \text{ and } j, \\
& \theta_1, \quad \alpha \in [0,1], \\
& x_{ij} \geqslant 0 \text{ and integer},
\end{aligned} \tag{7}$$

where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

The above problem can be solved by any existing mixed-integer software. If there is no feasible solution, some adjustments about the α cut-off value will be needed. In this case, an interactive procedure with the

Table 1
Parameters for a fuzzy MCF problem (with 8 nodes and 11 arcs for Example 1)

Node no.	Supply/demand	Arc no.	Fuzzy cost	Fuzzy capacity	Note
❶	10	①	(0.5, 1, 1.5, 2)	(0, 0, 9, 11)	x_{21}
❷	20	②	(0, 0, 0.5, 1)	(0, 1, 9, 13)	x_{23}
❸	0	③	(5, 6, 7, 8)	(0, 0, 9, 11)	x_{26}
❹	−5	④	(1.5, 2, 2.5, 3)	(0, 2, 15, 16)	x_{14}
❺	0	⑤	(0.5, 1, 1.5, 2)	(0, 0, 5, 9.5)	x_{34}
❻	0	⑥	(3, 4, 5, 6)	(0, 0, 10, 12)	x_{35}
❼	−15	⑦	(4, 5, 6, 7)	(0, 2, 10, 14.5)	x_{47}
❽	−10	⑧	(1.5, 2, 2.5, 3.5)	(0, 0, 20, 22)	x_{56}
		⑨	(6, 7, 8, 9)	(0, 0, 15, 17)	x_{57}
		⑩	(7, 8, 9, 10)	(0, 1, 10, 12)	x_{68}
		⑪	(8, 9, 10, 11.5)	(0, 0, 15, 16.5)	x_{78}



decision maker should be adapted. Notice that in the above expression there are only $2m$ arc capacity constraints instead of $3m$ constraints as in the Negi's formation [16]. Moreover, these $2m$ arc capacity constraints can be reduced to m constraints if all the lower bounds are zero bounds.

2.2. Example 1. Fuzzy minimum flow problem

This is a fuzzy MCF problem with 8 nodes and 11 arcs. The fuzzy data are summarized in Table 1 together with the structure of the network. The fuzzy MCF problem can be formulated as

$$\begin{aligned}\text{Min}\quad f1 = {} & (0.5,1,1.5,2)x_{21} + (0,0,0.5,1)x_{23} + (5,6,7,8)x_{26} + (1.5,2,2.5,3)x_{14} \\ & + (0.5,1,1.5,2)x_{34} + (3,4,5,6)x_{35} + (4,5,6,7)x_{47} + (1.5,2,2.5,3.5)x_{56} \\ & + (6,7,8,9)x_{57} + (7,8,9,10)x_{68} + (8,9,10,11.5)x_{78}\end{aligned}$$

$$\begin{aligned}\text{s.t.}\quad & x_{14} + x_{21} = 10, \quad x_{21} + x_{23} + x_{26} = 20, \quad x_{34} + x_{35} - x_{23} = 0, \quad x_{47} - x_{14} - x_{34} = -5, \\ & x_{56} + x_{57} - x_{35} = 0, \quad x_{68} - x_{56} - x_{26} = 0, \quad x_{78} - x_{47} - x_{57} = -15, \quad -x_{68} - x_{78} = -10, \\ & x_{21} \in (0,0,9,11), \; x_{23} \in (0,1,9,13), \quad x_{26} \in (0,0,9,11), \; x_{14} \in (0,2,15,16), \quad x_{34} \in (0,0,5,9.5),\end{aligned}$$

$x_{35} \in (0,0,10,12)$, $x_{47} \in (0,2,10,14.5)$, $x_{56} \in (0,0,20,22)$, $x_{57} \in (0,0,15,17)$, $x_{68} \in (0,1,10,12)$, $x_{78} \in (0,0,15,16.5)$, and all x_{ij} are integers in the arcs.

Using Eq. (7), the crisp linear programming problem can be formulated as

$$
\begin{aligned}
\text{Min} \quad & f1 \\
\text{s.t.} \quad & (f1 - 0.5x_{21} - 5x_{26} - 1.5x_{14} - 0.5x_{34} - 3x_{35} - 4x_{47} - 1.5x_{56} - 6x_{57} - 7x_{68} - 8x_{78})/\ (0.5x_{21} \\
& +x_{26} + 0.5x_{14} + 0.5x_{34} + x_{35} + x_{47} + x_{68} + x_{78}) \geqslant \alpha, \\
& 0.5x_{21} + 5x_{26} + 1.5x_{14} + 0.5x_{34} + 3x_{35} + 4x_{47} + 1.5x_{56} + 6x_{57} + 7x_{68} + 8x_{78} \leqslant f1, \\
& f1 \leqslant x_{21} + 6x_{26} + 2x_{14} + x_{34} + 4x_{35} + 5x_{47} + 2x_{56} + 7x_{57} + 8x_{68} + 9x_{78}, \\
& x_{14} - x_{21} = 10,\ x_{21} + x_{23} + x_{26} = 20,\ x_{34} + x_{35} - x_{23} = 0,\ x_{47} - x_{14} - x_{34} = -5, \\
& x_{56} + x_{57} - x_{35} = 0,\ x_{68} - x_{56} - x_{26} = 0,\ x_{78} - x_{47} - x_{57} = -15,\ -x_{68} - x_{78} = -10, \\
& x_{21} \leqslant 11 - 2\alpha,\ \alpha \leqslant x_{23} \leqslant 13 - 4\alpha,\ x_{26} \leqslant 11 - 2\alpha,\ 2\alpha \leqslant x_{14} \leqslant 16 - \alpha,\ x_{34} \leqslant 9.5 - 4.5\alpha, \\
& x_{35} \leqslant 12 - 2\alpha,\ 2\alpha \leqslant x_{47} \leqslant 14.5 - 4.5\alpha,\ x_{56} \leqslant 22 - 2\alpha,\ x_{57} \leqslant 17 - 2\alpha, \\
& \alpha \leqslant x_{68} \leqslant 12 - 2\alpha,\ x_{78} \leqslant 16.5 - 1.5\alpha,
\end{aligned}
$$

where $\alpha \in [0,1]$ and all x_{ij} are positive integers.

This problem was solved using the LINGO mixed-integer software with a cut-off value $\alpha = 0.5$. The numerical values for the optimal solution are $f1 = 236.5$, $x_{14} = 10$, $x_{34} = 7$, $x_{35} = 4$, $x_{47} = 12$, $x_{57} = 3$, $x_{68} = 10$, $x_{26} = 9$, $x_{23} = 11$, $x_{56} = 1$, and $x_{78} = x_{21} = 0$.

3. Fuzzy multiple objective MCF problems

For a problem with k objectives, Eq. (2) is replaced by

$$
\begin{aligned}
\text{Min/Max} \quad & f^k(\boldsymbol{x}) = \sum_{(I,j)\in A} c_{ij}^{k\mathrm{T}} x_{ij}, \quad k = 1,\ldots,K \\
\text{s.t.} \quad & \sum_{\{j:(i,j)\in A\}} x_{ij} - \sum_{\{j:(j,i)\in A\}} x_{ji} = b(i),\ \forall i \in \boldsymbol{N}, \\
& l_{ij} \leqslant x_{ij} \leqslant u_{ij}, \quad \forall (i,j) \in \boldsymbol{A}, \\
& x_{ij} \geqslant 0 \text{ and integer}, \quad \forall (i,j) \in \boldsymbol{A}.
\end{aligned} \tag{8}
$$

where $i = 1,\ldots,m$; $j = 1,\ldots,n$.

If the possibility concept is applied to handle the multiple objectives, the one which has the minimum objective will dominate the solution, which is obviously undesirable. Thus, some other approach is needed to handle the multiple objectives. We shall use the ideal solution and the anti-ideal solution concept [11, 21]. The ideal solution, or positive ideal solution (PIS) is obtained by solving each objective function independently subject to the constraints of the original problem. The anti-ideal solution, or negative ideal solution (NIS) is the worst solution and, if the objective is maximization, the problem is solved by minimizing the objective independently subject to the constraints of the original problem. Using this concept, the fuzzy multiple objective

MCF problem, Eq. (8), becomes

$$\begin{aligned}
\text{Min} \quad & \left\{ \sum_k [(f^k(\boldsymbol{x}^+) - f^k(\boldsymbol{x}))/(f^k(\boldsymbol{x}^+) - f^k(\boldsymbol{x}^-))]^p \right\}^{1/p}, \\
& k = 1,\dots,K \\
\text{s.t.} \quad & \sum_{\{j:(i,j)\in A\}} x_{ij} - \sum_{\{j:(j,i)\in A\}} x_{ji} = b(i), \quad \forall i \in \boldsymbol{N}, \\
& l_{ij} \leqslant x_{ij} \leqslant u_{ij}, \quad \forall (i,j) \in \boldsymbol{A}, \\
& x_{ij} \geqslant 0 \text{ and integer}, \quad \forall (i,j) \in \boldsymbol{A}.
\end{aligned} \tag{9}$$

where $i=1,\dots,m$; $j=1,\dots,n$. The reference functions $f^k(\boldsymbol{x}^+)$ and $f^k(\boldsymbol{x}^-)$ are the ideal and anti-ideal solutions, respectively. Using Eqs. (7) and (9), we can obtain the following desired crisp problem for the fuzzy multiple objective MCF problem:

$$\begin{aligned}
\text{Min} \quad & h(\boldsymbol{x}) = \left\{ \sum_k [(f^k(\boldsymbol{x}^+(-f^k(\boldsymbol{x}))/(f^k(\boldsymbol{x}^+) - f^k(\boldsymbol{x}^-))]^p \right\}^{1/p}, \quad k = 1,\dots,k \\
\text{s.t.} \quad & \alpha(l_{ij2} - l_{ij1}) + l_{ij1} \leqslant x_{ij} \leqslant u_{ij4} - \alpha(u_{ij2} - u_{ij1}), \quad \forall i \text{ and } j, \\
& \sum_i \sum_j c_{ij1}^{\mathrm{T}} x_{ij} \leqslant f^1 \leqslant \sum_i \sum_j c_{ij2}^{\mathrm{T}} x_{ij}, \quad \text{and } \theta_1^1 \geqslant \alpha, \quad \forall i \text{ and } j, \\
& \left(\sum_i \sum_j c_{ij3}^{\mathrm{T}} x_{ij} \leqslant f^1 \leqslant \sum_i \sum_j c_{ij4}^{\mathrm{T}} x_{ij}, \text{and} \quad \theta_2^1 \geqslant \alpha, \text{for maximization objective} \right) \\
& \dots \\
& \sum_i \sum_j c_{ij1}^{\mathrm{T}} x_{ij} \leqslant f^K \leqslant \sum_i \sum_j c_{ij2}^{\mathrm{T}} x_{ij}, \quad \text{and } \theta_1^K \geqslant \alpha, \quad \forall i \text{ and } j, \\
& \left(\sum_i \sum_j c_{ij3}^{\mathrm{T}} x_{ij} \leqslant f^K \leqslant \sum_i \sum_j c_{ij4}^{\mathrm{T}} x_{ij}, \text{ and} \quad \theta_2^K \geqslant \alpha, \text{for maximization objective} \right) \\
& x_{ij} \geqslant 0 \text{ and integer},
\end{aligned} \tag{10}$$

where $i=1,\dots,m$ and $j=1,\dots,n$.

Example 2. A multiple objective problem

Consider a two objectives problem with the same network flow structure as in Example 1. The first objective is the same as that in Example 1 and the second objective is to minimize the total passing time. The data used for this problem is listed in Table 2.

We must solve four independent problems first to obtain the ideal and the anti-ideal solutions. Let $f1$ and $f2$ be the ideal solutions of the first and the second objectives, respectively; and let $f3$ and $f4$ be the anti-ideal solutions of the first and second objectives, respectively. The first problem is the same as in

Table 2
Numerical parameters for Examples 2 and 3

Node no.	Supply/demand	Arc no.	*1st Objective* Fuzzy cost	*2nd Objective* Fuzzy time
❶	10	①	(0.5, 1, 1.5, 2)	(2, 2, 2.5, 3)
❷	20	②	(0, 0, 0.5, 1)	(1, 2, 2.5, 3)
❸	0	③	(5, 6, 7, 8)	(5, 6, 7, 8)
❹	−5	④	(1.5, 2, 2.5, 3)	(2, 2, 3, 3)
❺	0	⑤	(0.5, 1, 1.5, 2)	(1.2, 2, 2, 2.5)
❻	0	⑥	(3, 4, 5, 6)	(1.5, 2, 2, 2.5)
❼	−15	⑦	(4, 5, 6, 7)	(6, 7, 7.5, 8)
❽	−10	⑧	(1.5, 2, 2.5, 3.5)	(1, 2, 2.5, 3)
		⑨	(6, 7, 8, 9)	(1, 2, 2.5, 3)
		⑩	(7, 8, 9, 10)	(2, 2.5, 3, 3.5)
		⑪	(8, 9, 10, 11.5)	(2, 2.2, 3, 3.5)

Example 1. The remaining three problems are

$$\begin{aligned}
\text{Min}\quad & f2\\
\text{s.t.}\quad & (f2-2x_{21}-x_{23}-5x_{26}-2x_{14}-1.2x_{34}-1.5x_{35}-6x_{47}-x_{56}-x_{57}-2x_{68}-2x_{78})/\\
& (x_{23}+x_{26}+0.8x_{34}+0.5x_{35}+x_{47}+x_{56}+0.5x_{68}+0.2x_{78})\geqslant\alpha,\\
& 2x_{21}+x_{23}+5x_{26}+2x_{14}+1.2x_{34}+1.5x_{35}+6x_{47}+x_{56}+x_{57}+2x_{68}+2x_{78}\leqslant f2,\\
& f2\leqslant 2x_{21}+2x_{23}+6x_{26}+2x_{14}+2x_{34}+2x_{35}+7x_{47}+2x_{56}+2x_{57}+2.5x_{68}+2.2x_{78},\\
& x_{14}-x_{21}=10,\ x_{21}+x_{23}+x_{26}=20, x_{34}+x_{35}-x_{23}=0,\ x_{47}-x_{14}-x_{34}=-5,\\
& x_{56}+x_{57}-x_{35}=0,\ x_{68}-x_{56}-x_{26}=0, x_{78}-x_{47}-x_{57}=-15, -x_{68}-x_{78}=-10,
\end{aligned}$$

$$\begin{aligned}
x_{21} &\leqslant 11-2\alpha,\ \alpha\leqslant x_{23}\leqslant 13-4\alpha,\ x_{26}\leqslant 11-2\alpha, 2\alpha\\
&\leqslant x_{14}\leqslant 16-\alpha,\ x_{34}\leqslant 9.5-4.5\alpha, X_{35}\\
&\leqslant 12-2\alpha,\ 2\alpha\leqslant x_{47}\leqslant 14.5-4.5\alpha, X_{56}\\
&\leqslant 22-2\alpha,\ x_{57}\leqslant 17-2\alpha,\\
\alpha &\leqslant x_{68}\leqslant 12-2\alpha,\ x_{78}\leqslant 16.5-1.5\alpha,\ \alpha\in[0,1] \text{ and all } x_{ij} \text{ are positive integers.}
\end{aligned}$$

$$\begin{aligned}
\text{Max}\quad & f3\\
\text{s.t.}\quad & (2x_{21}+x_{23}+8x_{26}+3x_{14}+2x_{34}+6x_{35}+7x_{47}+3.5x_{56}+9x_{57}+10x_{68}+11.5x_{78}\\
& -f_3)/0.5x_{21}+0.5x_{23}+x_{26}+0.5x_{14}+0.5x_{34}+x_{35}+x_{47}+x_{56}+x_{57}+x_{68}+1.5x_{78})\geqslant\alpha,\\
& 1.5x_{21}+0.5x_{23}+7x_{26}+2.5x_{14}+1.5x_{34}+5x_{35}+6x_{47}+2.5x_{56}+8x_{57}+9x_{68}+10x_{78}\leqslant f3,\\
& f3\leqslant 2x_{21}+x_{23}+8x_{26}+3x_{14}+2x_{34}+6x_{35}+7x_{47}+3.5x_{56}+9x_{57}+10x_{68}+11.5x_{78},\\
& x_{14}-x_{21}=10, x_{21}+x_{23}+x_{26}=20, x_{34}+x_{35}-x_{23}=0, x_{47}-x_{14}-x_{34}=-5,\\
& x_{56}+x_{57}-x_{35}=0,\ x_{68}-x_{56}-x_{26}=0, x_{78}-x_{47}-x_{57}=-15, -x_{68}-x_{78}=-10,
\end{aligned}$$

Table 3
PIS/NIS sets for Example 2

	Min $f1$ (PIS)	Min $f2$ (PIS)	Max $f3$ (NIS)	Max $f4$ (NIS)	
Objective value	236.50	176.60	409.75	293.25	Note
①	0	0	5	5	x_{21}
②	11	11	11	11	x_{23}
③	9	9	4	4	x_{26}
Arc ④	10	10	15	15	x_{14}
no. ⑤	7	0	0	2	x_{34}
⑥	4	11	11	9	x_{35}
⑦	12	5	10	12	x_{47}
⑧	1	0	0	0	x_{56}
⑨	3	11	11	9	x_{57}
⑩	10	9	4	4	x_{68}
⑪	0	1	6	6	x_{78}

$$x_{21} \leqslant 11-2\alpha,\quad \alpha \leqslant x_{23} \leqslant 13-4\alpha, x_{26} \leqslant 11-2\alpha,\quad 2\alpha \leqslant x_{14} \leqslant 16-\alpha,$$
$$x_{34} \leqslant 9.5-4.5\alpha,\quad x_{35} \leqslant 12-2\alpha, 2\alpha \leqslant x_{47} \leqslant 14.5-4.5\alpha,\quad x_{56} \leqslant 22-2\alpha,$$
$$x_{57} \leqslant 17-2\alpha,\quad \alpha \leqslant x_{68} \leqslant 12-2\alpha,\quad x_{78} \leqslant 16.5-1.5\alpha,\quad \alpha \in [0,1]$$

and all x_{ij} are positive integers.

$$\begin{aligned}
\text{Max}\quad & f4\\
\text{s.t.}\quad & (f4 - 3x_{21} - 3x_{23} - 8x_{26} - 3x_{14} - 3x_{34} - 2.5x_{35} - 8x_{47} - 3x_{57} - 3.5x_{68} - 4x_{78})/\\
& (0.5x_{21} + 0.5x_{23} + x_{26} + 0.5x_{14} + 3x_{34} + 0.5x_{35} + 0.5x_{47} + 0.5x_{57} + 0.5x_{68} + x_{78}) \geqslant \alpha,\\
& 2.5x_{21} + 2.5x_{23} + 7x_{26} + 2.5x_{14} + 3x_{34} + 2x_{35} + 7.5x_{47} + 2.5x_{56} + 2.5x_{57} + 3x_{68} + 3x_{78} \leqslant f4,\\
& f4 \leqslant 3x_{21} + 3x_{23} + 8x_{26} + 3x_{14} + 3x_{34} + 2.5x_{35} + 8x_{47} + 3x_{56} + 3x_{57} + 3.5x_{68} + 4x_{78},\\
& x_{14} - x_{21} = 10,\ x_{21} + x_{23} + x_{26} = 20, x_{34} + x_{35} - x_{23} = 0,\ x_{47} - x_{14} - x_{34} = -5,\\
& x_{56} + x_{57} - x_{35} = 0,\ x_{68} - x_{56} - x_{26} = 0, x_{78} - x_{47} - x_{57} = -15,\ -x_{68} - x_{78} = -10,\\
& x_{21} \leqslant 11-2\alpha,\ \alpha \leqslant x_{23} \leqslant 13-4\alpha, x_{26} \leqslant 11-2\alpha,\ 2\alpha \leqslant x_{14} \leqslant 16-\alpha,\\
& x_{34} \leqslant 9.5-4.5\alpha,\ x_{35} \leqslant 12-2\alpha, 2\alpha \leqslant x_{47} \leqslant 14.5-4.5\alpha,\ x_{56} \leqslant 22-2\alpha,\\
& x_{57} \leqslant 17-2\alpha,\ \alpha \leqslant x_{68} \leqslant 12-2\alpha, x_{78} \leqslant 16.5-1.5\alpha,\ \alpha \in [0,1]
\end{aligned}$$

and all x_{ij} are positive integers.

With a cut-off value α=0.5, the optimal objectives obtained are $f1 = 236.50$ (NIS), $f2 = 176.60$ (PIS), $f3 = 409.75$ (anti-ideal solution or NIS), $f4 = 293.25$ (anti-ideal solution or NIS). The detailed solutions are listed in Table 3. Notice that the maximization problem uses the right-hand side of the trapezoidal fuzzy numbers instead of the left-hand side.

This problem is solved with the parameter $p = 1$. Using Eq. (10), we have

$$\begin{aligned}\text{Min}\quad h(\boldsymbol{x}) &= (f1(\boldsymbol{x}^+) - f1(\boldsymbol{x}))/(f1(\boldsymbol{x}^+) - f3(\boldsymbol{x}^-)) + (f2(\boldsymbol{x}^+) - f2(\boldsymbol{x}^-))/(f2(\boldsymbol{x}^+) - f4(\boldsymbol{x}^-))\\ &= 0.005772 f1(x) + 0.008573 f2(x) - 2.87901\end{aligned}$$

$$\begin{aligned}\text{s.t.}\quad & (f1 - 0.5x_{21} - 5x_{26} - 1.5x_{14} - 0.5x_{34} - 3x_{35} - 4x_{47} - 1.5x_{56} - 6x_{57} - 7x_{68}\\ &\quad - 8x_{78})/(0.5x_{21} + x_{26} + 0.5x_{14} + 0.5x_{34} + x_{35} + x_{47} + x_{68} + x_{78}) \geqslant \alpha,\\ & 0.5x_{21} + 5x_{26} + 1.5x_{14} + 0.5x_{34} + 3x_{35} + 4x_{47} + 1.5x_{56} + 6x_{57} + 7x_{68} + 8x_{78} \leqslant f1,\\ & f1 \leqslant x_{21} + 6x_{26} + 2x_{14} + x_{34} + 4x_{35} + 5x_{47} + 2x_{56} + 7x_{57} + 8x_{68} + 9x_{78},\\ & (f2 - 2x_{21} - x_{23} - 5x_{26} - 2x_{14} - 1.2x_{34} - 1.5x_{35} - 6x_{47} - x_{56} - x_{57} - 2x_{68} - 2x_{78})/\\ &\quad (x_{23} + x_{26} + 0.8x_{34} + 0.5x_{35} + x_{47} + x_{56} + 0.5x_{68} + 0.2x_{78}) \geqslant \alpha,\\ & 2x_{21} + x_{23} + 5x_{26} + 2x_{14} + 1.2x_{34} + 1.5x_{35} + 6x_{47} + x_{56} + x_{57} + 2x_{68} + 2x_{78} \leqslant f2,\\ & f2 \leqslant 2x_{21} + 2x_{23} + 6x_{26} + 2x_{14} + 2x_{34} + 2x_{35} + 7x_{47} + 2x_{56} + 2x_{57} + 2.5x_{68} + 2.2x_{78},\\ & x_{14} - x_{21} = 10,\ x_{21} + x_{23} + x_{26} = 20,\ x_{34} + x_{35} - x_{23} = 0,\ x_{47} - x_{14} - x_{34} = -5,\\ & x_{56} + x_{57} - x_{35} = 0,\ x_{68} - x_{56} - x_{26} = 0, x_{78} - x_{47} - x_{57} = -15, -x_{68} - x_{78} = -10,\\ & x_{21} \leqslant 11 - 2\alpha,\ \alpha \leqslant x_{23} \leqslant 13 - 4\alpha, x_{26} \leqslant 11 - 2\alpha,\ 2\alpha \leqslant x_{14} \leqslant 16 - \alpha,\\ & x_{34} \leqslant 9.5 - 4.5\alpha,\ x_{35} \leqslant 12 - 2\alpha, 2\alpha \leqslant x_{47} \leqslant 14.5 - 4.5\alpha,\ x_{56} \leqslant 22 - 2\alpha,\\ & x_{57} \leqslant 17 - 2\alpha, \alpha \leqslant x_{68} \leqslant 12 - 2\alpha, x_{78} \leqslant 16.5 - 1.5\alpha,\\ & \alpha \in [0, 1] \text{ and all } x_{ij} \text{ are positive integers.}\end{aligned}$$

With the cut-off value $\alpha = 0.5$, the above problem becomes a mixed integer problem which was solved by the LINGO mixed-integer code. Notice that the original two objective function spaces were added to the constraint set of the auxiliary problem. The optimal values are $x_{26} = 9$, $x_{14} = 10$, $x_{35} = 11$, $x_{47} = 5$, $x_{56} = 1, x_{57} = 10$, $x_{68} = 10$, $x_{23} = 11$, and $x_{34} = x_{78} = x_{21} = 0$, with $f1 = 269.75$ and $f2 = 176.75$. The solution is a compromise between the minimum cost and the minimum time. We also solved this multi-objective problem with the distance parameter, $p = \infty$. The optimum values are $x_{26} = 9$, $x_{14} = 10$, $x_{34} = 3$, $x_{35} = 8$, $x_{47} = 8, x_{56} = 1$, $x_{57} = 7$, $x_{68} = 10$, $x_{23} = 11$, and $x_{21} = x_{78} = 0$, with $f1 = 255.5$ and $f2 = 191.3$. Both solutions are located at the non-dominated boundary.

4. Fuzzy multiple level MCF problem

For more practical applications, let us consider a decentralized planning problem in which multiple agents with some interactions participate in the decision making process. The agents are located in a multiple level or hierarchy structure. Based on the work of Shih, et al. [19], a multi-level MCF problem can be represented as

$$\text{Min/Max}\quad f^1(\boldsymbol{x}) = \sum_{(i,j)\in A} c_{ij}^{1\mathrm{T}} x_{ij}, \quad \text{(1st level)} \tag{11}$$

where $x_{ij}^2, x_{ij}^3, \ldots, x_{ij}^K$ and solve

$$\text{Min/Max} f^2(\boldsymbol{x}) = \sum_{(i,j)\in A} c_{ij}^{2\mathrm{T}} x_{ij}, \quad \text{(2nd level)}$$

where $x_{ij}^3,\dots,x_{ij}^K$ and solve $\cdots$

$$\text{Min/Max} \quad f^k(x) = \sum_{(i,j)\in A} c_{ij}^{KT} x_{ij}, \quad (K\text{th level})$$

$$\text{s.t.} \quad \sum_{\{j:(i,j)\in A\}} x_{ij} - \sum_{\{j:(j,i)\in A\}} x_{ji} = b(i),\ \forall i\in \boldsymbol{N}, \quad l_{ij}\leqslant x_{ij}\leqslant u_{ij}, \quad \forall (i,j)\in \boldsymbol{A}, x_{ij}\geqslant 0 \text{ and integer}, \quad \forall (i,j)\in \boldsymbol{A}.$$

where $x_{ij} = x_{ij}^2 + x_{ij}^2 + x_{ij}^3 + \cdots + x_{ij}^k$; $i=1,\dots,m; j=1,\dots,n$; and $k=1,\dots,K$. Notice that there is only one objective which is either minimization or maximization at any one decision level.

The multi-level programming problem is a very difficult problem to solve and it has been proved to be an NP-hard problem. However, this problem has been solved under fuzzy assumptions [19]. This procedure is based on the concepts of tolerance membership functions. First, the upper-level decision maker defines his or her objectives with some tolerances which are described by fuzzy membership functions. The lower level decision maker makes his or her decision based on this tolerance. For the minimization MCF problem, the goal can be modified as

$$\mu_{f^1}(f^1(\boldsymbol{x})) = \begin{cases} 1, & \text{if } f^1(\boldsymbol{x}) < f^{1\prime}, \\ [f^{1\mathrm{U}} - f^1(\boldsymbol{x})]/[f^{1\mathrm{U}} - f^{1\prime}], & \text{if } f^{1\prime} \leqslant f^1(x) \leqslant f^{1\mathrm{U}}, \\ 0, & \text{if } f^1(x) > f^{1\mathrm{U}}, \end{cases} \tag{12}$$

where $f^{1\prime}$ and $f^{1\mathrm{U}}$ are the acceptable range for the goal for the upper-level decision maker.

The upper level decision maker also sets the acceptable tolerances for his or her decisions:

$$\mu_{xij}(x_{ij}) = \begin{cases} [x_{ij} - (x_{ij}^{\mathrm{U}} - p_{ij})]/p_{ij}, & \text{if } x_{ij}^{\mathrm{U}} - p_{ij} \leqslant x_{ij} \leqslant x_{ij}^{\mathrm{U}}, \\ [(x_{ij}^{\mathrm{U}} + p_{ij}) - x_{ij})]/p_{ij}, & \text{if } x_{ij}^{\mathrm{U}} < x_{ij} \leqslant x_{ij}^{\mathrm{U}} + p_{ij}, \\ 0, & \text{otherwise}, \end{cases} \tag{13}$$

where p_{ij} is the two-sided tolerance for the decision variable x_{ij}. The decision variables for the upper-level decision maker will be represented by x_{ij}^1, $\forall i$ and j.

These fuzzy informations then restrict the feasible space for the lower-level decision maker. For each possible solution available to the upper-level decision maker, the lower-level decision maker defines his or her goal as

$$\mu_{f^2}(f^2(\boldsymbol{x})) = \begin{cases} 1, & \text{if } f^2(\boldsymbol{x}) < f^{2\prime}, \\ [f^{2L} - f^2(\boldsymbol{x})]/[f^{2L} - f^{2\prime}], & \text{if } f^{2\prime} \leqslant f^2(\boldsymbol{x}) \leqslant f^{2\mathrm{L}}, \\ 0, & \text{if } f^2(\boldsymbol{x}) > f^{2\mathrm{L}}, \end{cases} \tag{14}$$

where $f^{2\prime}$ and $f^{2\mathrm{L}}$ are the acceptable range for the goal for the lower-level decision maker.

To formulate the fuzzy multi-level MCF problem, we must consider two different fuzzy aspects: the imprecise parameters which will be handled by the possibility concept and the fuzzy aspects due to the multi-level structure which will be handled by multi-level formation due to Shih et al. [19]. To simplify the discussion, we shall consider a fuzzy bi-level MCF problem. Using the possibility approach and trapezoidal fuzzy numbers, the imprecisions in the parameters can be handled by the following representation:

$$\text{Min/Max} \quad f^1(x_{ij}^1, x_{ij}^2) \tag{15}$$

where x_{ij}^2 solves

$$\begin{aligned}
\text{Min/Max}\quad & f^2(x_{ij}^1, x_{ij}^2)\\
\text{s.t.}\quad & \alpha(l_{ij2}-l_{ij1})+l_{ij1}\leqslant x_{ij}\leqslant u_{ij4}-\alpha(u_{ij2}-u_{ij1}),\quad \forall i \text{ and } j,\\
& \sum_i\sum_j c_{ij1}^{\mathrm{T}}x_{ij}\leqslant f^1\leqslant \sum_i\sum_j c_{ij2}^{\mathrm{T}}x_{ij},\ \text{and } \theta_1^1\geqslant\alpha,\quad \forall i \text{ and } j,\\
& \left(\sum_i\sum_j c_{ij3}^{\mathrm{T}}x_{ij}\leqslant f^1\leqslant \sum_i\sum_j c_{ij4}^{\mathrm{T}}x_{ij},\ \text{and } \theta_2^1\geqslant\alpha,\ \text{for maximization}\right)\\
& \sum_i\sum_j c_{ij1}^{\mathrm{T}}x_{ij}\leqslant f^2\leqslant \sum_i\sum_j c_{ij2}^{\mathrm{T}}x_{ij},\ \text{and } \theta_1^2\geqslant\alpha,\quad \forall i \text{ and } j,\\
& \left(\sum_i\sum_j c_{ij3}^{\mathrm{T}}x_{ij}\leqslant f^2\leqslant \sum_i\sum_j c_{ij4}^{\mathrm{T}}x_{ij},\ \text{and } \theta_2^2\geqslant\alpha,\ \text{for maximization}\right)\\
& \alpha\in[0,1],\ x_{ij}\geqslant 0 \text{ and integer},
\end{aligned}$$

where $x_{ij}=x_{ij}^1+x_{ij}^2$; $i=1,\dots,m$; and $j=1,\dots,n$. The two objectives are

$$f^1=\sum\sum c_{ij}^{11\mathrm{T}}x_{ij}^1+c_{ij}^{12\mathrm{T}}x_{ij}^2,\quad \forall i \text{ and } j$$

and

$$f^2=\sum\sum c_{ij}^{21\mathrm{T}}x_{ij}^1+c_{ij}^{22\mathrm{T}}x_{ij}^2,\quad \forall i \text{ and } j.$$

For the second fuzzy aspect, using Eqs. (12)–(14) we can reformulate Eq. (15) as

$$\begin{aligned}
\text{Min/Max}\quad & f^2(x_{ij}^1, x_{ij}^2) \qquad (16)\\
\text{s.t.}\quad & (f^1(x_{ij}^+)-f^1)/(f^1(x_{ij}^+)-f^{1\prime})\geqslant\lambda^2,\\
& [x_{ij}^1-(x_{ij}^{1\mathrm{U}}-p_{ij})]/p_{ij}\geqslant\lambda_{ij}^2, [(x_{ij}^{1\mathrm{U}}+p_{ij})-x_{ij}^1]/p_{ij}\geqslant\lambda_{ij}^2,\ \forall i \text{ and } j\\
& (f^2(x_{ij}^+)-f^2)/(f^2(x_{ij}^+)-f^{2\prime})\geqslant\lambda^3,\\
& ((f^2-f^{2\prime})/(f^2(x_{ij}^+)-f^{2\prime})\geqslant\lambda^3,\ \text{for maximization})\\
& \alpha(l_{ij2}-l_{ij1})+l_{ij1}\leqslant x_{ij}\leqslant u_{ij4}-\alpha(u_{ij2}-u_{ij1}),\quad \forall i \text{ and } j,\\
& \sum_i\sum_j c_{ij1}^{\mathrm{T}}x_{ij}\leqslant f^1\leqslant \sum_i\sum_j c_{ij2}^{\mathrm{T}}x_{ij},\ \text{and } \theta_1^1\geqslant\alpha,\quad \forall i \text{ and } j,\\
& \left(\sum_i\sum_j c_{ij3}^{\mathrm{T}}x_{ij}\leqslant f^1\leqslant \sum_i\sum_j c_{ij4}^{\mathrm{T}}x_{ij},\ \text{and } \theta_2^1\geqslant\alpha,\ \text{for maximization}\right)\\
& \sum_i\sum_j c_{ij1}^{\mathrm{T}}x_{ij}\leqslant f^2\leqslant \sum_i\sum_j c_{ij2}^{\mathrm{T}}x_{ij},\ \text{and } \theta_1^2\geqslant\alpha,\quad \forall i \text{ and } j,
\end{aligned}$$

$$\left(\sum_i \sum_j c_{ij3}^{\mathrm{T}} x_{ij} \leqslant f^2 \leqslant \sum_i \sum_j c_{ij4}^{\mathrm{T}} x_{ij}, \text{ and } \theta_2^2 \geqslant \alpha, \text{ for maximization}\right)$$

$$\lambda^1, \lambda_{ij}^2, \lambda^3, \alpha, \theta_1^1, \theta_1^2 \in [0,1], \quad \forall i \text{ and } j,$$

$$(\theta_2^1, \theta_2^2 \in [0,1] \text{ for maximization})$$

$$x_{ij} \geqslant 0 \text{ and integers,}$$

where $x_{ij} = x_{ij}^1 + x_{ij}^2$. The variables x_{ij}^1 and x_{ij}^2 are the decision variable sets for the upper-level and lower-level decision makers, respectively. p_{ij} is the two-sided tolerance for the decision variable x_{ij}. The two level objectives are $f^1 = \sum c_{ij}^{11\mathrm{T}} x_{ij}^1 + c_{ij}^{12\mathrm{T}} x_{ij}^2$ and $f^2 = \sum c_{ij}^{21\mathrm{T}} x_{ij}^1 + c_{ij}^{22\mathrm{T}} x_{ij}^2$, with $i = 1, \ldots, m$, $j = 1, \ldots, n$.

For the minimization of the total degree of satisfaction, i.e. $\lambda = \min\{\lambda_{ij}^1, \lambda^2, \lambda^3\}$, $\forall i = 1, \ldots, m$, $j = 1, \ldots, n$, Eq. (16) can be transformed into

$$\begin{aligned}
\text{Max} \quad & \lambda \qquad (17)\\
\text{s.t.} \quad & (f^1(x_{ij}^+) - f^1)/(f^1(x_{ij}^+) - f^{1\prime}) \geqslant \lambda,\\
& [x_{ij}^1 - (x_{ij}^{1\,\mathrm{U}} - p_{ij})]/p_{ij} \geqslant \lambda, [(x_{ij}^{1\,\mathrm{U}} + p_{ij}) - x_{ij}^1]/p_{ij} \geqslant \lambda, \quad \forall i \text{ and } j\\
& (f^2(x_{ij}^+) - f^2)/(f^2(x_{ij}^+) - f^{2\prime}) \geqslant \lambda,\\
& ((f^2 - f^{2\prime})/(f^2(x_{ij}^+) - f^{2\prime}) \geqslant \lambda, \text{ for maximization})\\
& \alpha(l_{ij2} - l_{ij1}) + l_{ij1} \leqslant x_{ij} \leqslant u_{ij4} - \alpha(u_{ij2} - u_{ij1}), \quad \forall i \text{ and } j,\\
& \sum_i \sum_j c_{ij1}^{\mathrm{T}} x_{ij} \leqslant f^1 \leqslant \sum_i \sum_j c_{ij2}^{\mathrm{T}} x_{ij}, \text{ and } \theta_1^1 \geqslant \alpha, \quad \forall i \text{ and } j,\\
& \left(\sum_i \sum_j c_{ij3}^{\mathrm{T}} x_{ij} \leqslant f^1 \leqslant \sum_i \sum_j c_{ij4}^{\mathrm{T}} x_{ij}, \text{ and } \theta_2^1 \geqslant \alpha, \text{ for maximization}\right)\\
& \sum_i \sum_j c_{ij1}^{\mathrm{T}} x_{ij} \leqslant f^2 \leqslant \sum_i \sum_j c_{ij2}^{\mathrm{T}} x_{ij}, \text{ and } \theta_1^2 \geqslant \alpha, \quad \forall i \text{ and } j,\\
& \left(\sum_i \sum_j c_{ij3}^{\mathrm{T}} x_{ij} \leqslant f^2 \leqslant \sum_i \sum_j c_{ij4}^{\mathrm{T}} x_{ij}, \text{ and } \theta_2^2 \geqslant \alpha, \text{ for maximization}\right)\\
& \lambda, \alpha, \theta_1^1, \theta_1^2 \in [0,1], \quad \forall i \text{ and } j,\\
& (\theta_2^1, \theta_2^2 \in [0,1] \text{ for maximization})\\
& x_{ij} \geqslant 0 \text{ and integers.}
\end{aligned}$$

Obviously, the approach can be extended to solve problems which have more than two levels

Example 3. A bi-level fuzzy MCF problem

Using the same data as that used in Example 2 and Table 2, a two level decision making problem was solved. The two objectives used in Example 2 are treated as in two levels. The problem is to minimize the total cost f^1 for the upper level decision maker and to minimize the passing time f^2 for the lower level decision maker within the tolerance of the upper level. Based on the information listed in Table 3, the fuzzy range for each objective can be established as: $f^1 \in [236.5, 409.75]$ and $f^2 \in [176.6, 293.25]$. The two control variables, x_{14} and x_{35}, are assumed to be within the control of the upper level decision maker. The value of the first decision, x_{14}, is around 10 with negative and positive side tolerances 6 and 5, respectively, and the value of the second, x_{35}, is around 4 with negative and positive side tolerances 4 and 7, respectively. Thus, additional constraints needed for the two level problem are:

$$x_{14} - 4 \geqslant 6\lambda_{14}^1, \qquad 15 - x_{14} \geqslant 5\lambda_{14}^1,$$

$$x_{35} \geqslant 4\lambda_{35}^1, \qquad 11 - x_{35} \geqslant 7\lambda_{35}^1.$$

Furthermore, the objective ranges of the two decision makers are

$$409.75 - f^1 \geqslant 173.25\lambda^2,$$

and

$$293.25 - f^2 \geqslant 116.65\lambda^3.$$

Substituting the above equations in Eq. (17) with $\lambda = \min\{\lambda_{14}^1, \lambda_{35}^1, \lambda^2, \lambda^3\}$, we obtain the following crisp mixed integer problem:

$$\begin{aligned}
\text{Max} \quad & \lambda \\
\text{s.t.} \quad & 15 - x_{14} \geqslant 5\lambda, x_{14} - 4 \geqslant 6\lambda, \\
& 11 - x_{35} \geqslant 7\lambda, x_{35} \geqslant 4\lambda, \\
& 409.75 - f^1 \geqslant 173.25\lambda, \\
& 239.25 - f^2 \geqslant 173.25\lambda, \\
& (f^1 - 0.5x_{21} - 5x_{26} - 1.5x_{14} - 0.5x_{34} - 3x_{35} - 4x_{47} - 1.5x_{56} - 6x_{57} - 7x_{68} - 8x_{78})/ \\
& (0.5x_{21} + x_{26} + 0.5x_{14} + 0.5x_{34} + x_{35} + x_{47} + x_{68} + x_{78}) \geqslant \alpha, \\
& 0.5x_{21} + 5x_{26} + 1.5x_{14} + 0.5x_{34} + 3x_{35} + 4x_{47} + 1.5x_{56} + 6x_{57} + 7x_{68} + 8x_{78} \leqslant f^1, \\
& f^1 \leqslant x_{21} + 6x_{26} + 2x_{14} + x_{34} + 4x_{35} + 5x_{47} + 2x_{56} + 7x_{57} + 8x_{68} + 9x_{78}, \\
& (f^2 - 2x_{21} - x_{23} - 5x_{26} - 2x_{14} - 1.2x_{34} - 1.5x_{35} - 6x_{47} - x_{56} - x_{57} - 2x_{68} - \\
& 2x_{78})/(x_{23} + x_{26} + 0.8x_{34} + 0.5x_{35} + x_{47} + x_{56} + 0.5x_{68} + 0.2x_{78}) \geqslant \alpha, \\
& 2x_{21} + x_{23} + 5x_{26} + 2x_{14} + 1.2x_{34} + 1.5x_{35} + 6x_{47} + x_{56} + x_{57} + 2x_{68} + 2x_{78} \leqslant f^2, \\
& f^2 \leqslant 2x_{21} + 2x_{23} + 6x_{26} + 2x_{14} + 2x_{34} + 2x_{35} + 7x_{47} + 2x_{56} + 2x_{57} + 2.5x_{68} + 2.2x_{78},
\end{aligned}$$

$$x_{14} - x_{21} = 10, x_{21} + x_{23} + x_{26} = 20, x_{34} + x_{35} - x_{23} = 0, x_{47} - x_{14} - x_{34} = -5,$$

$$x_{56} + x_{57} - x_{35} = 0, x_{68} - x_{56} - x_{26} = 0, x_{78} - x_{47} - x_{57} = -15, -x_{68} - x_{78} = -10,$$

$$x_{21} \leqslant 11 - 2\alpha, \quad \alpha \leqslant x_{23} \leqslant 13 - 4\alpha, x_{26} \leqslant 11 - 2\alpha,$$

$$2\alpha \leqslant x_{14} \leqslant 16 - \alpha, x_{34} \leqslant 9.5 - 4.5\alpha, \quad x_{35} \leqslant 12 - 2\alpha,$$

$$2\alpha \leqslant x_{47} \leqslant 14.5 - 4.5\alpha, \; x_{56} \leqslant 22 - 2\alpha, x_{57} \leqslant 17 - 2\alpha, \quad \alpha \leqslant x_{68} \leqslant 12 - 2\alpha, x_{78} \leqslant 16.5 - 1.5\alpha, \quad \lambda$$

and $\alpha \in [0,1]$, and all x_{ij} are positive integers.

Assuming the decision maker provided a cut-off value of 0.5, the above mixed integer problem was solved. The optimal degree of satisfaction is 0.429 with the optimal values for the objectives $f^1 = 260.5$ and $f^2 = 187.2$. The optimal flows are: $x_{14} = 10$, $x_{35} = 8$, $x_{26} = 10$, $x_{34} = 2$, $x_{47} = 7$, $x_{57} = 8$, $x_{68} = 10$, $x_{23} = 10$, and $x_{21} = x_{56} = x_{78} = 0$. This solution satisfies the upper level first and then, within the specified tolerance decided by the upper level, optimizes the decision of the second level. Thus, the solution should meet the requirements in the multiple level hierarchy structure.

5. Compensatory fuzzy multi-level MCF problem

Although the above max–min approach is most frequently used, it is not compensatory. The decisions of managements are usually compensatory. To overcome this problem, Zimmermann and Zysno [22] proposed a compensatory *and* operator, which is a combination of the product and the algebraic sum with the parameter γ. The aggregated membership function, μ_θ, by the aggregation of m elements are

$$\mu_\theta = \left(\prod_i \mu_i\right)^{1-\gamma} \left[1 - \prod_i (1 - \mu_i)\right]^{\gamma}, \quad 0 \leqslant \mu \leqslant 1, \; 0 \leqslant \gamma \leqslant 1, \tag{18}$$

where $i = 1, 2, \ldots, m$, and γ is defined as the grade of compensation.

Since this γ-model for the aggregation of different objectives will result in high non-linearity which cannot be solved easily, Luhandjula [14] suggested the following min-bounded sum operator by the use of convex combination:

$$\mu_\theta = \gamma \min_i(\mu_i) + (1-\gamma) \min\left(1, \sum_i \mu_i\right), \quad 0 \leqslant \mu \leqslant 1, \; 0 \leqslant \gamma \leqslant 1, \tag{19}$$

where $i = 1, 2, \ldots, m$, and $m =$ number of different elements to be aggregated.

Recently, Werners [20] proposed the following "fuzzy and" and "fuzzy or" operators which appear to have good results compared to the empirical data of Zimmermann and Zysno [23]:

$$\mu_{\text{and}} = \gamma \min_i(\mu_i) + (1-\gamma) \left(\sum_i \mu_i\right) \Big/ m, \quad 0 \leqslant \mu \leqslant 1, \; 0 \leqslant \gamma \leqslant 1, \tag{20}$$

$$\mu_{\text{or}} = \gamma \max_i(\mu_i) + (1-\gamma) \left(\sum_i \mu_i\right) \Big/ m, \quad 0 \leqslant \mu \leqslant 1, \; 0 \leqslant \gamma \leqslant 1. \tag{21}$$

Again, $i = 1, 2, \ldots, m$, and $m =$ number of elements to be aggregated. These two aggregators result in linear equations. The "fuzzy and" operator is similar to the two-phase approach of Lee and Li [11] which considers an average operator for compensation at the second phase. A problem in fuzzy multiple level decision making with Werners' "fuzzy and" compensatory operator was solved in the following:

Example 4. Fuzzy multi-level MCF problem with compensatory operator

The bi-level fuzzy MCF problem which was solved in Example 3 was solved again with the compensatory operator. The data used are the same as that used in Example 3. The crisp numerical problem can be represented by

$$
\begin{aligned}
\text{Max}\quad & \mu_{\text{and}} = \lambda + (1-\gamma)(\lambda^1_{14} + \lambda^1_{35} + \lambda^2 + \lambda^3)/4, \\
\text{s.t.}\quad & 15 - x_{14} \geqslant 5(\lambda + \lambda^1_{14}),\ x_{14} - 4 \geqslant 6(\lambda + \lambda^1_{14}), \\
& 11 - x_{35} \geqslant 7(\lambda + \lambda^1_{35}),\ x_{35} \geqslant 4(\lambda + \lambda^1_{35}), \\
& 409.75 - f^1 \geqslant 173.25(\lambda + \lambda^2), 239.25 - f^2 \geqslant 116.65(\lambda + \lambda^3), \\
& (f^1 - 0.5x_{21} - 5x_{26} - 1.5x_{14} - 0.5x_{34} - 3x_{35} - 4x_{47} - 1.5x_{56} - 6x_{57} - 7x_{68} - 8x_{78})/(0.5x_{21} + x_{26} \\
& + 0.5x_{14} + 0.5x_{34} + x_{35} + x_{47} + x_{68} + x_{78}) \geqslant \alpha, \\
& 0.5x_{21} + 5x_{26} + 1.5x_{14} + 0.5x_{34} + 3x_{35} + 4x_{47} + 1.5x_{56} + 6x_{57} + 7x_{68} + 8x_{78} \leqslant f^1, \\
& f^1 \leqslant x_{21} + 6x_{26} + 2x_{14} + x_{34} + 4x_{35} + 5x_{47} + 2x_{56} + 7x_{57} + 8x_{68} + 9x_{78}, \\
& (f^2 - 2x_{21} - x_{23} - 5x_{26} - 2x_{14} - 1.2x_{34} - 1.5x_{35} - 6x_{47} - x_{56} - x_{57} - 2x_{68} - 2x_{78}) \\
& /(x_{23} + x_{26} + 0.8x_{34}) + 0.5x_{35} + x_{47} + x_{56} + 0.5x_{68} + 0.2x_{78}) \geqslant \alpha, \\
& 2x_{21} + x_{23} + 5x_{26} + 2x_{14} + 1.2x_{34} + 1.5x_{35} + 6x_{47} + x_{56} + x_{57} + 2x_{68} + 2x_{78} \leqslant f^2, \\
& f^2 \leqslant 2x_{21} + 2x_{23} + 6x_{26} + 2x_{14} + 2x_{34} + 2x_{35} + 7x_{47} + 2x_{56} + 2x_{57} + 2.5x_{68} + 2.2x_{78}, \\
& x_{14} - x_{21} = 10,\ x_{21} + x_{23} + x_{26} = 20, x_{34} + x_{35} - x_{23} = 0,\ x_{47} - x_{14} - x_{34} = -5, x_{56} + x_{57} - x_{35} = 0, \\
& x_{68} - x_{56} - x_{26} = 0, x_{78} - x_{47} - x_{57} = -15,\ -x_{68} - x_{78} = -10, \\
& x_{21} \leqslant 11 - 2\alpha,\ \alpha \leqslant x_{23} \leqslant 13 - 4\alpha,\ x_{26} \leqslant 11 - 2\alpha, 2\alpha \leqslant x_{14} \leqslant 16 - \alpha,\ x_{34} \leqslant 9.5 - 4.5\alpha, \\
& x_{35} \leqslant 12 - 2\alpha,\ 2\alpha \leqslant x_{47} \leqslant 14.5 - 4.5\alpha,\ x_{56} \leqslant 22 - 2\alpha,\ x_{57} \leqslant 17 - 2\alpha,\ \alpha \leqslant x_{68} \leqslant 12 - 2\alpha,\ x_{78} \leqslant 16.5 - 1.5\alpha, \\
& \lambda^1_{14}, \lambda^1_{35}, \lambda^2, \lambda^3, \lambda \text{ and } \alpha \in [0,1], \text{ and all } x_{ij} \text{ are positive integers.}
\end{aligned}
$$

The numerical values obtained for the compromise solution are: $f^* = (f^{1*}, f^{2*}) = (255.75, 192.05), x_{14} = 10, x_{35} = 7, x_{26} = 10,\ x_{34} = 3,\ x_{47} = 8,\ x_{57} = 7, x_{68} = 10,\ x_{23} = 10,\ x_{21} = x_{56} = x_{78} = 0$ with the total degree of satisfaction $\mu_{\text{and}} = 0.560$ for a compensatory parameter value of $\gamma = 0.5$ and a fixed cut-off value $\alpha = 0.5$. To study the effect of compensation, we solved the problem with 11 different parameter values for γ with $\alpha = 0.5$. The results are summarized in Table 4.

6. Discussions

The concept of fuzzy tolerance, which is similiar to subjective possibility, and trapezoidal fuzzy numbers are used to handle the vagueness in the parameters of the MCF problem. In this way the crisp or auxiliary problem has at most $2m$ capacity constraints instead of the $3m$ constraints resulted in original approach of Negi [16]. The formulation is extended to fuzzy multiple objective problems and fuzzy multiple level systems

Table 4
"fuzzy and" operation, Example 4

Compensatory parameter γ	Degree of satisfaction μ_{mod}	λ (original)	Objective (f^1, f^2)	Note
0	0.813	0.280	(241.5, 206.6)	flow A
0.1	0.759	0.280	(241.5, 206.6)	
0.2	0.706	0.280	(241.5, 206.6)	
0.3	0.653	0.280	(241.5, 206.6)	
0.4	0.600	0.280	(241.5, 206.6)	
0.5	0.560	0.405	(255.75, 192.05)	flow B
0.6	0.531	0.429	(260.5, 187.2)	flow C
0.7	0.505	0.429	(260.5, 187.2)	
0.8	0.480	0.429	(260.5, 187.2)	
0.9	0.454	0.429	(260.5, 187.2)	
1	0.429	0.429	(260.5, 187.2)	

Flow sequences: $(x_{21}, x_{23}, x_{26}, x_{14}, x_{34}, x_{35}, x_{47}, x_{56}, x_{57}, x_{68}, x_{78})$. The numerical values are:
flow A: $(0, 10, 10, 10, 6, 4, 11, 0, 4, 10, 0)$,
flow B: $(0, 10, 10, 10, 3, 7, 8, 0, 7, 10, 0)$,
flow C: $(0, 10, 10, 10, 2, 8, 7, 0, 8, 10, 0)$.

which involves decentralized planning with interactions. These formulations and extensions should be very useful for attacking practical network problems.

Both the possibilistic approach and Zimmermann's preference-based model are used to solve the fuzzy multi-level MCF problem. The possibilistic approach is used for the modeling the parameter imprecision and the preference-based model is used for simplifying the multi-level structure. The concept of imprecise structure, due to Chanas et al. [8], appears to be a another useful approach for multi-level network problems.

The proposed approach can handle network problems in the fuzzy domain with the resulting problem being mixed-integer or nonlinear problem. One advantage of the approach is that the whole solution procedure is independent of the structure of the system. For example, we can use a nonlinear code to solve the resulting nonlinear programming problem instead of some complicated searching procedure to search the network.

All the examples in this paper are solved by the LINGO mix-integer code. However, this code may be inefficient for handling large scale problems. Many of the special codes can only handle all integer problems and they are not suited for mixed integer problem resulting from the MCF problem. Although we can process the non-integer variables by scaling, e.g., multiply all values by 1000 to make the variables integers [10], the approach is not a good method for handle fuzzy problems.

References

[1] R.K. Ahuja, T.L. Magnanti, J.B. Orlin, Network Flows, Prentice-Hall, Englewood Cliffs, NJ, 1993.
[2] J.F. Bard, Coordination of a multidivisional organization through two levels of management, Omega 11 (1983) 457–486.
[3] O. Ben-Ayed, Bilevel linear programming, Comput. Ops. Res. 20 (1993) 485–501.
[4] W.F. Bialas, M.H. Karwan, On two-level optimization, IEEE Trans. Autom. Control AC-27 (1982) 211–214.
[5] W.F. Bialas, M.K. Karwan, Two-level linear programming, Management Sci. 30 (1984) 1004–1020.
[6] J.J. Buckley, Possibilistic linear programming with triangular fuzzy numbers, Fuzzy Sets and Systems 20 (1) (1988) 135–138.
[7] J.J. Buckley, Solving possibilistic linear programming problems, Fuzzy Sets and Systems 31 (3) (1989) 329–341.
[8] S. Chanas, M. Delgado, J.L. Verdegay, M.A. Vila, Fuzzy optimal flow on imprecise structures, European J. Oper. Res. 83 (3) (1995) 568–580.
[9] D. Dubois, H. Prade, Fuzzy Sets and Systems, Academic Press, New York, 1980.
[10] J.L. Kennington, R.V. Helgason, Algorithms for Network Programming, Wiley, New York, 1980.
[11] E.S. Lee, R.J. Li, Fuzzy multiple objective programming and compromise programming with Pareto optimum, Fuzzy Sets and Systems 53 (1993) 275–288.

[12] R.J. Li, Multiple Objective Decision Making in a Fuzzy Environment. Ph.D. Dissertation, Dept. of Industrial Engineering, Kansas State University, Manhattan, KA, 1990.
[13] LINGO, Lindo System Inc., Chicago, 1992.
[14] M.K. Luhandjula, Compensatory operators in fuzzy linear programming with multiple objectives, Fuzzy Sets and Systems 8 (3) (1982) 245–252.
[15] M.K. Luhandjula, On possibilistic linear programming, Fuzzy Sets and Systems 18 (1) (1986) 15–30.
[16] D. Negi, Fuzzy Analysis and Optimization, Ph.D. Dissertation, Dept. of Industrial Engineering, Kansas State University, Manhattan, KA, 1989.
[17] D.S. Negi, E.S. Lee, Possibility programming by the comparison of fuzzy numbers, Comput. Math. Appl. 25 (9) (1993) 43–50.
[18] T.W. Ruefli, A generalized goal decomposition model, Management Sci. 17 (8) (1971) B505-B518.
[19] H.S. Shih, Y.J. Lai, E.S. Lee, Fuzzy approach for multi-level mathematical programming problems, Comput. and Oper. Res. 23 (1) (1996) 73–91.
[20] B.M. Werners, Aggregation models in mathematical programming, in: G. Mitra (Ed.), Mathematical Models for Decision Support, Springer, Berlin, 1988, pp. 295–305.
[21] M. Zeleny, Multiple Criteria Decision Making, McGraw-Hill, New York, 1982.
[22] H.-J. Zimmermann, P. Zysno, Latent connectives in human decision making, Fuzzy Sets and Systems 4 (1) (1980) 37–51.
[23] H.-J. Zimmermann, P. Zysno, Decision and evaluations by hierarchical aggregation of information, Fuzzy Sets and Systems 10 (1983) 243–260.