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An EOQ model for deteriorating items with time varying demand and partial backlogging

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In the classical economic order quantity model, it is often assumed that the shortages are either completely backlogged or completely lost. However, in some inventory systems, it is more reasonable to assume that the backlogging rate is dependent on the length of the waiting time for the next replenishment. The longer the waiting time is, the smaller the backlogging rate would be. In this paper, we focus on the effect of the backlogging rate on the economic order quantity decision. Numerical examples are presented to illustrate the model.

Keywords: finite time horizon; deteriorating items; time-varying demand; partial backlogging; inventory control

Introduction

In the last two decades, the models for inventory replenishment policies involving time-varying demand patterns have received the attention of several researchers. The fundamental result in the development of economic order quantity model with time-varying demand patterns is that of Donaldson[1] who established the classical no-shortage inventory model with a linear trend in demand over a known and finite horizon. However, his procedure was too complex and tedious in computing. The complexity of Donaldson's approach has led to the development of heuristic methods. Silver,[2] Phelps,[3] Ritchie,[4] and Teng,[5] derived simple heuristic procedures for Donaldson's problem.

Dave and Patel[6] first considered the inventory model for deteriorating items with time-varying demand. They considered a linear increasing demand rate over a finite horizon and a constant deterioration rate. Sachan[7] extended Dave and Patel's model to allow for shortages. Datta and Pal[8] presented an EOQ model for items with variable deterioration rate and power demand pattern. Researchers including Mudeshwar,[9] Goswami and Chaudhuri,[10] Goyal *et al,*[11] Hariga,[12] Chakrabarti and Chaudhuri,[13] Benkherouf,[14] and Hariga and Alyan[15] developed economic order quantity models that focused on deteriorating items with time-varying demand and shortages.

In practice, some customers would like to wait for backlogging during the shortage period, but the others would not. Consequently, the opportunity cost due to lost sales should be considered in the modeling. Many researchers[9–16] assumed that shortages are completely backlogged. A recent article[17] in the field of deteriorating items with shortages has revealed the economic order quantity with a known market demand rate. In this paper, the backlogging rate was assumed to be a fixed fraction of demand rate during the shortage period. However, in some inventory system, for many stock such as fashionable commodities, the length of the waiting time for the next replenishment becomes main factor for determining whether the backlogging will be accepted or not. The longer the waiting time is, the smaller the backlogging rate would be. Therefore, the backlogging rate is variable and is dependent on the waiting time for the next replenishment.

This present work attempts to model the situation where the demand rate is a time-continuous function and items deteriorate at a constant rate with partial backlogging.

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Assumptions and notations

The mathematical model in this paper is developed on the basis of the following notations and assumptions:

Notations

A = Ordering cost of inventory, \$/per order.
C_1 = Holding cost, \$/per unit/per unit time.
C_2 = Shortage cost, \$/per unit/per unit time.
C_3 = Opportunity cost due to lost sales, \$/per unit.
C_4 = Cost of the inventory item, \$/per unit.
R_i = The amount of inventory carried during the ith cycle.
D_i = The amount of deteriorated items during the ith cycle.

S_i = The amount of shortage during the ith cycle.
B_i = The amount of lost sales during the ith cycle.
θ = Deterioration rate, a fraction of the on-hand inventory.
$I(t)$ = The inventory level at time t.
n = The number of replenishment cycles during the planning horizon.
s_i = Time at which shortages start during the ith cycle, $i = 1, 2, \ldots, n-1$.
t_i = Time at which the ith replenishment is made, $i = 1, 2, \ldots, n$.
T_i = Length of the ith cycle, $i = 1, 2, \ldots, n$.

Assumptions

(1) A single item is considered with a constant rate of deterioration over a known and finite planning horizon of length H.
(2) The replenishment occurs instantaneously at an infinite rate.
(3) There is no repair or replacement of deteriorated units during the planning horizon. The items will be withdrawn from warehouse immediately as they become deterioration.
(4) The demand rate $f(t)$ is a time continuous and monotonic function, and $f(t)/f'(t)$ is non-decreasing in t, where $f'(t)$ denotes the first derivative $f(\cdot)$ with respect to t and $f'(t) \neq 0$ for all t.
(5) Shortages are allowed in all cycles and each cycle starts with shortages.
(6) During the shortage period, the backlogging rate is variable and is dependent on the length of the waiting time for the next replenishment. The longer the waiting time is, the smaller the backlogging rate would be. Hence, the proportion of customers who would like to accept backlogging at time t is decreasing with the waiting time $(t_i - t)$ waiting for the next replenishment. To take care of this situation we have defined the backlogging rate to be $1/1 + \alpha(t_i - t)$ when inventory is negative. The backlogging parameter α is a positive constant, $s_{i-1} \leqslant t < t_i$.

Model formulation

According to the notations and assumptions mentioned before, the behaviour of inventory system at any time t can be depicted in Figure 1. From Figure 1, it can be seen that the depletion of the inventory occurs due to the combined effect of the demand and the deterioration during the interval $[t_1, s_i]$ of the ith replenishment cycle. Hence, the variation of inventory with respect to time can be described by the following differential equation:

$$\frac{dI(t)}{dt} = -\theta I(t) - f(t), \qquad t_i \leqslant t < s_i, \tag{1}$$

with boundary condition $I(s_i) = 0, i = 1, 2, \ldots, n$. The solution of (1) may be represented by

$$I(t) = e^{-\theta t} \int_t^{s_i} e^{\theta u} f(u)du, \qquad t_i \leqslant t < s_i. \tag{2}$$

From (2), the amount of inventory carried during the ith cycle is given by

$$R_i = \int_{t_i}^{s_i} e^{-\theta t} \int_t^{s_i} e^{\theta u} f(u)du\, dt = \frac{1}{\theta} \int_{t_i}^{s_i} (e^{\theta(t-t_i)} - 1) f(t)dt. \tag{3}$$

For an inventory with a constant deterioration rate of θ, the amount of deteriorated items during the ith cycle is given by

$$D_i = \theta R_i = \int_{t_i}^{s_i} (e^{\theta(t-t_i)} - 1) f(t)dt. \tag{4}$$

In addition, the depletion of inventory occurs due to the demand backlogged during the interval $[s_{t-1}, t_i]$. The

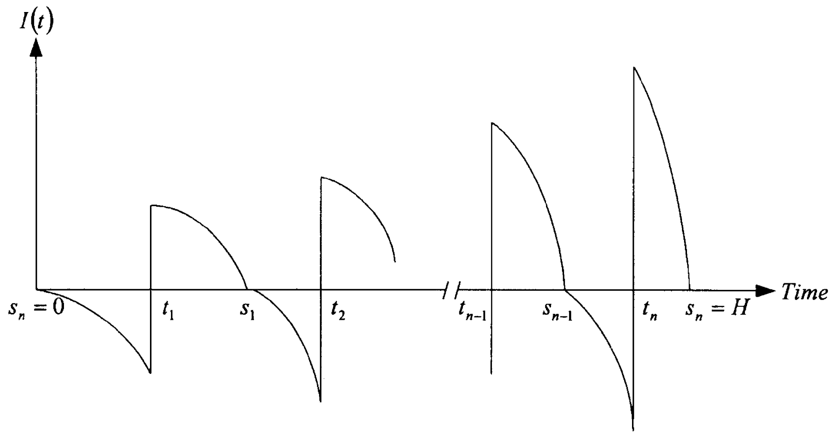

Figure 1 Inventory level *I(t)* vs time.

variation of inventory with respect to time can be described by the following differential equation:

$$\frac{dI(t)}{dt} = -\frac{f(t)}{1+\alpha(t_i - t)}, \qquad s_{i-1} \leqslant t < t_i, \tag{5}$$

with boundary condition $I(s_{i-1}) = 0, i = 1, 2, \ldots, n$. The solution of (5) is

$$I(t) = -\int_{s_{i-1}}^{t} \frac{f(u)}{1+\alpha(t_i - u)} du, \qquad s_{i-1} \leqslant t < t_i. \tag{6}$$

From (6), the amount of shortage during the ith cycle is given by

$$S_i = \int_{s_{i-1}}^{t_i} \int_{s_{i-1}}^{t} \frac{f(u)}{1+\alpha(t_i - u)} du\, dt = \int_{s_{i-1}}^{t_i} \frac{(t_i - t)}{1+\alpha(t_i - t)} f(u)du. \tag{7}$$

Moreover, the amount of lost sales during the ith cycle is given by

$$\begin{aligned} B_i &= \int_{s_{i-1}}^{t_i} \left[1 - \frac{1}{1+\alpha(t_i - t)}\right] f(t)dt \\ &= \alpha \int_{s_{i-1}}^{t_i} \frac{(t_i - t)}{1+\alpha(t_i - t)} f(t)dt. \end{aligned} \tag{8}$$

As shown above, we can formulate the total inventory cost as the sum of the ordering cost, holding cost, deterioration cost, shortage cost and opportunity cost due to lost sales as follows:

$$\begin{aligned} TC &= nA + C_1 \sum_{i=0}^{n-1} R_{i+1} + C_4 \sum_{i=0}^{n-1} D_{i+1} \\ &\quad + C_2 \sum_{i=0}^{n-1} S_{i+1} + C_3 \sum_{i=0}^{n-1} B_{i+1} \\ &= nA + \frac{C_1 + C_4\theta}{\theta} \sum_{i=0}^{n-1} \int_{t_{i+1}}^{s_{i+1}} (e^{\theta(t - t_{i+1})} - 1) f(t)dt \\ &\quad + (C_2 + C_3\alpha) \sum_{i=0}^{n-1} \int_{s_i}^{t_{i+1}} \frac{(t_{i+1} - t)}{1+\alpha(t_{i+1} - t)} f(t)dt \end{aligned} \tag{9}$$

The objective of this paper is to determine the optimal number of replenishments n^*, the optimal replenishment points t_i^*, and the optimal shortage points s_i^* to minimise the total cost of the inventory system.

For a given positive integer n, the necessary conditions for TC to be minimum are

$$\begin{aligned} \frac{\partial TC}{\partial t_i} &= (C_2 + C_3\alpha) \int_{s_{i-1}}^{t_i} \frac{1}{[1+\alpha(t_i - t)]^2} f(t)dt \\ &\quad - (C_1 + C_4\theta) \int_{t_i}^{s_i} e^{\theta(t - t_i)} f(t)dt = 0, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{10}$$

$$\begin{aligned} \frac{\partial TC}{\partial s_i} &= (C_2 + C_3\alpha) \frac{(t_{i+1} - s_i)}{1+\alpha(t_{i+1} - s_i)} \\ &\quad - \frac{(C_1 + C_4\theta)}{\theta} (e^{\theta(s_i - t_i)} - 1) = 0, \\ &\qquad i = 1, 2, \ldots, n-1. \end{aligned} \tag{11}$$

It is obvious to see that once t_1 is known, $s_1(t_1)$, $t_2(t_1)$, $s_2(t_1)$, $\ldots$, $t_n(t_1)$, and $s_n(t_1)$ can be obtained successively from (10) and (11) for increasing demand patterns. Similarly, once t_n is known, we can use a numerical search method to obtain $s_{n-1}(t_n)$ from (10). Then $t_{n-1}(t_n)$ can be easily solved from (11). By repeating this procedure mentioned above, $s_{n-1}(t_n)$, $t_{n-2}(t_n)$, $\ldots$, $t_1(t_n)$, and $s_0(t_n)$ can be determined orderly from (10) and (11) for decreasing demand patterns.

To acquire the optimal replenishment policy, for a given value of n, that minimizes the total inventory cost, the value of t_1 should be selected to satisfy $s_n(t_1) = \mathrm{H}$ for increasing demand patterns and the value of t_n should be selected to satisfy $s_0(t_n) = 0$ for decreasing demand patterns.

We can show that $s_n(t_1)$ is increasing in t_1. Besides, it is not difficult to check from (10) and (11) that $s_n(0) < \mathrm{H}$ and that $s_n(\mathrm{H}) > \mathrm{H}$. Therefore, there exists a unique solution t_1^* satisfying $s_n(t_1^*) = \mathrm{H}$ in the interval [0, H] for increasing demand patterns. On the other hand, since $s_0(t_n)$ is increasing in t_n, from (10) and (11), we have $s_0(0) < 0$ and $s_0(\mathrm{H}) > 0$. Hence, the uniqueness of the optimal replenishment policy for decreasing demand patterns can be provided. (see Appendix A for details).

Moreover, the optimal replenishment policy has the following useful properties:

(i) If $f(t)$ is an increasing function, then $T_1 > T_2 > \cdots > T_n$.

(ii) If $f(t)$ is a decreasing function, then $T_1 < T_2 < \cdots < T_n$. (see Appendix B for details).

Numerical example

To illustrate the preceding theory, the following examples are considered.

Exponential demand patterns

Example 1

$$f(t) = 10e^{0.98t}, \quad A = 250, \quad C_1 = 40, \quad C_2 = 80, \quad C_3 = 30, \quad C_4 = 200, \quad \theta = 0.08, \quad H = 4, \quad \alpha = 20.$$

From (10), (11) and $s_n(t_1) = H$, the total cost TC can be found for different values of n. Computed results and the optimal replenishment policy are shown in Table 1. The optimal values of n and TC are $n^* = 11$ and $TC^* = 6190.6$.

Example 2

$$f(t) = 500e^{-0.98t}, \quad A = 250, \quad C_1 = 40, \quad C_2 = 80, \quad C_3 = 30, \quad C_4 = 200, \quad \theta = 0.08, \quad H = 4, \quad \alpha = 20.$$

From (10), (11) and $s_0(t_n) = 0$, the total cost TC can be found for different values of n. Computed results and the optimal replenishment policy are shown in Table 2. The optimal values of n and TC are $n^* = 12$ and $TC^* = 6082.5$.

Linear demand patterns

Example 3

$$f(t) = 40 + 3t, \quad A = 250, \quad C_1 = 40, \quad C_2 = 80, \quad C_3 = 30, \quad C_4 = 200, \quad \theta = 0.08, \quad H = 4, \quad \alpha = 20.$$

Computed results and the optimal replenishment policy are shown in Table 3. The optimal values of n and TC are $n^* = 8$ and $TC^* = 4231.4$.

Table 1 Optimal solution of Example 1

n	TC	i	t_i	s_i
1	12216.3	1	1.1273	1.5454
2	10150.5	2	1.6623	2.0033
3	8844.5	3	2.0693	2.3580
4	7964.8	4	2.4043	2.6548
5	7354.7	5	2.6905	2.9118
6	6926.1	6	2.9408	3.1390
7	6627.6	7	3.1636	3.3431
8	6424.6	8	3.3643	3.5283
9	6293.9	9	3.5470	3.6980
10	6219.8	10	3.7147	3.8546
11	6190.6*	11	3.8697	4.0000
12	6197.5			
13	6234.0			
14	6295.1			
15	6376.9			
TC^*	6190.6			
n^*	11			

Table 2 Optimal solution of Example 2

n	TC	i	t_i	s_i
1	16540.7	1	0.0127	0.1437
2	12240.4	2	0.1576	0.2987
3	9973.3	3	0.3140	0.4670
4	8596.5	4	0.4840	0.6510
5	7701.5	5	0.6702	0.8542
6	7099.8	6	0.8761	1.0811
7	6690.9	7	1.1068	1.3382
8	6416.3	8	1.3695	1.6355
9	6239.3	9	1.6752	1.9887
10	6134.7	10	2.0435	2.4266
11	6086.5	11	2.5157	3.0126
12	6082.5*	12	3.2655	4.0000
13	6113.7			
14	6173.4			
15	6256.5			
TC^*	6082.5			
n^*	12			

Example 4

$$f(t) = 50 - 3t, \quad A = 250, \quad C_1 = 40, \quad C_2 = 80, \quad C_3 = 30, \quad C_4 = 200, \quad \theta = 0.08, \quad H = 4, \quad \alpha = 20.$$

Computed results and the optimal replenishment policy are shown in Table 4. The optimal values of n and TC are $n^* = 8$ and $TC^* = 4124.6$.

Sensitivity analysis

In this section, we examine the effects of change in the backlogging parameter α on the optimal total cost and the optimal number of replenishments. A sensitivity analysis is performed by considering the same numerical examples. Seven different values of α are adopted, $\alpha = 1, 2.5, 5, 10,$ 25 and 50. Computed results are shown in Table 5.

Table 3 Optimal solution of Example 3

n	TC	i	t_i	s_i
1	5645.1	1	0.1240	0.5417
2	5241.5	2	0.6582	1.0690
3	4915.5	3	1.1791	1.5833
4	4654.3	4	1.6879	2.0860
5	4456.0	5	2.1859	2.5782
6	4321.6	6	2.6738	3.0608
7	4249.0	7	3.1526	3.5345
8	4231.4*	8	3.6229	4.0000
9	4259.8			
10	4325.2			
TC^*	4231.4			
n^*	8			

Table 4 Optimal solution of Example 4

n	TC	i	t_i	s_i
1	5534.2	1	0.0844	0.4649
2	5114.0	2	0.5523	0.9381
3	4779.5	3	1.0290	1.4204
4	4515.9	4	1.5152	1.9125
5	4320.9	5	2.0117	2.4153
6	4194.2	6	2.5195	2.9299
7	4131.6	7	3.0398	3.4576
8	4124.6*	8	3.5742	4.0000
9	4162.9			
10	4237.1			
TC^*	4124.6			
n^*	8			

The following inferences can be made from the results in Table 5.

(i) Increasing the value of α will result in an increase in the optimal total cost and the optimal replenishment times.
(ii) As the value of α increases, the optimal total cost becomes close to the optimal total cost without shortage.
(iii) The optimal total cost with partial backlogging is more sensitive to α when it's value is small.

Conclusions

In this paper, we develop an EOQ model for deteriorating items with time-varying demand and partial backlogging. In particular, the backlogging rate is considered to be a decreasing function of the waiting time for the next replenishment. This assumption is more realistic in the market. As α increases, the results indicate that the optimal total cost increases and becomes close to the optimal total cost without shortage. Moreover, the total cost is more sensitive to α when the value is small.

Appendix A

(i) For increasing demand patterns, $s_n(t_1)$ is increasing in t_1 and there exists a unique real solution $t_1^* \in [0, \mathrm{H}]$ satisfying $s_n(t_1^*) = \mathrm{H}$.

(ii) For decreasing demand patterns, $s_0(t_n)$ is increasing in t_n and there exists a unique real solution $t_n^* \in [0, \mathrm{H}]$ satisfying $s_0(t_n^*) = 0$.

Proof

(i) If $f(t)/f'(t)$ is non-decreasing in t for $t_i \leqslant t \leqslant s_i$, it implies that $f(t_i)/f'(t_i) \leqslant f(t)/f'(t)$. Using the result, we obtain $f'(t) \leqslant [f'(t_i)/f(t_i)]f(t)$. Multiply both sides by $e^{\theta(t-t_i)}$, since $e^{\theta(t-t_i)} > 0$ and $\theta e^{\theta(t-t_i)} f(t) > 0$ for $t_i \leqslant t \leqslant s_i$, we have

$$e^{\theta(t-t_i)}f'(t) + \theta e^{\theta(t-t_i)}f(t) \leqslant e^{\theta(t-t_i)}\frac{f'(t_i)}{f(t_i)}f(t) + \theta e^{\theta(t-t_i)}f(t). \tag{12}$$

In (12), multiply both sides by $(C_1 + C_4\theta)$ and integrate with respect to t between t_i and s_i, the previous inequity can be written as follows:

$$(C_1 + C_4\theta)\left[e^{\theta(s_i-t_i)}f(s_i) - f(t_i)\right] \leqslant (C_1 + C_4\theta)\left(\frac{f'(t_i)}{f(t_i)} + \theta\right)\int_{t_i}^{s_i} e^{\theta(t-t_i)}f(t)dt. \tag{13}$$

Recalling (10) and using the fact that $f(t_i)/f'(t_i) \geqslant f(t)/f'(t)$ for $s_{i-1} \leqslant t \leqslant t_i$, we have

$$(C_1 + C_4\theta)\left[e^{\theta(s_i-t_i)}f(s_i) - f(t_i) - \theta\int_{t_i}^{s_i} e^{\theta(t-t_i)}f(t)dt\right] \leqslant (C_2 + C_3\alpha)\int_{s_{i-1}}^{t_i}\frac{f'(t)}{[1+\alpha(t_i - t)]^2}dt.$$

After integrating by parts in the right hand side, the last inequality can be rewritten as

$$(C_1 + C_4\theta)\left[e^{\theta(s_i-t_i)}f(s_i) - f(t_i) - \theta\int_{t_i}^{s_i} e^{\theta(t-t_i)}f(t)dt\right] \leqslant (C_2 + C_3\alpha)\left(\int_{s_{i-1}}^{t_i}\frac{-2\alpha}{[1+\alpha(t_i - t)]^3}f(t)dt + f(t_i) - \frac{f(s_{i-1})}{[1+\alpha(t_i - s_{i-1})]^2}\right). \tag{14}$$

Table 5 Sensitivity analysis

		Complete backlogging	α						Without shortage
			1	2.5	5	10	25	50	
Example 1	TC^*	5078.6	5312.4	5537.7	5761.9	5993.4	6243.1	6368.5	6772.4
	n^*	10	10	11	11	11	11	12	14
Example 2	TC^*	5003.4	5227.0	5449.0	5664.1	5890.5	6131.0	6247.5	6425.5
	n^*	10	10	11	11	11	12	12	13
Example 3	TC^*	3490.1	3622.3	3760.2	3909.7	4077.9	4276.0	4393.9	4575.2
	n^*	7	7	7	7	8	8	8	9
Example 4	TC^*	3410.7	3563.7	3668.0	3810.2	3972.9	4166.4	4276.2	4451.7
	n^*	7	7	7	7	7	8	8	9

Substituting $M_i = s_i - t_i$ and $K_i = t_i - s_{i-1}$ into (10), and then differentiating (10) with respect to t_1 yields

$$(C_1 + C_4\theta)e^{\theta M_i}f(s_i)\frac{\partial M_i}{\partial t_1} + (C_1 + C_4\theta) \times \left(e^{\theta M_i}f(s_i) - f(t_i) - \theta\int_{t_i}^{s_i} e^{\theta(t-t_i)}f(t)dt\right)t_i' = (C_2 + C_3\alpha)\left(\int_{s_{i-1}}^{t_i}\frac{-2\alpha}{[1+\alpha(t_i - t)]^3}f(t)dt + f(t_i) - \frac{f(s_{i-1})}{[1+\alpha(t_i - s_{i-1})]^2}\right)t_i' + (C_2 + C_3\alpha)\frac{f(s_{i-1})}{[1+\alpha(t_i - s_{i-1})]^2}\frac{\partial K_i}{\partial t_1}, \quad i = 1, 2, \ldots, n. \quad (15)$$

From the analysis carried out so far, is easily shown that $(\partial K_1/\partial t_1) = 1$ and $(\partial M_1/\partial t_1) > 0$. Next, differentiating (11) with respect to t_1 yields

$$(C_1 + C_4\theta)e^{\theta(s_i - t_i)}\frac{\partial M_i}{\partial t_1} = (C_2 + C_3\alpha)\frac{1}{[1+\alpha(t_{i+1} - s_i)]^2}\frac{\partial K_{i+1}}{\partial t_1}, \quad i = 1, 2, \ldots, n-1. \quad (16)$$

Using the fact that $(\partial M_1/\partial t_1) > 0$, we can obtain $(\partial K_2/\partial t_1) > 0$ from (16). By repeating this procedure mentioned above, we can show that $(\partial K_i/\partial t_1) > 0$ and $(\partial M_i/\partial t_1) > 0$ for $i = 1, 2, \ldots, n$ from (15) and (16).

Moreover, since $s_n(t_1) = \sum_{i=1}^{n}(M_i + K_i)$, it is easily shown that

$$s_n'(t_1) = \frac{\partial s_n(t_1)}{\partial t_1} = \sum_{i=1}^{n}\left(\frac{\partial M_i}{\partial t_1} + \frac{\partial K_i}{\partial t_1}\right) > 0.$$

Besides, it is not difficult to check from (10) and (11) that $s_n(0) < \mathrm{H}$ and that $s_n(\mathrm{H}) > \mathrm{H}$. Therefore, there exists a unique solution t_1^* to $s_n(t_1^*) = \mathrm{H}$ in the interval [0, H].

(ii) The case of decreasing demand patterns is also similar. Therefore the proof is complete.

Appendix B

(i) If $f(t)$ is an increasing function, then $T_1 > T_2 > \cdots > T_n$.

(ii) If $f(t)$ is a decreasing function, then $T_1 < T_2 < \cdots < T_n$.

Proof

First, by applying mean value theorem to integral in (10), we have

$$\frac{(C_1 + C_4\theta)}{\theta}f(x_1)(e^{\theta(s_i - t_i)} - 1) = (C_2 + C_3\alpha)f(x_2)\frac{(t_i - s_{i-1})}{1+\alpha(t_i - s_{i-1})}, \quad (17)$$

where $t_i < x_1 < s_i, s_{i-1} < x_2 < t_i$.

(i) If $f(t)$ is an increasing function, it is clear that

$$\frac{(C_1 + C_4\theta)}{\theta}(e^{\theta(s_i - t_i)} - 1) < (C_2 + C_3\alpha)\frac{(t_i - s_{i-1})}{1+\alpha(t_i - s_{i-1})}, \quad i = 1, 2, \ldots, n. \quad (18)$$

Moreover, since

$$\frac{(C_1 + C_4\theta)}{\theta}(e^{\theta(s_i - t_i)} - 1) = (C_2 + C_3\alpha)\frac{(t_{i+1} - s_i)}{1+\alpha(t_{i+1} - s_i)},$$

(18) can be rewritten as

$$\frac{(t_{i+1} - s_i)}{1+\alpha(t_{i+1} - s_i)} < \frac{(t_i - s_{i-1})}{1+\alpha(t_i - s_{i-1})}.$$

Suppose that $g(x) = x/1 + \alpha x, x \geqslant 0$ and $\alpha \geqslant 0$. Taking the first derivative for $g(x)$ with respect to x, we get

$$\frac{d}{dx}\left(\frac{x}{1+\alpha x}\right) = \frac{1}{(1+\alpha x)^2} > 0, \qquad \forall x \geqslant 0.$$

Hence, $g(x)$ is a strictly increasing function. We then have

$$g(x_n) < g(x_{n-1}) < \cdots < g(x_3) < g(x_2) < g(x_1),$$

$$0 < x_n < x_{n-1} < \cdots < x_3 < x_2 < x_1.$$

Let $x_i = t_i - s_{i-1}, i = 1, 2, \ldots, n$, it is obvious to see that

$$\frac{(t_{i+1} - s_i)}{1+\alpha(t_{i+1} - s_i)} < \frac{(t_i - s_{i-1})}{1+\alpha(t_i - s_{i-1})}$$

if and only if $0 < (t_{i+1} - s_i) < (t_i - s_{i-1})$. Applying this, we have $K_i > K_{i+1}, i = 1, 2, \ldots, n-1$. From (11) and (18), it is clear that

$$\frac{(C_1 + C_4\theta)}{\theta}(e^{\theta(s_{i-1} - t_{i-1})} - 1) > \frac{(C_1 + C_4\theta)}{\theta}(e^{\theta(s_i - t_i)} - 1)$$

and $(s_{i-1} - t_{i-1}) > (s_i - t_i), i = 2, 3, \ldots, n$. We have $M_{i-1} > M_i, i = 2, 3, \ldots, n$. As discussed earlier, since $T_i = K_i + M_i$, we can conclude that $T_i > T_{i+1}, i = 1, 2, \ldots, n-1$.

(ii) The case of decreasing demand patterns is also similar. Therefore, the proof is complete.

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