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Fuzzy Sets and Systems 114 (2000) 71–87

FUZZY sets and systems

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Compensatory fuzzy multiple level decision making

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Received August 1997; received in revised form August 1998

Abstract

Fuzzy set theory has been shown to be an effective tool to overcome the computational difficulties encountered in solving large multiple level programming problems (Shih et al., 1996). In this paper, compensatory operators are introduced for adjusting the decision making process between the different levels and also between the decision makers of the same level. After a brief consideration of the bi-level and three level systems, the large decentralized organizations with both equal and unequal goals are investigated. Various numerical examples are given to compare the influences of compensation and to illustrate the approaches. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Mathematical programming; Compensatory operator; Multiple level programming; Fuzzy decision making

1. Introduction

Multiple level programming techniques [1–6,8,9,12–15,19–23] are ideal approaches for solving large decentralized planning problems with multiple decision makers (DMs) such as the frequently encountered hierarchical organizations of large companies or the decentralized systems of nonprofit and government organizations. The approach explicitly assigns each DM a unique objective, a set of decision variables, and a set of common constraints that affect all DMs. But, unfortunately, the multiple level programming problems (MLPP) are difficult to solve and have been proved to be NP-hard. Some existing numerical techniques such as the extreme point search approach, the procedure based on the Karush–Kuhn–Tucker condition, and the descent methods [5,6,9,13] are effective only for solving very simple problems.

To overcome the above mentioned difficulties, Shih et al. [19] developed a fairly effective fuzzy approach by using the concept of tolerance membership functions and multiple objective decision making. The idea is to use the basic fuzziness and vagueness nature of such large hierarchy systems to make the complexity tractable. The upper level defines his or her tolerances by the use of membership function which constrains the lower level DM' s feasible space. The resulting iterative procedure, instead of the usually used extreme point search, relies on the change of membership functions which expresses the degree of satisfaction of the solutions to both the upper- and the lower-level DMs. It should be emphasized that the approach explores the inherent

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PII: S0165-0114(98)00409-6

vagueness of the system and thus generates no significant additional constraints. In fact it significantly reduces the amount of computation required for large multiple level decentralized programming problems (MLDPP).

However, the earlier work [19] used the non-compensatory max–min aggregation operator, which is not the usual practice for solving real-world decision making problems. Managerial decisions almost always allow some compensation between different achievements so that a balance of the objectives can be obtained. Furthermore, in real world applications, the objectives of the DMs in the same hierarchy level can also be different. Thus, in the present approach, multiple decision units in the same level can have unequal objectives.

In the following sections, the compensatory fuzzy operators are first considered. By using these compensatory operators, the solution procedures for the various types of multiple level decision problems are formulated. Numerical examples are solved to illustrate the approaches.

2. Compensatory fuzzy aggregation operators

There exist various fuzzy aggregation operators. Max–min operator simulates the classic logical "exclusive or" and "and", and thus are the earliest aggregator proposed [7]. Zimmermann and Zysno [28,29] studied empirically the operators "min", "product", "max", "weighted geometric mean", and "γ-operator". They concluded that the "γ-operator" is especially suited for human decision making. Various other investigators also proposed and studied various compensatory and non-compensatory operators (see, e.g. [10,11,16,25]). More recently, Yager [26] introduced the ordered weighted averaging operators.

Although Zimmermann and Zysno [28] have shown that the Hamacher γ-operator predicts human judgment very well, but it is a nonlinear operator and increases the computational difficulties tremendously. Based on the γ-operator, Werners [24] introduced the following compensatory fuzzy "or" and "and" operators which are the convex combinations of max and arithmetical mean, and min and arithmetical mean, respectively:

$$\begin{aligned}\mu_{\text{or}} &= \gamma\ \max_i(\mu_i) + (1-\gamma)\left(\sum_i \mu_i\right)\Big/ m,\\ \mu_{\text{and}} &= \gamma\ \min_i(\mu_i) + (1-\gamma)\left(\sum_i \mu_i\right)\Big/ m,\end{aligned} \tag{1}$$

where $0 \leqslant \mu_i \leqslant 1$, $i = 1,2,\ldots,m$, and the magnitude of $\gamma \in [0,1]$ represents the grade of compensation.

Obviously, when $\gamma = 1$, Eq. (1) reduces to $\mu_{\text{or}} = \max$ and $\mu_{\text{and}} = \min$. The combination of these two operators forms the generalized "or" and "and" operators. For illustrative purposes, the fuzzy "and" operator will be used in this investigation and Eq. (1) is further simplified in the following for multi-level decision making problems:

Let $\lambda = \min_i(\mu_i)$, then the fuzzy "and" operator becomes:

$$\begin{aligned}\mu_{\text{and}} &= \gamma\lambda + (1-\gamma)\left(\sum_i \mu_i\right)\Big/ m = \gamma\lambda + (1-\gamma)\sum_i((\mu_i\lambda) + \lambda)/m\\ &= \gamma\lambda + (1-\gamma)m\lambda/m + (1-\gamma)\sum_i(\mu_i - \lambda)/m = \lambda + (1-\gamma)\sum_i(\mu_i - \lambda)/m,\\ &\quad 0 \leqslant \mu_i \leqslant 1,\ i = 1,2,\ldots,m,\ 0 \leqslant \lambda \leqslant 1,\ 0 \leqslant \gamma \leqslant 1.\end{aligned}$$

After letting $\lambda_i = \mu_i - \lambda$, the above expression reduces to

$$\mu_{\text{and}} = \lambda + (1-\gamma)\sum_i \lambda_i/m, \tag{2}$$

where m represents the number of the fuzzy membership functions under consideration. For multi-level decision making problems, these fuzzy membership functions are the tolerances of the decisions and the objectives of the DMs in the various levels.

3. Compensation in bi-level programming problems

The bi-level programming problem (BLPP) has two DMs at two different hierarchical levels and is similar to the static Stackelberg game – a special case of a two-person, non-zero sum, non-cooperative game, with full information. This BLPP can be represented by the following equations, where the top level has control over the vector $\boldsymbol{x}_1$ and the bottom level has control over the vector $\boldsymbol{x}_2$ [22]:

$$\max_{\boldsymbol{x}1} \quad f_1(\boldsymbol{x}_1,\boldsymbol{x}_2)=\boldsymbol{c}_{11}^{\mathrm{T}}\boldsymbol{x}_1+\boldsymbol{c}_{12}^{\mathrm{T}}\boldsymbol{x}_2 \qquad \text{(upper level)} \tag{3}$$

where $\boldsymbol{x}_2$ solves

$$\max_{\boldsymbol{x}2} \quad f_2(\boldsymbol{x}_1,\boldsymbol{x}_2)=\boldsymbol{c}_{21}^{\mathrm{T}}\boldsymbol{x}_1+\boldsymbol{c}_{22}^{\mathrm{T}}\,\boldsymbol{x}_2 \qquad \text{(lower level)} \tag{4}$$

$$\text{s.t.} \quad (\boldsymbol{x}_1,\boldsymbol{x}_2)\in \boldsymbol{F}_2=\{(\boldsymbol{x}_1,\boldsymbol{x}_2)|\boldsymbol{A}_1\boldsymbol{x}_1+\boldsymbol{A}_2\boldsymbol{x}_2\leqslant \boldsymbol{b},\ \text{and}\ \boldsymbol{x}_1,\boldsymbol{x}_2\geqslant \boldsymbol{0}\},$$

where $\boldsymbol{c}_{11}$, $\boldsymbol{c}_{12}$, $\boldsymbol{c}_{21}$, $\boldsymbol{c}_{22}$, and $\boldsymbol{b}$ are linear vectors and $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are matrices. The functions f_1 and f_2 for the two decision makers are assumed to be linear and bounded.

For a given $\boldsymbol{x}_1$, let $\boldsymbol{G}(\boldsymbol{x})$ denote the set of optimal solutions to the following lower-level problem:

$$\max_{x2\,\in\,\boldsymbol{F}1(x1)} f_2^*(\boldsymbol{x}_2)=\boldsymbol{c}_{22}^{\mathrm{T}}\boldsymbol{x}_2,$$

where $\boldsymbol{F}_1(\boldsymbol{x}_1)=\{\boldsymbol{x}_2\,|\,\boldsymbol{A}_2\boldsymbol{x}_2\leqslant \boldsymbol{b}-\boldsymbol{A}_1\boldsymbol{x}_1\}$ represents the upper-level DM's feasible decision space. The set of rational reactions f_2 over $\boldsymbol{F}_2$ can also be defined as

$$\boldsymbol{S}_{f2}(\boldsymbol{F}_2)=\{(\boldsymbol{x}_1,\boldsymbol{x}_2)\,|\,(\boldsymbol{x}_1,\boldsymbol{x}_2)\in \boldsymbol{F}_2 \text{ and } \boldsymbol{x}_1\in \boldsymbol{G}(\boldsymbol{x}_1)\}.$$

In the earlier paper [19], the above BLPP was solved by the use of the max–min fuzzy operator. Following Zimmermann [27], the tolerances of the DMs are treated as fuzzy membership functions. The following auxiliary linear programming problem can be obtained:

$$\begin{aligned}
\text{Max} \quad & \lambda \\
\text{s.t.} \quad & \boldsymbol{A}_1\boldsymbol{x}_1+\boldsymbol{A}_2\boldsymbol{x}_2\leqslant \boldsymbol{b}, \\
& \mu_{f1}(f_1(\boldsymbol{x}))\geqslant\lambda, \\
& \mu_{\boldsymbol{x}1}(\boldsymbol{x}_1)\geqslant\lambda\boldsymbol{I}, \\
& \mu_{f2}(f_2(\boldsymbol{x}))\geqslant\lambda, \\
& \boldsymbol{x}_1,\boldsymbol{x}_2\geqslant\boldsymbol{0}, \\
& \lambda\in[0,1]
\end{aligned}$$

or

$$\begin{aligned}
\text{Max} \quad & \lambda \\
\text{s.t.} \quad & \boldsymbol{A}_1\boldsymbol{x}_1+\boldsymbol{A}_2\boldsymbol{x}_2\leqslant \boldsymbol{b},
\end{aligned}$$

$$\begin{aligned}
&\mu_{f1}(f_1(\boldsymbol{x})) = [f_1(\boldsymbol{x}) - f_1']/[f_1^{\mathrm{T}} - f_1'] \geqslant \lambda,\\
&[(\boldsymbol{x}_1^{\mathrm{T}} + \boldsymbol{p}_1) - \boldsymbol{x}_1]/\boldsymbol{p}_1 \geqslant \lambda \boldsymbol{I},\\
&[\boldsymbol{x}_1 - (\boldsymbol{x}_1^{\mathrm{T}} - \boldsymbol{p}_1)]/\boldsymbol{p}_1 \geqslant \lambda \boldsymbol{I},\\
&\mu_{f2}(f_2(\boldsymbol{x})) = [f_2(\boldsymbol{x}) - f_2']/[f_2^{\mathrm{T}} - f_2'] \geqslant \lambda,\\
&\boldsymbol{x}_1, \boldsymbol{x}_2 \geqslant \boldsymbol{0},\\
&\lambda \in [0,1],
\end{aligned} \tag{5}$$

where $\boldsymbol{I}$ is a column vector with the same dimension as $\boldsymbol{x}_1$ and with all its elements equal to 1. $\boldsymbol{x}_1^{\mathrm{T}}$ is the preferred value of $\boldsymbol{x}_1$ and $\boldsymbol{p}_1$ is the tolerance of the two sides of $\boldsymbol{x}_1$. f_1^{T} and f_1' are the upper and lower bounds of f_1, respectively.

If we use the compensatory aggregator, Eq. (2), Eq. (5) can be reformulated as

$$\begin{aligned}
\text{Max}\quad & \mu_{\text{and}} = \lambda + (1-\gamma)(\lambda_1 + \lambda_2 + \lambda_3)/3\\
\text{s.t.}\quad & \boldsymbol{A}_1\boldsymbol{x}_1 + \boldsymbol{A}_2\boldsymbol{x}_2 \leqslant \boldsymbol{b},\\
& \boldsymbol{x}_1, \boldsymbol{x}_2 \geqslant \boldsymbol{0},\\
& \mu_{f1}(f_1(\boldsymbol{x})) = [f_1(\boldsymbol{x}) - f_1']/[f_1^{\mathrm{T}} - f_1'] \geqslant (\lambda + \lambda_1),\\
& [(\boldsymbol{x}_1^{\mathrm{T}} + \boldsymbol{p}_1) - \boldsymbol{x}_1]/\boldsymbol{p}_1 \geqslant (\lambda + \lambda_2)\boldsymbol{I},\\
& [\boldsymbol{x}_1 - (\boldsymbol{x}_1^{\mathrm{T}} - \boldsymbol{p}_1)]/\boldsymbol{p}_1 \geqslant (\lambda + \lambda_2)\boldsymbol{I},\\
& \mu_{f2}(f_2(\boldsymbol{x})) = [f_2(\boldsymbol{x}) - f_2']/[f_2^{\mathrm{T}} - f_2'] \geqslant (\lambda + \lambda_3),\\
& \lambda + \lambda_i \leqslant 1, \quad i = 1,2,3\\
& \gamma \in [0,1],\\
& \lambda, \lambda_1, \lambda_2, \text{ and } \lambda_3 \in [0,1],
\end{aligned} \tag{6}$$

where γ represents the grade of compensation.

The three membership functions, which are implicitly represented by λ_1, λ_2 and λ_3 in Eq. (6), represent the tolerances allowed for the upper level decision variable $\boldsymbol{x}_1$, the upper level goal f_1, and the lower level goal f_2, respectively. Thus, expression (6) is flexible and compensatory. The flexibility depends on the range of the decision information offered by the DM and the compensation represents a balance between the goals of the two levels and the decision variable of the upper level. This transformed model is linear and thus can be solved easily. Furthermore, an interactive iterative procedure can be formulated to obtain better tolerances to satisfy both the DMs. In this way the degree of satisfaction, which is represented by the membership functions, can be adjusted according to the preference of the DMs.

Let us consider the following example used in the earlier paper [19]:

Example 1. A bi-level programming problem for compensating between export balance and profit.

$$\underset{x1}{\text{Max}}\, f_1 = 2x_1 - x_2, \quad \text{(balancing export trade, upper-level)}$$

where x_2 solves

$$\begin{aligned}
\max_{x_2}\quad & f_2 = x_1 + 2x_2 && \text{(profit, lower-level)}\\
\text{s.t.}\quad & 3x_1 - 5x_2 \leqslant 15 && \text{(capacity)}\\
& 3x_1 - x_2 \leqslant 21 && \text{(management)}\\
& 3x_1 + x_2 \leqslant 27 && \text{(space)}\\
& 3x_1 + 4x_2 \leqslant 45 && \text{(material)}\\
& x_1 + 3x_2 \leqslant 30 && \text{(labor hours)}\\
& x_1,\ x_2 \geqslant 0.
\end{aligned}$$

In the earlier approach, the optimal solutions for the objective functions were first obtained by solving Eqs. (3) and (4) individually with the above constraint set. The results obtained were: $\boldsymbol{x}^{\mathrm{U}} = (x_1^{\mathrm{U}},\ x_2^{\mathrm{U}}) = (7.5, 1.5)$ for $f_1^{\mathrm{U}} = 13.5$ and $\boldsymbol{x}^{\mathrm{L}} = (x_1^{\mathrm{L}},\ x_2^{\mathrm{L}}) = (3, 9)$ for $f_2^{\mathrm{L}} = 21$. These results were used as the reference set. Thus let $f_1^{\mathrm{U}} = 13.5$ and $f_2' = 10.5$. The meaningful value for f_1' cannot be -3 because minus values have no meaning and cannot be allowed, thus set $f_1 = 0$. Let the upper-level DM's control decision x_1 be around 7.5 with the negative and positive-sides tolerances of 4.5 and 0.5, respectively. The membership functions for $\mu_{x1}(\bullet)$, $\mu_{f1}(\bullet)$, and $\mu_{f2}(\bullet)$ can now be obtained based on these assumed numerical values. The lower-level auxiliary model then becomes

$$\begin{aligned}
\text{Max}\quad & \lambda\\
\text{s.t.}\quad & \boldsymbol{x} \in \boldsymbol{X},\\
& \mu_{f1}(f_1(\boldsymbol{x})) = f_1/(13.5 - 0) \geqslant \lambda,\\
& \mu_{x1}(x_1) = (x_1 - 4.5)/(7.5 - 4.5) \geqslant \lambda,\\
& \mu_{x1}(x_1) = (8 - x_1)/(8 - 7.5) \geqslant \lambda,\\
& \mu_{f2}(f_2(\boldsymbol{x})) = (f_2 - 10.5)/(21 - 10.5) \geqslant \lambda,\\
& \lambda \in [0, 1]
\end{aligned}$$

where $\boldsymbol{X}$ represents the original crisp constraint set.

Using Eq. (6) and the numerical values obtained above, the desired compensatory model can be obtained as

$$\begin{aligned}
\text{Max}\quad & \mu_{\text{and}} = \lambda + (1 - \gamma)(\lambda_1 + \lambda_2 + \lambda_3)/3\\
\text{s.t.}\quad & \boldsymbol{x} \in \boldsymbol{X},\\
& (f_1 - 0)/(13.5 - 0) \geqslant (\lambda + \lambda_1),\\
& (x_1 - 4.5)/(7.5 - 4.5) \geqslant (\lambda + \lambda_2),\\
& (8 - x_1)/(8 - 7.5) \geqslant (\lambda + \lambda_2),\\
& (f_2 - 10.5)/(21 - 10.5) \geqslant (\lambda + \lambda_3),\\
& \lambda, \lambda_1, \lambda_2,\ \text{and}\ \lambda_3 \in [0, 1],
\end{aligned}$$

where $\gamma \in [0, 1]$ represents the grade of compensation.

This compensatory linear programming model can be solved easily by the use of LINDO [18]. The compromise solution is $\boldsymbol{f}^* = (f_1^*, f_2^*) = (9.28, 17.72)$ at $\boldsymbol{x}^* = (x_1^*, x_2^*) = (7.26, 5.23)$ with a total degree of satisfaction

Table 1
Results based on the degree of compensation of Example 1

Degree of compensation γ	Degree of satisfaction μ_{and}	(λ)	Solution (x_1, x_2)	Objective (f_1, f_2)	Note
0	0.78	(0.57)	(7.5, 4.5)	(10.5, 16.5)	
0.1	0.77	(0.69)	(7.26, 5.23)	(9.28, 17.72)	Change of the solution set
0.2	0.76	(0.69)	(7.26, 5.23)	(9.28, 17.72)	
0.3	0.75	(0.69)	(7.26, 5.23)	(9.28, 17.72)	
0.4	0.74	(0.69)	(7.26, 5.23)	(9.28, 17.72)	
0.5	0.73	(0.69)	(7.26, 5.23)	(9.28, 17.72)	
0.6	0.72	(0.69)	(7.26, 5.23)	(9.28, 17.72)	
0.7	0.71	(0.69)	(7.26, 5.23)	(9.28, 17.72)	
0.8	0.70	(0.69)	(7.26, 5.23)	(9.28, 17.72)	
0.9	0.70	(0.69)	(7.26, 5.23)	(9.28, 17.72)	
1	0.69	(0.69)	(7.26, 5.23)	(9.28, 17.72)	

Note: (1) Werners' fuzzy "and" operator is $\mu_{and} = \gamma \min_i(\mu_i) + (1-\gamma)(\Sigma_i \mu_i)/m$, $0 \leqslant \mu \leqslant 1$, $0 \leqslant \gamma \leqslant 1$, with m membership functions, $i = 1, \ldots, m$. Here $m = 3$ for the bi-level programming problem of Example 1.

(2) When $\gamma \geqslant 0.1$, the solution is changed from one set to another set.

(3) λ is the degree of satisfaction for non-compensatory operation, i.e. max–min operation. The compensatory solution and non-compensatory solution are equal when $\gamma = 1$.

$\mu_{and} = 0.73$ at $\gamma = 0.5$. Compared to $\lambda = 0.69$ with the same objectives $\boldsymbol{f}^*$ and decisions $\boldsymbol{x}^*$ in Shih et al. [19], the degree of satisfaction has been improved after compensation. To investigate the effect of different degrees of compensations, 11 cases with different values of compensations were solved and the results are listed in Table 1. For comparison purposes, it is interesting to note that the solution for the non-fuzzy (crisp) results are $\boldsymbol{x}^* = (8, 3)$ and $\boldsymbol{f}^* = (13, 14)$ and the degree of satisfaction for upper-level and lower-level DMs are 0 and 0.33, respectively.

4. Compensation in three-level programming problems

According to the previous discussions, the three level programming problem (TLPP) can be defined as a three-person, non-zero sum game with perfect information in which each player moves sequentially [3]. The TLPP can be represented as

$$\max_{x1} \quad f_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = \boldsymbol{c}_{11}^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_{12}^{\mathrm{T}}\boldsymbol{x}_2 + \boldsymbol{c}_{13}^{\mathrm{T}}\boldsymbol{x}_3, \quad \text{(1st level)} \tag{7}$$

where $\boldsymbol{x}_2, \boldsymbol{x}_3$ solve

$$\max_{x2} f_2 \quad (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = \boldsymbol{c}_{21}^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_{22}^{\mathrm{T}}\boldsymbol{x}_2 + \boldsymbol{c}_{23}^{\mathrm{T}}\boldsymbol{x}_3, \quad \text{(2nd level)} \tag{8}$$

where $\boldsymbol{x}_3$ solves

$$\max_{x3} \quad f_3(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = \boldsymbol{c}_{31}^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{c}_{32}^{\mathrm{T}}\boldsymbol{x}_2 + \boldsymbol{c}_{33}^{\mathrm{T}}\boldsymbol{x}_3, \quad \text{(3rd level)} \tag{9}$$

$$\text{s.t.} \quad \boldsymbol{A}_1\boldsymbol{x}_1 + \boldsymbol{A}_2\boldsymbol{x}_2 + \boldsymbol{A}_3\boldsymbol{x}_3 \leqslant \boldsymbol{b},$$
$$\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \geqslant \boldsymbol{0},$$

where $\boldsymbol{x}_1, \boldsymbol{x}_2$, and $\boldsymbol{x}_3$ are the control or decision variables for the 1st, 2nd, and 3rd levels, respectively.

Although both the vertex enumeration [21] and the KKT condition [1] approaches have been modified to solve the TLPP problems, these numerical approaches are very inefficient.

The supervised search procedure for the fuzzy TLPP problems are carried out from top to bottom [14]. The top level or the first level DM provides his or her preferred ranges of f_1 and $\boldsymbol{x}_1$ to the second level and the second level solves his or her problem with this additional preference information from the top level. The results obtained by the second level are presented to the top level. If the results are not acceptable, some modifications of the membership functions are made. This process can be continued until a satisfactory solution is reached. The interactions of the third level with the top and the second levels can be handled essentially in the same manner.

The auxiliary model of the fuzzy approach for the TLPP can be represented as:

$$\begin{aligned}
\text{Max} \quad & \lambda \\
\text{s.t.} \quad & \boldsymbol{A}_1\boldsymbol{x}_1 + \boldsymbol{A}_2\boldsymbol{x}_2 + \boldsymbol{A}_3\boldsymbol{x}_3 \leqslant \boldsymbol{b}, \\
& \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \geqslant \boldsymbol{0}, \\
& \mu_{f1}(f_1(\boldsymbol{x})) \geqslant \lambda, \\
& \mu_{x1}(\boldsymbol{x}_1) \geqslant \lambda \boldsymbol{I}, \\
& \mu_{f2}(f_2(\boldsymbol{x})) \geqslant \lambda, \\
& \mu_{x2}(\boldsymbol{x}_2) \geqslant \lambda \boldsymbol{I}, \\
& \mu_{f3}(f_3(\boldsymbol{x})) \geqslant \lambda, \\
& \lambda \in [0, 1].
\end{aligned}$$

or

$$\begin{aligned}
\text{Max} \quad & \lambda \\
\text{s.t.} \quad & \boldsymbol{A}_1\boldsymbol{x}_1 + \boldsymbol{A}_2\boldsymbol{x}_2 + \boldsymbol{A}_3\boldsymbol{x}_3 \leqslant \boldsymbol{b}, \\
& \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \geqslant \boldsymbol{0}, \\
& \mu_{f1}(f_1(\boldsymbol{x})) = [f_1(\boldsymbol{x}) - f_1']/[f_1^{\mathrm{T}} - f_1'] \geqslant \lambda, \\
& [(\boldsymbol{x}_1^{\mathrm{T}} + \boldsymbol{p}_1) - \boldsymbol{x}_1]/\boldsymbol{p}_1 \geqslant \lambda \boldsymbol{I}, \\
& [\boldsymbol{x}_1 - (\boldsymbol{x}_1^{\mathrm{T}} - \boldsymbol{p}_1)]/\boldsymbol{p}_1 \geqslant \lambda \boldsymbol{I}, \\
& \mu_{f2}(f_2(\boldsymbol{x})) = [f_2(\boldsymbol{x}) - f_2']/[f_2^{\mathrm{T}} - f_2'] \geqslant \lambda, \\
& [(\boldsymbol{x}_2^{\mathrm{T}} + \boldsymbol{p}_2) - \boldsymbol{x}_2]/\boldsymbol{p}_2 \geqslant \lambda \boldsymbol{I}, \\
& [\boldsymbol{x}_2 - (\boldsymbol{x}_2^{\mathrm{T}} - \boldsymbol{p}_2)]/\boldsymbol{p}_2 \geqslant \lambda \boldsymbol{I}, \\
& \mu_{f3}(f_3(\boldsymbol{x})) = [f_3(\boldsymbol{x}) - f_3']/[f_3^{\mathrm{T}} - f_3'] \geqslant \lambda,
\end{aligned} \tag{10}$$

where the nomenclature is the same as that used in the bi-level problem except with the addition of the third level

By using Eq. (2), the above expression can be transformed into the following compensatory form:

$$\begin{aligned}
\text{Max}\quad & \mu_{\text{and}} = \lambda + (1-\gamma)(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)/5 \\
\text{s.t.}\quad & \boldsymbol{A}_1\boldsymbol{x}_1 + \boldsymbol{A}_2\boldsymbol{x}_2 + \boldsymbol{A}_3\boldsymbol{x}_3 \leqslant \boldsymbol{b}, \\
& \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \geqslant \boldsymbol{0}, \\
& \mu_{f1}(f_1(\boldsymbol{x})) = [f_1(\boldsymbol{x}) - f_1']/[f_1^{\mathrm{T}} - f_1'] \geqslant (\lambda + \lambda_1), \\
& [(\boldsymbol{x}_1^{\mathrm{T}} + \boldsymbol{p}_1) - \boldsymbol{x}_1]/\boldsymbol{p}_1 \geqslant (\lambda + \lambda_2)\boldsymbol{I}, \\
& [\boldsymbol{x}_1 - (\boldsymbol{x}_1^{\mathrm{T}} - \boldsymbol{p}_1)]/\boldsymbol{p}_1 \geqslant (\lambda + \lambda_2)\boldsymbol{I}, \\
& \mu_{f2}(f_2(\boldsymbol{x})) = [f_2(\boldsymbol{x}) - f_2']/[f_2^{\mathrm{T}} - f_2'] \geqslant (\lambda + \lambda_3), \\
& [(\boldsymbol{x}_2^{\mathrm{T}} + \boldsymbol{p}_2) - \boldsymbol{x}_2]/\boldsymbol{p}_2 \geqslant (\lambda + \lambda_4)\boldsymbol{I}, \\
& [\boldsymbol{x}_2 - (\boldsymbol{x}_2^{\mathrm{T}} - \boldsymbol{p}_2)]/\boldsymbol{p}_2 \geqslant (\lambda + \lambda_4)\boldsymbol{I}, \\
& \mu_{f3}(f_3(\boldsymbol{x})) = [f_3(\boldsymbol{x}) - f_3']/[f_3^{\mathrm{T}} - f_3'] \geqslant (\lambda + \lambda_5), \\
& \lambda + \lambda_i \leqslant 1, \;\; i = 1,2,3,4,5 \\
& \lambda,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4, \text{ and } \lambda_5 \in [0,1].
\end{aligned} \tag{11}$$

Example 2. Consider the following TLPP [1]:

$$\underset{x1}{\text{Max}}\quad f_1 = 7x_1 + 3x_2 - 4x_3,$$

where x_2, x_3 solve

$$\underset{x2}{\text{Max}}\quad f_2 = x_2,$$

where x_3 solves

$$\begin{aligned}
\underset{x3}{\text{Max}}\quad & f_3 = x_3 \\
\text{s.t.}\quad & x_1 + x_2 + x_3 \leqslant 3, \;\; x_1 + x_2 - x_3 \leqslant 1, \\
& x_1 + x_2 + x_3 \geqslant 1, \;\; -x_1 + x_2 + x_3 \leqslant 1, \\
& x_3 \leqslant 0.5, \\
& x1, x_2, \text{ and } x_3 \geqslant 0.
\end{aligned}$$

The individual solutions for each level are $f_1^{1\mathrm{T}} = 8.5$ at $\boldsymbol{x}_1^{1\mathrm{T}} = (1.5, 0, 0.5)$, $f_2^{1\mathrm{T}} = 1$ at $\boldsymbol{x}_2^{1\mathrm{T}} = (0, 1, 0)$ or $(0, 1, 0.5)$, and $f_3^{1\mathrm{T}} = 0.5$ at $\boldsymbol{x}_3^{1\mathrm{T}} = (1.5, 0, 0.5)$, $(0.5, 1, 0.5)$, or $(0, 0.5, 0.5)$ (see Table 2). In addition, $f_1' = 0$ instead of -0.5 and $f_2' = f_3' = 0$. Remember that only positive values are meaningful values. Also, assume that the control decision x_1 should be around 1.5 with 1.5 for tolerances on both positive and negative sides. The

Table 2
Individual solution for each objective at each level of Example 2

Vertex	Objective for each level			Note
	f_1	f_2	f_3	
Reference data for the first problem:				
$A(1.5,0,0.5)$	8.5*	0	0.5#	Level 1 optimum
$B(0,1,0)$	3	1*	0#	Level 2 optimum
$C(0.5,1,0.5)$	4.5	1*	0.5#	Alternative optimum of level 2
Reference data for the second problem:				
$D(0.9,0.6,0.5)$	6.18	0.58	0.5#	Compromise optimum of level 1 and level 2
$E(1.5,0,0.5)$	8.5	0	0.5*	Level 3 optimum
$F(0.5,1,0.5)$	4.5	1	0.5*	Alternative optimum of level 3
$G(0,0.5,0.5)$	−0.5	0.5	0.5*	Alternative optimum of level 3
Modified fuzzy range:				
	$n=1$	$n=2$	$n=3$	
f_n	$[0,6.1)$	$[0,0.6)$	$[0,0.5)$	Objective
x_n	$[0,0.9,1.8]$	$[0,0.6,2]$	—	Decision variables

Note: (1) f_1, f_2, and f_3 represents the objective of the first-level, the second level, and third level, respectively.
(2) f_3 values marked with "//" are only for reference there. These values will not use for the process.
(3) f_n values marked with "#" represent the optimal solution corresponding to the related level n.
(4) Modified fuzzy ranges are based on the solutions of the second problem. We choose one-sided fuzzy numbers for objectives and triangular fuzzy numbers for decision variables.

compensatory auxiliary problem for the second level can be represented as

$$
\begin{aligned}
\text{Max}\quad & \lambda + (1-\gamma)(\lambda_1+\lambda_2+\lambda_3)/3 \\
\text{s.t.}\quad & \boldsymbol{x}\in \boldsymbol{X}, \\
& 7x_1 + 3x_2 - 4x_3 \geqslant 8.5(\lambda+\lambda_1), \\
& x_1 \geqslant 1.5(\lambda+\lambda_2), \\
& x_1 + 1.5(\lambda+\lambda_2) \leqslant 3, \\
& f_2 = x_2 \geqslant (\lambda+\lambda_3), \\
& \lambda+\lambda_i \leqslant 1,\ \ i=1,2,3, \\
& \lambda,\lambda_1,\lambda_2,\lambda_3 \in [0,1].
\end{aligned}
$$

where $\boldsymbol{X}$ is the constraint set of the original problem and γ is the grade of compensation.

The compromise solutions for the first and second levels are: the degree of satisfaction is $\mu_{\text{and}}=0.62$, the objectives are $\boldsymbol{f}^{2\text{T}}=(f_1^{2\text{T}},\ f_2^{2\text{T}})=(6.1,0.6)$ at decisions $\boldsymbol{x}^{2\text{T}}=(x_1^{2\text{T}},\ x_2^{2\text{T}},\ x_3^{2\text{T}})=(0.9,0.6,0.5)$ with $\gamma=0.5$. Assume that the top two levels are satisfied with this solution, we then can solve the bottom level or the third level by using the results of the top two levels. Since some information has been compromised, some modifications are needed to obtain the optimum for the entire organization. Let us modify the decision variables and goals by the following triangular fuzzy numbers: $x_1=[0,0.9,1.8]$ and $x_2=[0,0.6,2]$, and let $f_1=[0,6.1)$, $f_2=[0,0.6)$ and $f_3=[0,0.5)$ (see Table 2).

The third-level problem can be considered as a hierarchical system. The auxiliary problem of the third level is thus formulated as

$$\begin{aligned}
\text{Max}\quad & \mu_{\text{and}} = \lambda + (1-\gamma)(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)/5 \\
\text{s.t.}\quad & \boldsymbol{x} \in \boldsymbol{X}, \\
& 7x_1 + 3x_2 - 4x_1 \geqslant 6.1(\lambda + \lambda_1), \\
& x_1 \geqslant 0.9(\lambda + \lambda_2), \\
& x_1 + 0.9(\lambda + \lambda_2) \leqslant 1.8, \\
& f_2 = x_2 \geqslant 0.6(\lambda + \lambda_3), \\
& x_2 + (\lambda + \lambda_4) \leqslant 2, \\
& f_3 = x_3 \geqslant 0.5(\lambda + \lambda_5), \\
& \lambda + \lambda_i \leqslant 1, \ \ i = 1,2,3,4,5 \\
& \lambda, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in [0,1].
\end{aligned}$$

The solution of the third-level auxiliary problem is $\boldsymbol{f}^* = (f_1^*, f_2^*, f_3^*) = (6.1, 0.6, 0.5)$ at decisions $\boldsymbol{x}^* = (x_1^*, x_2^*, x_3^*) = (0.9, 0.6, 0.5)$ and $\gamma = 0.5$ with a satisfaction level of 1.0, in other words, full satisfaction for all DMs. The crisp solution is $\boldsymbol{f}^* = (4.5, 1, 0.5)$ at $\boldsymbol{x}^* = (0.5, 1, 0.5)$ with satisfaction level: $(1.0, 0, 1.0)$.

Obviously, the above procedure of compensatory fuzzy approach can be extended easily to kth level programming problems, with $k \geqslant 4$. In this case, we must first solve $(k-1)$ auxiliary problems. Furthermore, if the solution obtained is not satisfactory, some interactive iteration will be needed to obtain an overall satisfaction.

5. Bi-level decentralized programming problem with equally important goals

A bi-level decentralized programming problem (BLDPP) is characterized by one decision center at the top level and p decision units at the bottom level with the assumption that the decision units in the lower level are independent of each other and under the control of the upper level. Let $f_{ki}(\boldsymbol{x})$ represents the objective function of the ith division at the kth level and c_{kij} represent the cost coefficient of the decision vector x_j for the ith division at the kth level. Thus, in a bi-level decentralized organization, $k = 1$ represents the upper level and $k = 2$ represents the lower level, and $i = 1, 2, \ldots, s_k$. $s_1 = 1$ indicates that there is only one decision unit at the upper level and $s_2 = s$ indicates that there are s decision units at the second or lower level. The equations for the BLDPP can now be represented by

$$\underset{x11}{\text{Max}} \quad f_{11}(\boldsymbol{x}) = \sum_j \boldsymbol{c}_{11j}^{\mathrm{T}} \boldsymbol{x}_{\mathrm{j}}, \quad \text{(upper level)} \tag{12}$$

where $\boldsymbol{x}_{21}, \boldsymbol{x}_{22}, \ldots, \boldsymbol{x}_{2s}$ solve

$$\left\{\begin{array}{l}
\underset{x21}{\text{Max}} \quad f_{21}(\boldsymbol{x}) = \sum_j \boldsymbol{c}_{21j}^{\mathrm{T}} \boldsymbol{x}_j \\
\cdots \\
\underset{2s}{\text{Max}} \quad f_{2s}(\boldsymbol{x}) = \sum_j \boldsymbol{c}_{2sj}^{\mathrm{T}} \boldsymbol{x}_j
\end{array}\right. \quad \text{(lower level)} \tag{13}$$

$$\text{s.t.} \quad \sum_{\forall k,i,j} \boldsymbol{A}_{ki}\boldsymbol{x}_j \leqslant \boldsymbol{b},$$
$$k = 1,2 \text{ and } i = 1,2,\ldots,s_k,$$
$$\boldsymbol{x}_j \geqslant \boldsymbol{0}, \; j = 1,2,\ldots,n.$$

where $j = 1,2,\ldots,n$ represents the jth decision variable, the constraint set is represented by $\boldsymbol{X}$, and $j = s_1 + s_2$. The decision variable $\boldsymbol{x}_j$, for $j = 2,3,\ldots,n$ is the decision of the separate decision unit of the second level.

To solve the BLDPP, Anandalingam [1] first proposed a procedure based on the KKT condition. Later, Anandalingam and Apprey [2] proposed a new approach to the multi-agent system based on the concept of penalty function. This approach has been used to solve a conflict resolution problem of an international river. Based on Anandalingam's structure, Wang et al. [20] recently provided a method for generating non-dominated solutions for a multi-objective, three-level decentralized system.

In this investigation, the fuzzy approach [19] will be used to solve this problem. Eqs. (12) and (13) will be solved by a procedure of supervised search. First the degree of satisfaction of the upper-level is transferred to the lower level. The lower level solves its problem based on the demand of the upper level. To obtain the initial values of the degrees of satisfaction in terms of fuzzy membership functions, we solve the following optimization problems independently:

$$f_{ki}^* = f_{ki}(\boldsymbol{x}_{ki}^*) = \max_{\boldsymbol{x}\in\boldsymbol{X}} f_{ki}(\boldsymbol{x}), \quad \forall k \text{ and } i.$$

Then,

$$f_{ki}' = \min_{\boldsymbol{x}\in\boldsymbol{X}} f_{ki}(\boldsymbol{x}), \quad \forall k \text{ and } i.$$

Assume that the tolerance vector is given by the upper-level DM. The membership functions $\mu_{x1}(\boldsymbol{x}_1)$ and $\mu_{ki}(f_{ki}(\boldsymbol{x}))$ can be formulated.

In a similar manner, the lower-level DMs can also formulate their degrees of satisfaction. Observe that each division or decision unit at the same level has his or her own objective. In this section, we shall assume that each decision unit in the same level has equally important objectives, that is, each unit has the same decision power from the viewpoint of the upper-level DM. Based on max–min (non-compensatory) operator, the auxiliary model of the lower-level problem is:

$$\begin{aligned}
\text{Max} \quad & \lambda \\
\text{s.t.} \quad & \boldsymbol{x} \in \boldsymbol{X}, \\
& \mu_{f1}(f_1(\boldsymbol{x})) = [f_1(\boldsymbol{x}) - f_1']/[f_1^{\mathrm{T}} - f_1'] \geqslant \lambda, \\
& [(\boldsymbol{x}_1^{\mathrm{T}} + \boldsymbol{p}_1) - \boldsymbol{x}_1]/\boldsymbol{p}_1 \geqslant \lambda \boldsymbol{I}, \\
& [\boldsymbol{x}_1 - (\boldsymbol{x}_1^{\mathrm{T}} - \boldsymbol{p}_1)]/\boldsymbol{p}_1 \geqslant \lambda \boldsymbol{I}, \\
& \mu_{f2i}(f_{2i}(\boldsymbol{x})) = [f_{2i}(\boldsymbol{x}) - f_{2i}']/[f_{2i}^{\mathrm{T}} - f_{2i}'] \geqslant \lambda \boldsymbol{I}, \; i = 1,2,\ldots,s \\
& \lambda \in [0,1].
\end{aligned} \tag{14}$$

For the compensatory model, the above equation can be transformed into:

$$
\begin{aligned}
\text{Max} \quad & \mu_{\text{and}} = \lambda + (1-\gamma)(\lambda_1 + \lambda_2 + \cdots + \lambda_{s+2})/(s+2) \\
\text{s.t.} \quad & \boldsymbol{x} \in \boldsymbol{X}, \\
& \mu_{f1}(f_1(\boldsymbol{x})) = [f_1(\boldsymbol{x}) - f_1']/[f_1^{\mathrm{T}} - f_1'] \geqslant \lambda + \lambda_1, \\
& [(\boldsymbol{x}_1^{\mathrm{T}} + \boldsymbol{p}_1) - \boldsymbol{x}_1]/\boldsymbol{p}_1 \geqslant (\lambda + \lambda_2)\boldsymbol{I}, \\
& [\boldsymbol{x}_1 - (\boldsymbol{x}_1^{\mathrm{T}} - \boldsymbol{p}_1)]/\boldsymbol{p}_1 \geqslant (\lambda + \lambda_2)\boldsymbol{I}, \\
& \mu_{f21}(f_{21}(\boldsymbol{x})) = [f_{21}(\boldsymbol{x}) - f_{21}']/[f_{21}^{\mathrm{T}} - f_{21}'] \geqslant (\lambda + \lambda_3)\boldsymbol{I}, \\
& \ldots \\
& \mu_{f2s}(f_{2s}(\boldsymbol{x})) = [f_{2s}(\boldsymbol{x}) - f_{2s}']/[f_{2s}^{\mathrm{T}} - f_{2s}'] \geqslant (\lambda + \lambda_{s+2})\boldsymbol{I}, \\
& \lambda + \lambda_i \leqslant 1, \ i = 1, 2, \ldots, s+2 \\
& \lambda, \lambda_1, \lambda_2, \ldots, \lambda_{s+2} \in [0, 1]
\end{aligned}
\tag{15}
$$

where the nomenclature has been defined before.

Example 3. Consider the following BLDPP [1]

$$\underset{x1}{\text{Max}} \quad f_1 = x_1 + y_1 + 2y_2 + y_3,$$

where y_1, y_2, and y_3 solve

$$
\begin{aligned}
\underset{y1}{\text{Max}} \quad & f_{21} = -x_1 + 3y_1 - 2y_2 - y_3 \\
\underset{y2}{\text{Max}} \quad & f_{22} = -x_1 - y_1 + 3y_2 - y_3 \\
\underset{y3}{\text{Max}} \quad & f_{23} = -x_1 - y_1 - y_2 + 3y_3 \\
\text{s.t.} \quad & 3x_1 + 3y_1 \leqslant 30, \ 2x_1 + y_1 \leqslant 20, \\
& y_2 \leqslant 10, \ y_2 + y_3 \leqslant 15, \\
& y_3 \leqslant 10, \ x_1 + 2y_1 + 2y_2 + y_3 \leqslant 40, \\
& x_1, y_1, y_2, \text{ and } y_3 \geqslant 0.
\end{aligned}
$$

Individual solutions of the top and bottom levels are $f_{11}^{\mathrm{T}} = 35$ at $\boldsymbol{x}_{11}^{\mathrm{T}} = (x_1^{\mathrm{T}}, y_1^{\mathrm{T}}, y_2^{\mathrm{T}}, y_3^{\mathrm{T}}) = (10, 0, 10, 0)$ or $(5, 5, 10, 3)$, $f_{21}^{\mathrm{T}} = 30$ at $\boldsymbol{x}_{21}^{\mathrm{T}} = (0, 10, 0, 0)$, $f_{22}^{\mathrm{T}} = 30$ at $\boldsymbol{x}_{22}^{\mathrm{T}} = (0, 0, 10, 0)$ and $f_{23}^{\mathrm{T}} = 30$ at $\boldsymbol{x}_{23}^{\mathrm{T}} = (0, 0, 0, 10)$ (see Table 3). Assume that $f_{11}' = f_{21}' = f_{22}' = f_{23}' = 0$. The tolerance for x_1 is that the upper-level DM is fully satisfied if the values for x_1 lie between 5 and 10. The required membership functions can thus be established.

Table 3
Individual solution for each objective of Example 3

Vertex	Objective for each decision unit				Note
	f_1	f_{21}	f_{22}	f_{23}	
$A(10,0,10,5)$	35*	−25	15	−5	Level 1 optimum
$B(5,5,10,5)$	35*	−5	15	−5	Alternative optimum
$C(0,10,0,0)$	10	30*	−10	−10	Level 2, division 2 optimum
$D(0,0,10,0)$	20	−10	30*	−10	Level 2, division 2 optimum
$E(0,0,0,10)$	10	−10	−10	30*	Level 2, division 3 optimum
Modified fuzzy range:					
	$n=1$	$n=21$	$n=22$	$n=23$	
f_n	$[10,35)$	$[0,30)$	$[0,30)$	$[0,30)$	
x_n	$[0,5,10]$	—	—	—	

Note: (1) f_1 represents the objective of the upper-level DM; f_{21}, f_{22}, and f_{23} represent the three DMs of the lower-level.
(2) "$*$" represents the optimal objective corresponding to the related divisions of 1st level and 2nd level.
(3) Modified fuzzy ranges are based on the solutions of individual problems. We choose one-sided fuzzy numbers for objectives and triangular fuzzy numbers for decision variables.

Using Eq. (15), we have

$$\begin{aligned}
\text{Max} \quad & \mu_{\text{and}} = \lambda + (1-\gamma)(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)/5 \\
\text{s.t.} \quad & \boldsymbol{x} \in \boldsymbol{X}, \\
& \mu_{f11}(f_1(\boldsymbol{x})) = [f_{11}(\boldsymbol{x}) - f'_{11}]/[f_{11}^{\mathrm{T}} - f'_{11}] = (x_1 + y_1 + 2y_2 + y_3)/35 \geqslant \lambda + \lambda_1, \\
& [x_1 - (x_1^{\mathrm{T}} - p_1)]/p_1 = x_1/5 \geqslant (\lambda + \lambda_2), \\
& x_1 \leqslant 10, \\
& \mu_{f21}(f_{21}(\boldsymbol{x})) = [f_{21}(\boldsymbol{x}) - f'_{21}]/[f_{21}^{\mathrm{T}} - f'_{21}] = (-x_1 + 3y_1 - 2y_2 - y_3)/30 \geqslant (\lambda + \lambda_3), \\
& \mu_{f22}(f_{22}(\boldsymbol{x})) = [f_{22}(\boldsymbol{x}) - f'_{22}]/[f_{22}^{\mathrm{T}} - f'_{22}] = (-x_1 - y_1 + 3y_2 - y_3)/30 \geqslant (\lambda + \lambda_4), \\
& \mu_{f23}(f_{23}(\boldsymbol{x})) = [f_{23}(\boldsymbol{x}) - f'_{23}]/[f_{23}^{\mathrm{T}} - f'_{23}] = (-x_1 - y_1 - y_2 + 3y_3)/30 \geqslant (\lambda + \lambda_5), \\
& \lambda + \lambda_i \leqslant 1, \ \ i = 1,2,3,4,5 \\
& \lambda \in [0,1].
\end{aligned}$$

The above auxiliary compensatory model was solved and the results are: $\boldsymbol{f}^* = (f_{11}^*, f_{21}^*, f_{22}^*, f_{23}^*) = (31.07, 6.43, 6.43, 6.43)$ with the optimal satisfaction $\mu_{\text{and}} = 0.28$ at $\boldsymbol{x}^* = (1.07, 7.5, 7.5, 7.5)$ and with $\gamma = 0.5$. In addition, 11 cases for different values of the compensatory parameter γ were also obtained and the results are listed in Table 4. The crisp solution is $\boldsymbol{f}^* = (35, -5, 15, -5)$ with satisfaction $\boldsymbol{\lambda} = (1, 0, 0.5, 0)$ at $\boldsymbol{x}^* = (5, 5, 10, 5)$ [1].

Table 4
Results based on the degree of compensation of Example 3

Degree of compensation γ	Degree of satisfaction μ_{and} (λ)	Solution (x_1, y_1, y_2, y_3)	Objective $(f_1, f_{21}, f_{22}, f_{23})$	Note
0	0.41 (0)	(3.75, 6.25, 8, 75, 6.25)	(33.75, 0, 10, 0)	
0.1	0.37 (0)	(3.75, 6.25, 8.75, 6.25)	(33.75, 0, 10, 0)	Change of the solution set
0.2	0.34 (0.17)	(2.5, 7.5, 7.5, 7.5)	(32.5, 5, 5, 5)	
0.3	0.32 (0.17)	(2.5, 7.5, 7.5, 7.5)	(32.5, 5, 5, 5)	
0.4	0.30 (0.17)	(2.5, 7.5, 7.5, 7.5)	(32.5, 5, 5, 5)	Change of the solution set
0.5	0.28 (0.21)	(1.07, 7.5, 7.5, 7.5)	(31.07, 9.28, 17.72)	
0.6	0.27 (0.21)	(1.07, 7.5, 7.5, 7.5)	(31.07, 9.28, 17.72)	
0.7	0.25 (0.21)	(1.07, 7.5, 7.5, 7.5)	(31.07, 9.28, 17.72)	
0.8	0.24 (0.21)	(1.07, 7.5, 7.5, 7.5)	(31.07, 9.28, 17.72)	
0.9	0.23 (0.21)	(1.07, 7.5, 7.5, 7.5)	(31.07, 9.28, 17.72)	
1	0.21 (0.21)	(1.07, 7.5, 7.5, 7.5)	(31.07, 9.28, 17.72)	

Note: (1) Werners' fuzzy "and" operator is $\mu_{\text{and}} = \gamma \min_i(\mu_i) + (1-\gamma)(\Sigma_i \mu_i)/m$, $0 \leqslant \mu \leqslant 1$, $0 \leqslant \gamma \leqslant 1$, with m membership functions, $i = 1, \ldots, m$. Here $m = 5$ for the bi-level decentralized programming problem of Example 3.

(2) When $0 \leqslant \gamma \leqslant 0.1, 0.2 \leqslant \gamma \leqslant 0.4$, and $0.5 \leqslant \gamma \leqslant 1$, the solution is belong to a different set, respectively.

(3) λ is the degree of satisfaction for non-compensatory operation, i.e. max–min operation. The compensatory solution and non-compensatory solution are equal when $\gamma = 1$.

6. Bi-level decentralized programming problem with unequal goals

In the previous section, the objectives or the goals in the same level are treated as equally important. In practice, this is not the case. In fact, unequal objectives or goals at the same level are more common. In this section, a convex combination of the different weights of the objectives will be used to handle the unequal goals.

Assume that the second level has s decision units or objectives. Assign weights $w_{21}, w_{22}, \ldots, w_{2s}$ to objectives $f_{21}, f_{22}, \ldots, f_{2s}$, respectively, where the summation of all the weights are equal to one. Using the weighted expressions in Eq. (15), we obtain the following new expression for the lower level:

$$\begin{array}{ll} \text{Max} & \mu^{+}_{\text{and}} = \lambda + (1-\gamma)[\lambda_1 + \lambda_2 + s(w_{21}\lambda_3 + w_{22}\lambda_4 + \cdots + w_{2s}\lambda_{s+2})]/(s+2) \\ \text{s.t.} & \boldsymbol{x} \in \boldsymbol{Y}, \end{array} \tag{16}$$

where $\boldsymbol{Y}$ represents the constraint set.

Example 4. Consider the same example used in Example 3 except with the addition of weights.

Suppose that the upper-level DM feels that the first objective f_{21} at the lower level is three times more important than either the second objective f_{22} or the third objective f_{23}. Also, suppose that the second objective is twice as important as the third objective at the lower level. Then, the concept of pairwise comparison of importance of the different objectives can be used and the relative weights can be obtained by the eignvector method [17]:

$$\boldsymbol{A} = [a_{ij}] = \begin{bmatrix} w_{21}/w_{21} & w_{21}/w_{22} & w_{21}/w_{23} \\ w_{22}/w_{21} & w_{21}/w_{22} & w_{22}/w_{23} \\ w_{23}/w_{21} & w_{23}/w_{22} & w_{23}/w_{23} \end{bmatrix} = \begin{bmatrix} 1/1 & 3/1 & 3/1 \\ 1/3 & 1/1 & 2/1 \\ 1/3 & 1/2 & 1/1 \end{bmatrix},$$

where objective i is a_{ij} times more important than objective j.

The desired weights can be obtained by first obtaining the eigenvalues from $(\boldsymbol{A} - \lambda \boldsymbol{I})w = \boldsymbol{0}$ and then solving a set of three simultaneous equations. The weights obtained are: $(w_{21}, w_{22}, w_{23}) = (0.59, 0.25, 0.16)$. After substituting these weights into Eq. (16), the objective function of the lower-level auxiliary model with compensation is

$$\text{Max } \mu_{\text{and}}^{+} = \lambda + (1-\gamma)[\lambda_1 + \lambda_2 + 3(0.59\lambda_3 + 0.25\lambda_4 + 0.16\lambda_5)]/5.$$

This auxiliary problem with the above objective function and the same set of constraints as that listed for Example 3 was solved and the results are: $\boldsymbol{f}^* = (f_{11}^*, f_{21}^*, f_{22}^*, f_{23}^*) = (31.73, 8.85, 5.77, 5.77)$. The optimal satisfaction level is: $\mu_{\text{and}}^{+} = 0.28$ at $\boldsymbol{x}^* = (0.96, 8.27, 7.5, 7.5)$ with $\gamma = 0.5$. Compared to the solutions obtained in Example 3, the effect of unequal weight is obvious.

7. Multiple level decentralized programming problems

Obviously, the approach can be expanded easily to multiple level decentralized programming problems (MLDPP), which is composed of a decision center at the top level and several divisions or decision units at each of the lower levels. It is essentially the multiple level hierarchy decentralized system proposed by Anandalingam [1]. Fig. 1 illustrates the general structure. This is a very difficult problem and there exists no explicit relationships among the decision units at the same level. In this section, the fuzzy approach will be used to solve the MLDPPs with equal and unequal importance in the goals at the same level. Using the fuzzy or tolerance nature, this large complicated problem becomes tractable with the fuzzy approach.

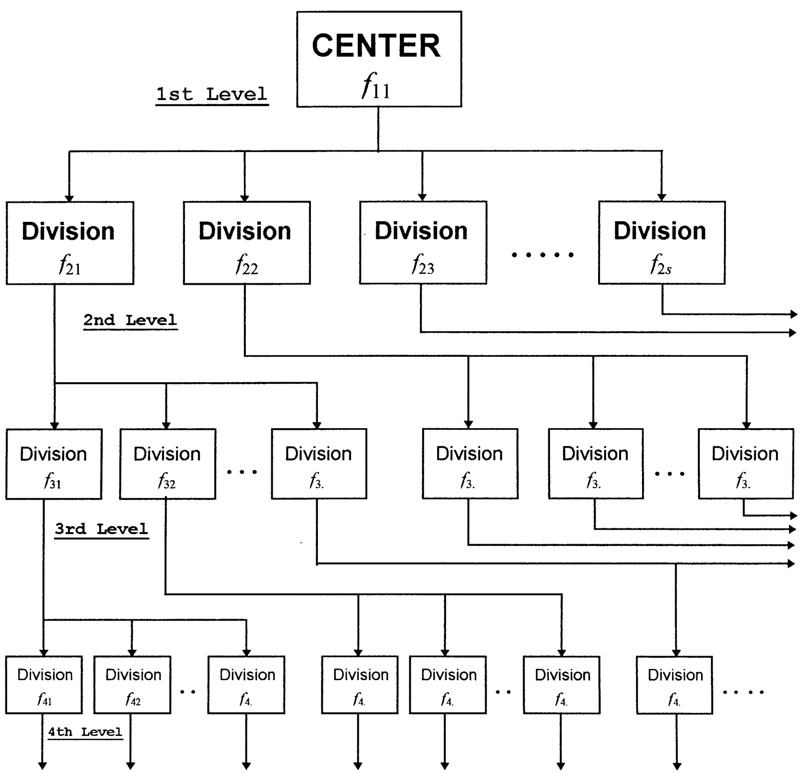

Fig. 1. A decentralized hierarchical structure.

The procedure of compensatory fuzzy approach for MLDPPs is essentially the same as that for BLDPPs. Assume that there are s decision units at the second level and q decision units at the third level, the objective functions for this third-level problem with equally important objectives at each level is

$$\mu_{\text{and}}^{+} = \lambda + (1-\gamma)[\lambda_1 + \lambda_2 + (\lambda_3 + \cdots + \lambda_{s+2}) + (\lambda_{s+3} + \cdots + \lambda_{2s+2}) + (\lambda_{2s+3} + \cdots + \lambda_{2s+q+2})]/(2s+q+2), \tag{17}$$

where $(\lambda_3, \ldots, \lambda_{s+2})$ represent the goals of the second level, $(\lambda_{s+3} + \cdots + \lambda_{2s+2})$ represent the decisions of the second level, and $(\lambda_{2s+3} + \cdots + \lambda_{2s+q+2})$ represent the goals of the third level.

For unequal goals, assuming the weights $w_{21}, w_{22}, \ldots, w_{2s}$ are assigned to objectives $f_{21}, f_{22}, \ldots, f_{2s}$, respectively, for the second level, and $w_{31}, w_{32}, \ldots, w_{3q}$ are assigned to objectives $f_{31}, f_{32}, \ldots, f_{3q}$, respectively, for the third level. Then, the objective function for the third-level auxiliary model can be represented as

$$\mu_{\text{and}}^{+} = \lambda + (1-\gamma)[\lambda_1 + \lambda_2 + s(w_{21}\lambda_3 + \cdots + w_{2s}\lambda_{s+2})/2 + s(w_{21}\lambda_{s+3} + \cdots + w_{2s}\lambda_{2s+2})/2 + q(w_{31}\lambda_{2s+3} + \cdots + w_{3q}\lambda_{2s+q+2})]/(2s+q+2), \tag{18}$$

where $(\lambda_3, \ldots, \lambda_{s+2})$ represent the goals of the second level, $(\lambda_{s+3} + \cdots + \lambda_{2s+2})$ represent the decisions of the second level, and $(\lambda_{2s+3} + \cdots + \lambda_{2s+q+2})$ represent the goals of the third levels.

Eqs. (17) and (18) are the auxiliary objectives for the third-level, and their constraint sets should include the decision information from the first and the second levels as discussed previously. The approach can be extended easily to kth level decentralized programming problems, with $k \geqslant 4$. If the current solution is not satisfactory, interactive iteration can be added to modify the degree of satisfaction until a satisfactory solution is obtained.

8. Conclusions and discussions

The fuzzy approach is an effective and powerful compromise technique for solving multiple level programming problems. The advantage of this approach is that it exploits the inherent vagueness and uncertainty of a large multiple hierarchical system so that this system becomes tractable and, in a way, much more simplified. Because of this exploitation of the vagueness and also because we use membership function to represent the degree of satisfaction, the very complex large decentralized multiple level problems can be solved effectively by this approach

Although this and the earlier paper [19] have shown the effectiveness of the fuzzy approach, only a start has been made. Many areas need to be explored and developed in this direction. For example, only the compensatory operator proposed by Werners [24] has been used. There are many other compensatory aggregators to be investigated. The concept of ordered weighted averaging aggregator due to Yager [26] appears to be a promising one. Another direction is in the development of more effective interactive iterative procedure.

Acknowledgements

We would like to thank Professor U.P. Wen at National Tsin Hwa University in Taiwan (ROC) for his valuable suggestions.

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