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A minimax distribution free procedure for mixed inventory models involving variable lead time with fuzzy lost sales

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Abstract

Recently, the continuous review inventory models with allowable shortages have been extended to include variable lead time, where a fixed fraction of the demand during the stockout period is backordered. However, in practice, the backorder (or lost sales) rate may change slightly due to some uncertainties. To incorporate this reality, this article attempts to apply the fuzzy set concepts to deal with the uncertain lost sales rate. For a situation where information about the lead time demand distribution is partial, we utilize the minimax distribution free procedure to find the optimal inventory strategy in the fuzzy sense. Two numerical examples are given to illustrate the results.

Keywords: Inventory; Fuzzy numbers; Membership function; Extension principle; Minimax distribution free procedure

1. Introduction

Traditionally, economic order quantity (EOQ) models dealing with continuous review inventory problems often assume that the demand during the stockout period is either completely backordered or completely lost; and the lead time is viewed as a prescribed constant or a random variable, which therefore is not subject to control [1,2]. However, these are not always true; for example, in real markets, it can be observed that, when the inventory system is out of stock, some of the customers may be willing to wait for their demand, while others may fill their demand from another source. And hence, many researchers extended the continuous review inventory models to include the partial backorder situation (see, e.g., [3–8]).

On the other hand, lead time usually consists of the following components [9]: order preparation, order transit, supplier lead time, delivery time and setup time. In many practical situations, lead time can be reduced at an added crashing cost; in other words, it is controllable. Moreover, the Japanese successful experiences of using Just-In-Time (JIT) production has evidenced that there are substantial advantages and benefits that can be obtained through various efforts of reducing lead time. Recently, lead time reduction has received a lot of interest by several researchers (see, e.g., [5–8,10,11]). Specifically, under various settings, Ouyang et al. [5], Ouyang and Wu [6], Moon and Choi [7] and Hariga and Ben-Daya [8]

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investigated the lead time reduction on continuous review inventory models, in which the partial backorder situation is also incorporated.

The underlying assumption in the above partial backorder models, no matter whether lead time reduction is involved [5–8] or not [3,4], is that the fraction of excess demand backordered (or lost) is a fixed constant. However, we can see from real markets that many products such as clothes, shoes and vegetables, whose backorder rate (or equivalently, lost sales rate) may be influenced by substitute, brand loyalty, customers' preference and waiting patience, etc. In other words, the lost sales rate may change slightly due to these potential factors, and it is difficult to measure an exact value for lost sales rate. Therefore, in this article, we attempt to apply the fuzzy set concepts to deal with the ambiguous lost sales rate. We believe that if we express the fuzzy lost sales rate as the neighborhood of the fixed lost sales rate, then it may match the real situation better. In fact, the application of fuzzy set concepts on EOQ inventory models have been proposed by many authors (e.g., [12–16]). However, their studies are almost concentrated on the simple EOQ models in which restrictive assumptions, such as demand is known with certainty and lead time is constant, are included so that they have few applications in real inventory systems.

The purpose of this paper is to modify Moon and Choi's [7] stochastic continuous review inventory model with variable lead time and partial backorders to capture the reality of uncertain backorders (lost sales). Specifically, in this paper we introduce two fuzzinesses of lost sales rate. We first fuzzify the lost sales rate to a triangular fuzzy number. Then, we utilize statistical methods to obtain the confidence interval for lost sales rate, and employ it to get the statistic-fuzzy number. For each case, under the assumption that only partial information about the lead time demand distribution is given, we apply the minimax distribution-free approach to solve the problem and develop an algorithm to find the optimal solution. Note that the minimax distribution-free approach was originally proposed by Scarf [17] to solve the newsboy problem in a situation where only the mean and standard deviation of the stochastic demand are known. Recently, Gallego and Moon [18] presented a new and very compact proof of the optimality of Scarf's ordering rule for the newsboy problem and extended the analysis to several cases. The applications of this approach to other production/inventory models, see for examples, [6–8,19–21].

2. Membership function of the fuzzy total cost

Under the assumptions that:

(i) the lead time L has n mutually independent components each having a different crashing cost for reducing lead time;
(ii) a fraction b $(0 \leqslant b \leqslant 1)$ of the demand during the stockout period can be backordered, and the remaining fraction $1-b$ is lost.

Moon and Choi [7] extended Ouyang et al.'s [5] model by simultaneously optimizing the order quantity, reorder point and lead time. Specifically, the expected annual total cost which is composed of ordering cost, inventory holding cost, stockout cost and lead time crashing cost is expressed by

$$\begin{aligned}\mathrm{EAC}(Q,\ r,\ L) &= A\frac{D}{Q}+h\left[\frac{Q}{2}+r-DL+(1-b)B(r)\right]+\frac{D}{Q}[\pi+\pi_0(1-b)]B(r)+\frac{D}{Q}R(L)\\ &= A\frac{D}{Q}+h\left[\frac{Q}{2}+r-DL+(1-b)B(r)\right]+\frac{D}{Q}[\pi+\pi_0(1-b)]B(r)\\ &\quad+\frac{D}{Q}\left\{c_i(L_{i-1}-L)+\sum_{j=1}^{i-1}c_j(v_j-u_j)\right\},\ \ L\in[L_i,\ L_{i-1}],\end{aligned} \tag{1}$$

where D is the average demand per year, A the fixed ordering cost per order, h the inventory holding cost per unit per year, π the fixed penalty cost per unit short, π_0 the marginal profit per unit, Q the order quantity, L the length of lead time, X the lead time demand which has a distribution function (d.f.) F with finite mean DL and standard deviation $\sigma\sqrt{L}$, where σ denotes the standard deviation of the demand per unit time, r the reorder point; $r = DL + k\sigma\sqrt{L}$, where k is the safety factor, $B(r)$ the expected demand shortage at the end of cycle; $B(r) = \int_r^\infty (x-r)\,\mathrm{d}F(x)$, u_i, v_i, c_i are the ith component of lead time L has a minimum duration u_i and normal duration v_i, and a crashing cost per unit time c_i, where $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_n$, L_i the length of lead time with components $1, 2, \ldots, i$ crashed to their minimum duration; $L_i = \sum_{j=1}^n v_j \sum_{j=1}^i (v_j - u_j)$, $i = 1, 2, \ldots, n$. Further, let $L_0 = \sum_{j=1}^n v_j$ and $R(L)$ be the lead time crashing cost per cycle; $R(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(v_j - u_j)$, $L \in [L_i, L_{i-1}]$ and $R(L_0) = 0$.

It is noted that the backorder rate, b, in model (1) is viewed as a fixed constant. However, as mentioned previously, in many practical situations, the backorder rate may change slightly due to various uncertainties. In order to match the realistic situation better, we therefore attempt to modify model (1) by fuzzifying the backorder rate (or equivalently, by fuzzifying the lost sales rate) to a fuzzy number.

In what follows, for convenience, we let $a \equiv 1 - b$ denote the lost sales rate. Therefore, for any $Q > 0$, $r > 0$ and $L > 0$, we may express the expected annual total cost function (1) as

$$G_{(Q,r,L)}(a) \equiv \mathrm{EAC}(Q,\ r,\ L) = [A + R(L) + \pi B(r)]\frac{D}{Q} + h\left(\frac{Q}{2} + r - DL\right) + a\left(h + \pi_0\frac{D}{Q}\right)B(r). \tag{2}$$

We now replace the lost sales rate, a, by the fuzzy number $\tilde{a}$, and consider the fuzzy number $\tilde{a}$ as the triangular fuzzy number, $\tilde{a} = (a - \Delta_1,\ a,\ a + \Delta_2)$, where $0 < \Delta_1 < a$ and $0 < \Delta_2 \leqslant 1 - a$, with the following membership function:

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x - a + \Delta_1}{\Delta_1} & \text{if } a - \Delta_1 \leqslant x \leqslant a, \\ \dfrac{a + \Delta_2 - x}{\Delta_2} & \text{if } a \leqslant x \leqslant a + \Delta_2, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The picture is shown in Fig. 1.

For any $Q > 0$, $r > 0$ and $L > 0$, we let $G_{(Q,r,L)}(x) = y (> 0)$. By extension principle [22,23], the membership function of the fuzzy cost $G_{(Q,r,L)}(\tilde{a})$ is given by

$$\mu_{G_{(Q,r,L)}(\tilde{a})}(y) = \begin{cases} \sup_{x \in G^{-1}_{(Q,r,L)}(y)} \mu_{\tilde{a}}(x) & \text{if } G^{-1}_{(Q,r,L)}(y) \neq \emptyset, \\ 0 & \text{if } G^{-1}_{(Q,r,L)}(y) = \emptyset. \end{cases} \tag{4}$$

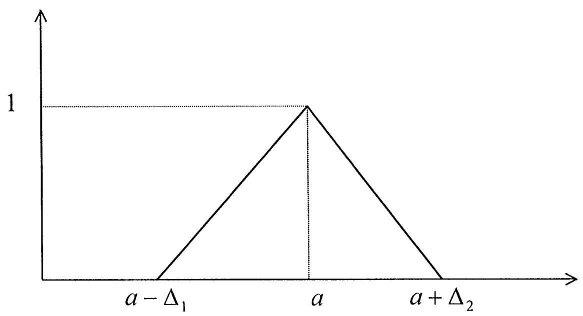

Fig. 1. Triangular fuzzy number $\tilde{a}$.

From $G_{(Q,r,L)}(x) = y$ and Eq. (2), we get

$$[A + R(L) + \pi B(r)]\frac{D}{Q} + h\left(\frac{Q}{2} + r - DL\right) + x\left(h + \pi_0\frac{D}{Q}\right)B(r) = y. \tag{5}$$

Consequently,

$$x = \frac{yQ - \left[WD + hQ\left(Q/2 + r - DL\right)\right]}{(hQ + \pi_0 D)B(r)}, \tag{6}$$

where $W = A + R(L) + \pi B(r)$.

Therefore, from (3) and (6), the membership function of $G_{(Q,r,L)}(\tilde{a})$ can be written as

$$\mu_{G_{(Q,r,L)}(\tilde{a})}(y) = \begin{cases} \dfrac{yQ - \left[WD + h\left(Q/2 + r - DL\right)\right]}{(hQ + \pi_0 D)B(r)\Delta_1} - \dfrac{a - \Delta_1}{\Delta_1}, & y_1 \leqslant y \leqslant y_2, \\ \dfrac{a + \Delta_2}{\Delta_2} + \dfrac{\left[WD + hQ\left(Q/2 + r - DL\right)\right] - yQ}{(hQ + \pi_0 D)B(r)\Delta_2}, & y_2 \leqslant y \leqslant y_3, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where

$$y_1 = W\frac{D}{Q} + h\left(\frac{Q}{2} + r - DL\right) + (a - \Delta_1)\left(h + \pi_0\frac{D}{Q}\right)B(r),$$

$$y_2 = W\frac{D}{Q} + h\left(\frac{Q}{2} + r - DL\right) + a\left(h + \pi_0\frac{D}{Q}\right)B(r),$$

and

$$y_3 = W\frac{D}{Q} + h\left(\frac{Q}{2} + r - DL\right) + (a + \Delta_2)\left(h + \pi_0\frac{D}{Q}\right)B(r).$$

The picture of the membership function of $G_{(Q,r,L)}(\tilde{a})$ is shown in Fig. 2.

We now derive the centroid (for more details, see e.g. [24, p.336]) of $\mu_{G_{(Q,r,L)}(\tilde{a})}(y)$ as follows:

$$\begin{aligned} M(Q,\, r,\, L) = \frac{\int_{-\infty}^{\infty} y\mu_{G_{(Q,r,L)}(\tilde{a})}(y)\,\mathrm{d}y}{\int_{-\infty}^{\infty} \mu_{G_{(Q,r,L)}(\tilde{a})}(y)\,\mathrm{d}y} &= \frac{1}{3}(y_1 + y_2 + y_3) \\ &= \mathrm{EAC}(Q,\, r,\, L) + \frac{(\Delta_2 - \Delta_1)}{3}\left(h + \pi_0\frac{D}{Q}\right)B(r), \end{aligned} \tag{8}$$

which is an estimate of the expected annual total cost in the fuzzy sense.

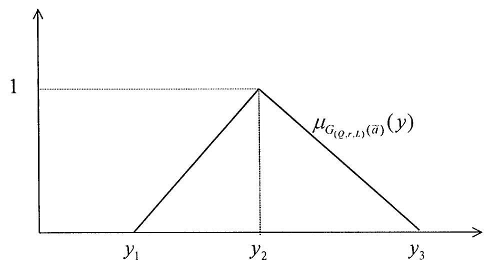

Fig. 2. Triangular fuzzy number of $G_{(Q,r,L)}(\tilde{a})$.

Remark 1. In the traditional inventory models with a mixture of backorders and lost sales, the backorder (or lost sales) rate is viewed as a given constant. However, in practical situations, the backorder (or lost sales) rate may change slightly for some uncertainties. Hence, it is unrealistic to regard the crisp value $\mathrm{EAC}(Q, r, L)$ as a true expected annual total cost. In order to match the realistic situation, we replace here the fixed lost sales rate, a, by the triangular fuzzy number, $\tilde{a} = (a - \Delta_1, a, a + \Delta_2)$, where $0 < \Delta_1 < a$ and $0 < \Delta_2 \leqslant 1 - a$, and use the value $M(Q, r, L)$ as the estimate of the expected annual total cost. Note that the absolute difference value between $M(Q, r, L)$ and $\mathrm{EAC}(Q, r, L)$ is equal to $|\Delta_2 - \Delta_1|/3\left(h + \pi_0 D/Q\right)B(r)$. This result can be viewed in Eq. (8).

Remark 2. If $\Delta_1 = \Delta_2$, then Fig. 1 is an isosceles triangle and Eq. (8) reduces to $M(Q, r, L) = \mathrm{EAC}(Q, r, L)$, which implies that the fuzzy case becomes the crisp case; i.e., the fixed lost sales inventory model is a special case of our new fuzzy lost sales inventory model. If $\Delta_1 < \Delta_2$, then the triangle in Fig. 1 is skewed to the right, and in this case $M(Q, r, L) > \mathrm{EAC}(Q, r, L)$. If $\Delta_1 > \Delta_2$, then the triangle in Fig. 1 is skewed to the left, and in this case $M(Q, r, L) < \mathrm{EAC}(Q, r, L)$.

3. Optimal solution

In this section, we investigate the optimal inventory strategy in the fuzzy sense for a situation where only the mean and standard deviation of lead time demand are known, but the distributional form of lead time demand is unknown. In this case, the exact value of the expected demand shortage at the end of the cycle $B(r)$ cannot be found, and hence, the optimal value of (Q, r, L) which minimize the fuzzy expected annual total cost $M(Q, r, L)$ cannot be obtained.

Now, we attempt to utilize the minimax distribution-free procedure to solve this problem. For convenience, we let $\mathbf{F}$ denote the class of d.f. F of X which has finite mean DL and standard deviation $\sigma\sqrt{L}$. Then the minimax distribution-free approach for this problem is to find the least favorable d.f. F in $\mathbf{F}$ for each (Q, r, L), and then to minimize the fuzzy expected annual total cost over Q, r and L. In mathematical symbolization, our problem is to solve

$$\underset{Q>0,\, r>0,\, L>0}{\mathrm{Min}} \ \underset{F\in\mathbf{F}}{\mathrm{Max}} \ M(Q, r, L). \tag{9}$$

To this end, we need the following proposition, whose proof can easily be obtained by using $r = DL + k\sigma\sqrt{L}$ and Lemma 1 in Gallego and Moon [18].

Proposition 1. *For any* $F \in \mathbf{F}$,

$$B(r) \leqslant \tfrac{1}{2}\sigma\sqrt{L}(\sqrt{1+k^2} - k). \tag{10}$$

Moreover, the upper bound (10) *is tight*.

Next, using (10), (8) and (1), and allowing the safety factor k as a decision variable instead of the reorder point r (because $r = DL + k\sigma\sqrt{L}$), problem (9) is reduced to minimize

$$\begin{aligned} M^u(Q, k, L) &= \mathrm{EAC}^u(Q, k, L) + \frac{(\Delta_2 - \Delta_1)}{3}\left(h + \pi_0\frac{D}{Q}\right)\frac{1}{2}\sigma\sqrt{L}(\sqrt{1+k^2} - k) \\ &= [A + R(L)]\frac{D}{Q} + h\left(\frac{Q}{2} + k\sigma\sqrt{L}\right) + \frac{1}{2}\sigma\sqrt{L}(\sqrt{1+k^2} - k) \\ &\quad \times \left[\pi\frac{D}{Q} + \left(a + \frac{\Delta_2 - \Delta_1}{3}\right)\left(h + \pi_0\frac{D}{Q}\right)\right], \quad L \in [L_i, L_{i-1}], \end{aligned} \tag{11}$$

where $M^u(\cdot)$ and $\mathrm{EAC}^u(\cdot)$ denote the least upper bound of the fuzzy expected annual total cost $M(\cdot)$ and the crisp expected annual total cost $\mathrm{EAC}(\cdot)$, respectively.

For fixed Q and k, we can show that $M^u(Q, k, L)$ is concave in $L \in [L_i, L_{i-1}]$; hence, when Q and k are given, the minimum expected annual total cost in the fuzzy sense will occur at the end points of the interval $[L_i, L_{i-1}]$. On the other hand, for fixed $L \in [L_i, L_{i-1}]$, it can be shown that $M^u(Q, k, L)$ is convex in (Q, k). Therefore, for a given $L \in [L_i, L_{i-1}]$, the minimum value of $M^u(Q, k, L)$ will occur at the point, say (Q^*, k^*), which satisfies the first-order conditions $\partial M^u(Q, k, L)/\partial Q = 0$ and $\partial M^u(Q, k, L)/\partial k = 0$, simultaneously; i.e., the point (Q^*, k^*) satisfies the following equations:

$$Q = \sqrt{\frac{2D}{h}\left\{A + R(L) + \frac{1}{2}\sigma\sqrt{L}(\sqrt{1+k^2} - k)\left[\pi + \pi_0\left(a + \frac{\Delta_2 - \Delta_1}{3}\right)\right]\right\}} \tag{12}$$

and

$$\frac{k}{\sqrt{1+k^2}} = 1 - \frac{2hQ}{\pi D + (hQ + \pi_0 D)\left(a + \dfrac{\Delta_2 - \Delta_1}{3}\right)}. \tag{13}$$

From Eqs. (12) and (13), we note that it is difficult to find the closed-form solutions of (Q^*, k^*). Therefore, we develop the following algorithm to find the optimal solutions for the order quantity, safety factor and lead time.

Algorithm 1.
Step 1. For given, L_i, $i = 0, 1, 2, \ldots, n$, perform (i)–(iv).

(i) Start with $k_{i1} = 0$.
(ii) Substituting k_{i1} into (12) evaluates Q_{i1}.
(iii) Utilizing Q_{i1} determines k_{i2} from (13).
(iv) Repeat (ii) to (iii) until no change occurs in the values of Q_i and k_i. Denote the solution by (Q_i^*, k_i^*).

Step 2. For each (Q_i^*, k_i^*, L_i), $i = 0, 1, 2, \ldots, n$, calculate the corresponding fuzzy expected annual total cost $M^u(Q_i^*, k_i^*, L_i)$ by utilizing (11).
Step 3. Find $\min_{i=0,1,2,\ldots,n} M^u(Q_i^*, k_i^*, L_i)$.
If $M^u(Q_{\tilde{a}}, k_{\tilde{a}}, L_{\tilde{a}}) = \min_{i=0,1,2,\ldots,n} M^u(Q_i^*, k_i^*, L_i)$, then $(Q_{\tilde{a}}, k_{\tilde{a}}, L_{\tilde{a}})$ is the optimal solution in the fuzzy sense.

Note that once $k_{\tilde{a}}$ and $L_{\tilde{a}}$ are obtained, the optimal reorder point $r_{\tilde{a}} = DL_{\tilde{a}} + k_{\tilde{a}}\sigma\sqrt{L_{\tilde{a}}}$ follows.

Example 1. In order to illustrate the above solution procedure, let us consider an inventory system with the data used in Moon and Choi ([7], which is the same as in Ouyang et al. [5]): $D = 600$ units per year, A =\$200 per order, h =\$20 per unit per year, π =\$50 per unit short, π_0 =\$150 per unit lost, $\sigma = 7$ units per week, and the lead time has three components with the data shown in Table 1.

Here, we consider three cases: $(\Delta_1, \Delta_2) = (0.2, 0.2)$, $(\Delta_1, \Delta_2) = (0.1, 0.4)$, and $(\Delta_1, \Delta_2) = (0.4, 0.1)$. We solve each case for lost sales rate $a = 0.5$. The results of the solution procedure are summarized in Table 2.

From Table 2, when $\Delta_1 = \Delta_2 = 0.2$ (in this situation, the fuzzy case becomes the crisp case), by comparing $M^u(Q_i^*, r_i^*, L_i)$, $i = 0, 1, 2, 3$, we obtain the optimal solution $(Q_{\tilde{a}}, r_{\tilde{a}}, L_{\tilde{a}}) = (158, 63, 3)$ and the minimum expected annual total cost in the fuzzy sense $M^u(Q_{\tilde{a}}, r_{\tilde{a}}, L_{\tilde{a}})$= \$3726.30, which are the same as shown in Moon and Choi [7]. Moreover, when $\Delta_1 = 0.1$ and $\Delta_2 = 0.4$, i.e., the fuzzy number

Table 1
Lead time data

Lead time component i	Normal duration v_i (days)	Minimum duration u_i (days)	Unit crashing cost c_i (\$/day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

Table 2
Solution procedure of Algorithm 1 ($L_i$in weeks)

Δ_1	Δ_2	i	L_i	$R(L_i)$	Q_i^*	r_i^* (k_i^*)	$M^u(Q_i^*, r_i^*, L_i)$
0.2	0.2	0	8	0.0	167	137 (2.2373)	\$4243.97
		1	6	5.6	161	108 (2.2856)	4013.37
		2	4	22.4	155	79 (2.3279)	3773.82
		3	3	57.4	158	63 (2.3089)	**3726.30**
0.1	0.4	0	8	0.0	170	139 (2.3645)	4358.10
		1	6	5.6	163	111 (2.4171)	4113.99
		2	4	22.4	158	81 (2.4647)	3857.27
		3	3	57.4	160	64 (2.4479)	**3798.11**
0.4	0.1	0	8	0.0	164	134 (2.0988)	4121.28
		1	6	5.6	158	106 (2.1428)	3905.31
		2	4	22.4	153	77 (2.1797)	3684.32
		3	3	57.4	156	61 (2.1584)	**3649.34**

$\tilde{a} = (0.4,\ 0.5,\ 0.9)$, we have $(Q_{\tilde{a}},\ r_{\tilde{a}},\ L_{\tilde{a}}) = (160,\ 64,\ 3)$ and $M^u(Q_{\tilde{a}},\ r_{\tilde{a}},\ L_{\tilde{a}})=$ \$3798.11. Here we note that $\mathrm{EAC}^u(Q^u,\ r^u,\ L^u)=$ \$3726.30 is the corresponding minimum expected annual total cost in the crisp case. Therefore, the absolute relative variation for the expected annual total cost between fuzzy case and crisp case can be measured and given by

$$\delta_M = \frac{|M^u(Q_{\tilde{a}}, r_{\tilde{a}}, L_{\tilde{a}}) - \mathrm{EAC}^u(Q^u, r^u, L^u)|}{\mathrm{EAC}^u(Q^u, r^u, L^u)} \times 100\% = \frac{|3798.11 - 3726.30|}{3726.30} \times 100\% = 1.93\%.$$

Similarly, for the case $\Delta_1 = 0.4$ and $\Delta_2 = 0.1$, i.e., the fuzzy number $\tilde{a} = (0.1,\ 0.5,\ 0.6)$, we have $(Q_{\tilde{a}},\ r_{\tilde{a}},\ L_{\tilde{a}}) = (156,\ 61,\ 3)$ and $M^u(Q_{\tilde{a}},\ r_{\tilde{a}},\ L_{\tilde{a}})=$ \$3649.34, and the absolute relative variation for the expected annual total cost between fuzzy case and crisp case $\delta_M = |3649.34 - 3726.30|/3726.30 \times 100\% = 2.06\%$.

Moreover, we examine the performance of the distribution-free approach against the normal distribution in the fuzzy sense. Consider a situation where $\Delta_1 = 0.1$ and $\Delta_2 = 0.4$. Using procedures similar to Algorithm 1, the optimal solution for normal distribution case is $(Q_N,\ r_N,\ L_N) = (121,\ 73,\ 4)$, and the corresponding total cost $M^N(Q_N,\ r_N,\ L_N)=$ \$2954.09. If one uses $(Q_{\tilde{a}},\ r_{\tilde{a}},\ L_{\tilde{a}}) = (160,\ 64,\ 3)$ (obtained by distribution free procedure) instead of using $(Q_N,\ r_N,\ L_N) = (121,\ 73,\ 4)$ for a normal distribution, then the added cost is $M^N(160,\ 64,\ 3) - M^N(121,\ 73,\ 4) =$ \$3174.15 − 2954.09 = \$220.06. This is the largest amount that we would be willing to pay for the knowledge of d.f. F, and such a quantity can be regarded as the expected value of additional information (EVAI) (see, e.g. [18,19]). We note that the concept of EVAI is analogous to EVPI (expected value of perfect information) discussed in the decision theory. Here when it is certain that the information is perfect (i.e., this specific form of d.f. F of lead time demand is going to occur in future), then the EVAI can be viewed as EVPI (for more details about EVPI, see e.g. [25]).

4. Using the sample data to fuzzify the lost sales rate

We now turn our attention to another possible situation where the actual lost sales rate, a, is unknown. When the actual lost sales rate a is unknown, we cannot utilize Eqs. (12) and (13) to determine the optimal inventory strategy. In order to estimate the value for the actual lost sales rate a, an intuitive method is to collect the random sample data of lost sales rate during past time and compute the mean of the sample measurements. Although the sample mean is a good point estimator of a, the error of estimation cannot be found. Consequently, here we attempt to utilize the technique known as *confidence interval estimation* to construct a confidence interval for the actual lost sales rate a. Then we employ the obtained confidence interval to find an estimate value for a in the fuzzy sense. The procedures are as follows.

Suppose that we have collected m random sample data of lost sales rate during past time, say $a_1, a_2, \ldots, a_m$. Then the sample mean is $\bar{a} = (1/m)\sum_{i=1}^{m} a_i$ and the sample variance is $s^2 = [1/(m-1)]\sum_{i=1}^{m}(a_i - \bar{a})^2$. It can be shown that $\bar{a}$ is a good point estimator of the lost sales rate a. Furthermore, suppose these sample data satisfy certain statistical assumptions. Then using the statistical method, we get a $(1-\alpha)\times 100\%$ confidence interval for a as follows:

$$\left[\bar{a} - t_{m-1}(\alpha_1)\frac{s}{\sqrt{m}},\quad \bar{a} + t_{m-1}(\alpha_2)\frac{s}{\sqrt{m}}\right], \tag{14}$$

where, $\alpha_1, \alpha_2 > 0$, $\alpha_1 + \alpha_2 = \alpha$, and $t_{m-1}(\alpha_i)$, $i = 1, 2$, is the tabulated upper α_i point of the t-distribution with $m-1$ degrees of freedom; that is, if T be a random variable distributed as t-distribution with $m-1$ degrees of freedom, then $t_{m-1}(\alpha_i)$ is the value that satisfies the following condition:

$$P[T > t_{m-1}(\alpha_i)] = \alpha_i, \quad i = 1, 2. \tag{15}$$

Once we obtain a $(1-\alpha)\times 100\%$ confidence interval of lost sales rate a, we can employ it to express the statistic-fuzzy lost sales rate $\tilde{a}^*$ as the following level $1-\alpha$ triangular fuzzy number:

$$\tilde{a}^* = \left[\bar{a} - t_{m-1}(\alpha_1)\frac{s}{\sqrt{m}},\quad \bar{a},\ \bar{a} + t_{m-1}(\alpha_2)\frac{s}{\sqrt{m}};\ 1-\alpha\right], \quad \text{where } \alpha_1 + \alpha_2 = \alpha. \tag{16}$$

And the membership function of statistic-fuzzy lost sales rate $\tilde{a}^*$ is given by

$$\mu_{\tilde{a}^*}(x) = \begin{cases} \dfrac{(1-\alpha)\left[x - \bar{a} + t_{m-1}(\alpha_1)\dfrac{s}{\sqrt{m}}\right]}{t_{m-1}(\alpha_1)\dfrac{s}{\sqrt{m}}} & \text{if } \bar{a} - t_{m-1}(\alpha_1)\dfrac{s}{\sqrt{m}} \leqslant x \leqslant \bar{a}, \\[2ex] \dfrac{(1-\alpha)\left[\bar{a} + t_{m-1}(\alpha_2)\dfrac{s}{\sqrt{m}} - x\right]}{t_{m-1}(\alpha_2)\dfrac{s}{\sqrt{m}}} & \text{if } \bar{a} \leqslant x \leqslant \bar{a} + t_{m-1}(\alpha_2)\dfrac{s}{\sqrt{m}}, \\[2ex] 0 & \text{otherwise.} \end{cases} \tag{17}$$

The picture is shown in Fig. 3.

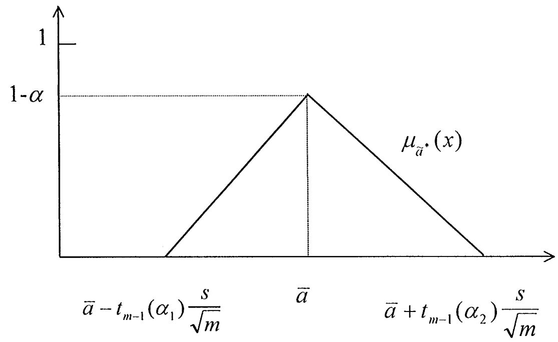

Fig. 3. Level $1-\alpha$ triangular fuzzy number $\tilde{a}^*$.

We now let $G_{(Q,r,L)}(x)=z$, where $G_{(Q,r,L)}(\cdot)$ be defined as in (2). By extension principle [22,23], the membership function of the fuzzy cost $G_{(Q,r,L)}(\tilde{a}^*)$ is

$$\mu_{G_{(Q,r,L)}(\tilde{a}^*)}(z)=\begin{cases}\sup_{x\in G^{-1}_{(Q,r,L)}(z)}\ \mu_{\tilde{a}^*}(x) & \text{if } G^{-1}_{(Q,r,L)}(z)\neq\emptyset,\\ 0 & \text{if } G^{-1}_{(Q,r,L)}(z)=\emptyset.\end{cases}\tag{18}$$

From $G_{(Q,r,L)}(x)=z$ and Eqs. (2), (17) and (18), we obtain the membership function of $G_{(Q,r,L)}(\tilde{a}^*)$ as follows:

$$\mu_{G_{(Q,r,L)}(\tilde{a}^*)}(z)=\begin{cases}\dfrac{(1-\alpha)\{zQ-[WD+hQ(Q/2+r-DL)]\}}{(hQ+\pi_0D)B(r)t_{m-1}(\alpha_1)s/\sqrt{m}}-\dfrac{(1-\alpha)[\bar{a}-t_{m-1}(\alpha_1)s/\sqrt{m}]}{t_{m-1}(\alpha_1)s/\sqrt{m}} & \text{if } z_1\leqslant z\leqslant z_2,\\ \dfrac{(1-\alpha)[\bar{a}+t_{m-1}(\alpha_2)s/\sqrt{m}]}{t_{m-1}(\alpha_2)s/\sqrt{m}}+\dfrac{(1-\alpha)\{[WD+hQ(Q/2+r-DL)]-zQ\}}{(hQ+\pi_0D)B(r)t_{m-1}(\alpha_2)s/\sqrt{m}} & \text{if } z_2\leqslant z\leqslant z_3\\ 0 & \text{otherwise,}\end{cases}\tag{19}$$

where

$$z_1=W\frac{D}{Q}+h\left(\frac{Q}{2}+r-DL\right)+\left[\bar{a}-t_{m-1}(\alpha_1)\frac{s}{\sqrt{m}}\right]\left(h+\pi_0\frac{D}{Q}\right)B(r),$$

$$z_2=W\frac{D}{Q}+h\left(\frac{Q}{2}+r-DL\right)+\bar{a}\left(h+\pi_0\frac{D}{Q}\right)B(r)$$

and

$$z_3=W\frac{D}{Q}+h\left(\frac{Q}{2}+r-DL\right)+\left[\bar{a}+t_{m-1}(\alpha_2)\frac{s}{\sqrt{m}}\right]\left(h+\pi_0\frac{D}{Q}\right)B(r).$$

Therefore, the centroid of $\mu_{G_{(Q,r,L)}(\tilde{a}^*)}(z)$ can be obtained by a formula, and is given by

$$\begin{aligned}V(Q,\ r,\ L)&=\frac{\int_{-\infty}^{\infty}z\mu_{G_{(Q,r,L)}(\tilde{a}^*)}(z)\,\mathrm{d}z}{\int_{-\infty}^{\infty}\mu_{G_{(Q,r,L)}(\tilde{a}^*)}(z)\,\mathrm{d}z}\\&=\frac{1}{3}(z_1+z_2+z_3)\\&=\mathrm{EAC}^*(Q,\ r,\ L)+\frac{1}{3}[t_{m-1}(\alpha_2)-t_{m-1}(\alpha_1)]\frac{s}{\sqrt{m}}\left(h+\pi_0\frac{D}{Q}\right)B(r),\end{aligned}\tag{20}$$

where

$$\mathrm{EAC}^*(Q,\ r,\ L) = [A + R(L) + \pi B(r)]\frac{D}{Q} + h\left(\frac{Q}{2} + r - DL\right) + \bar{a}\left(h + \pi_0\frac{D}{Q}\right)B(r).$$

Note that the only difference between $\mathrm{EAC}^*(Q,\ r,\ L)$ and $\mathrm{EAC}(Q,\ r,\ L)$ (Eq. (2)) is that the lost sales rate a in (2) is replaced by the sample mean $\bar{a}$.

We regard the value $V(Q, r, L)$ as an estimate of the expected annual total cost in the fuzzy sense.

Remark 3. If $\alpha_1 = \alpha_2 = \alpha/2$, then $t_{m-1}(\alpha_1) = t_{m-1}(\alpha_2)$, and hence the triangle in Fig. 3 becomes an isosceles triangle; this implies $V(Q,\ r,\ L) = \mathrm{EAC}^*(Q,\ r,\ L)$. If $\alpha_1 > \alpha_2$, then $t_{m-1}(\alpha_1) < t_{m-1}(\alpha_2)$, and hence the triangle is skewed to the right; it implies $V(Q,\ r,\ L) > \mathrm{EAC}^*(Q,\ r,\ L)$. If $\alpha_1 < \alpha_2$, then $t_{m-1}(\alpha_1) > t_{m-1}(\alpha_2)$, and hence the triangle is skewed to the left; it implies $V(Q,\ r,\ L) < \mathrm{EAC}^*(Q,\ r,\ L)$.

Again, we investigate the optimal inventory strategy in the fuzzy sense for the case where the distributional form of lead time demand X is unknown. Using similar arguments as discussed in Section 3, we first obtain the least upper bound of expected annual total cost $V(Q,\ r,\ L)$ in the fuzzy sense as follows:

$$V^u(Q,\ k,\ L) = [A + R(L)]\frac{D}{Q} + h\left(\frac{Q}{2} + k\sigma\sqrt{L}\right) + \frac{1}{2}\sigma\sqrt{L}(\sqrt{1+k^2} - k) \times \left[\pi\frac{D}{Q} + \left(\bar{a} + \frac{t_{m-1}(\alpha_2) - t_{m-1}(\alpha_1)}{3}\frac{s}{\sqrt{m}}\right)\left(h + \pi_0\frac{D}{Q}\right)\right], \quad L \in [L_i,\ L_{i-1}]. \tag{21}$$

Then, we seek to minimize $V^u(Q,\ k,\ L)$ by optimizing over Q, k and L. As discussed in previous section, we can show that $V^u(Q,\ k,\ L)$ is concave in $L \in [L_i,\ L_{i-1}]$ for fixed $(Q,\ k)$. Hence, for fixed $(Q,\ k)$, the minimum expected annual total cost in the fuzzy sense will occur at the end points of the interval $[L_i,\ L_{i-1}]$. Moreover, it can be shown that $V^u(Q,\ k,\ L)$ is convex in $(Q,\ k)$ for fixed $L \in [L_i,\ L_{i-1}]$. Then upon setting $\partial V^u(Q,\ k,\ L)/\partial Q = 0$ and $\partial V^u(Q,\ k,\ L)/\partial k = 0$, we obtain

$$Q = \sqrt{\frac{2D}{h}\left\{A + R(L) + \frac{1}{2}\sigma\sqrt{L}(\sqrt{1+k^2} - k)\left[\pi + \pi_0\left(\bar{a} + \frac{t_{m-1}(\alpha_2) - t_{m-1}(\alpha_1)}{3}\frac{s}{\sqrt{m}}\right)\right]\right\}} \tag{22}$$

and

$$\frac{k}{\sqrt{1+k^2}} = 1 - \frac{2hQ}{\pi D + (hQ + \pi_0 D)\left(\bar{a} + [t_{m-1}(\alpha_2) - t_{m-1}(\alpha_1)]/3\ \left(s/\sqrt{m}\right)\right)}. \tag{23}$$

Thus, for given $\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_1 + \alpha_2 = \alpha$, we can establish the following algorithm to find the optimal solutions for order quantity, safety factor and lead time.

Algorithm 2.
Step 1. Collect m sample data of lost sales rate, say $a_1, a_2, \ldots, a_m$, and then evaluate sample mean $\bar{a} = (1/m)\sum_{i=1}^{m} a_i$ and sample standard deviation $s = \sqrt{[1/(m-1)]\sum_{i=1}^{m}(a_i - \bar{a})^2}$. In addition, for given α_1 and α_2 $(\alpha_1 + \alpha_2 = \alpha)$, consult the t-distribution table to find the values of $t_{m-1}(\alpha_1)$ and $t_{m-1}(\alpha_2)$, where $t_{m-1}(\alpha_i)$ is the upper α_i point of the t-distribution with $m-1$ degrees of freedom, $i = 1, 2$.

Table 3
Solution procedure of Algorithm 2 (L_i in weeks)

i	L_i	$R(L_i)$	$\hat{Q}_i^*$	$\hat{r}_i^*\,(\hat{k}_i^*)$	$V^u(\hat{Q}_i^*, L_i)$
0	8	0	167	137 (2.2561)	$4260.78
1	6	5.6	161	109 (2.3051)	4028.18
2	4	22.4	156	79 (2.3481)	3786.10
3	3	57.4	158	63 (2.3294)	**3736.86**

Step 2. For given, L_i, $i = 0, 1, 2, \ldots, n$, perform (i)–(iv).

(i) Start with $k_{i1} = 0$.
(ii) Substituting k_{i1} into (22) evaluates Q_{i1}.
(iii) Utilizing Q_{i1} determines k_{i2} from (23).
(iv) Repeat (ii) to (iii) until no change occurs in the values of Q_i and k_i Denote the solution by $(\hat{Q}_i^*, \hat{k}_i^*)$.

Step 3. For each $(\hat{Q}_i^*, \hat{k}_i^*, L_i)$, $i = 0, 1, 2, \ldots, n$, calculate the corresponding fuzzy expected annual total cost $V(\hat{Q}_i^*, \hat{k}_i^*, L_i)$ by utilizing (21).
Step 4. Find $\min_{i=0,1,2,\ldots,n} V^u(\hat{Q}_i^*, \hat{k}_i^*, L_i)$. If $V^u(Q_{\tilde{a}^*}, k_{\tilde{a}^*}, L_{\tilde{a}^*}) = \min_{i=0,1,2,\ldots,n} V^u(\hat{Q}_i^*, \hat{k}_i^*, L_i)$, then $(Q_{\tilde{a}^*}, k_{\tilde{a}^*}, L_{\tilde{a}^*})$ is the optimal solution in the fuzzy sense.

Once again, when $k_{\tilde{a}^*}$ and $L_{\tilde{a}^*}$ are obtained, the optimal reorder point $r_{\tilde{a}^*} = DL_{\tilde{a}^*} + k_{\tilde{a}^*}\sigma\sqrt{L_{\tilde{a}^*}}$ follows.

Example 2. We use the same data as in Example 1, but the random sample of size 6 yields the sample mean of lost sales rate $\bar{a} = 0.5$ and sample standard deviation $s = 0.195$. We determine the optimal inventory strategy in the fuzzy sense for the case where $\alpha_1 = 0.1$ and $\alpha_2 = 0.05$. Consulting the t-distribution table, we find $t_5(0.1) = 1.476$ and $t_5(0.05) = 2.015$. The results of the solution procedure are summarized in Table 3.

From Table 3, by comparing $V^u(\hat{Q}_i^*, \hat{r}_i^*, L_i)$, $i = 0, 1, 2, 3$, we find that the optimal strategy $(Q_{\tilde{a}^*}, r_{\tilde{a}^*}, L_{\tilde{a}^*}) = (158, 63, 3)$, which leads to the minimum expected annual total cost, in the fuzzy sense, of $3736.86.

5. Concluding remarks

In this paper, we modify the continuous review inventory models involving variable lead time with a mixture of backorders and lost sales by fuzzifying the lost sales rate. Two fuzzinesses of lost sales rates are introduced. In Section 2, we discuss how to apply the fuzzy set concepts to deal with the problem in which no statistical data can be used. Moreover, when statistical data are available, we discuss how to combine the statistical and fuzzy technologies to deal with such a problem in Section 4. In each fuzzy case, we investigate a computing schema for the modified inventory model where information about the lead time demand distribution is partial. We solve the problem by utilizing the minimax distribution-free procedure and develop an algorithm procedure to find the optimal order quantity, reorder point and lead time. Furthermore, two numerical examples are given to illustrate the results.

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