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Some maximal inequalities and complete convergences of negatively associated random sequences

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1. Introduction

The concept of positively associated random variables (PA) was introduced by Lehmann (1966) and was generalized and studied by Esary et al. (1967). Since then there have been many papers published on this subject, and also its extensions and some related applications, for instance, by Karlin and Rinott (1980a, b), Shaked (1982), Newman and Wright (1982), Wood (1983), Burton et al. (1986), Cox and Grimmett (1984), Roussas (1991, 1994) and Newman (1980) among others.

The concept of negatively associated random sequence (NA) was introduced by Joag-Dev and Proschan (1983) although a very special case was first introduced by Lehmann (1966). The former derived several important properties about NA sequences and also discussed some applications in statistics. Compared to PA, the study of NA sequence has received less attention in the literature. There are some applications of NA in the areas of Probability, reliability and multivariate analysis. Readers may refer to Karlin and Rinott (1980b), Ebrahimi and Ghosh (1981), Block et al. (1982), Newman (1984), Joag-Dev (1990), Joag-Dev and Proschan (1983), Matula (1992) and Roussas (1994) among others.

It is to be noted that NA is related to but is not simply the dual of PA. There are also several kinds of negative dependence between random variables. However, as it was pointed out by Joag-Dev and Proschan (1983), NA has one distinct advantage over the other known types of negative dependence: some closure property holds for NA, but not for the other three types of negative dependence. They have also derived other important properties and studied some useful applications.

Recently, some authors focused on the problem of limiting behavior of partial sums of NA sequences. Su et al. (1996) derived some moment inequalities of partial sums and a weak convergence

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for a strong stationary NA sequence. Lin (1997) set up an invariance principal for NA sequences. Su and Qin (1997) also studied some limiting results for NA sequences. More recently, Liang and Su (1999) and Liang (2000) considered some complete convergence for weighted sums of NA sequences. The case of Cesaro sums was also considered.

Those results, especially some moment inequality by Yang (2000), undoubtedly propose important contributions to both theoretical developments and practical applications for the NA sequence. However, there still exists some difficulty in this area. For instance, so far we do not have Levy-type maximal inequality for the NA sequence, but we do so for the PA sequence, which can be stated as follows:

$$P\left\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant cs_n\right\}\leqslant 2P\{|S_n|\geqslant (c-\sqrt{2})s_n\},$$

where $c>\sqrt{2}, s_n^2=\operatorname{Var} S_n$.

As a part of the contribution for the development of such an important inequality, in this paper, we construct a maximal inequality for the partial sum of NA sequence. Through this maximal inequality, it is not only expected that some known results can be improved, but also that a useful result of complete convergence of NA partial sum can be obtained.

2. Main results

Random variables $X_1, X_2, \ldots, X_n$ are said to be negatively associated (NA) if for every pair of disjoint subsets T_1, T_2 of $\{1,2,\ldots,n\}$,

$$\operatorname{Cov}(f_1(X_i, i\in T_1), f_2(X_j, j\in T_2))\leqslant 0,$$

where f_1 and f_2 increase (or decreasing) for every variable, such that this covariance exists. A sequence of random variables $X_j, j\in J$ is said to be negatively associated if every finite subfamily of it is negatively associated.

For convenience, let $\ell(x)$ and $\beta(x)$ both denote non-negative monotone non-decreasing functions throughout the paper. Denote $\operatorname{inv}\beta(x)\equiv\inf\{u:\beta(u)=x\}$. Let the partial sums be denoted by $S_i=\sum_{j=1}^{i}(X_j-EX_j),\ i=1,\ldots,n$.

Theorem 2.1. *Let $X_1, X_2, \ldots, X_n$ be an identically distributed* NA *sequence. If, for $p\geqslant 2$*

(A1) $E|X_1|^p<\infty$,

(A2) $x\geqslant C_0(n(\log n)^{p^2})^{1/2}, n\geqslant \max\{\log^{2p^2+p}+\ n, \exp C_0^{-1}16^{1/p}E^{1/p}|X_1|^p\}$ *for some constant C_0, then,*

$$\begin{aligned}P\left\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant x\right\}&\leqslant C_3 n(x\log n)^{-p}E|X_1|^p I(|X_1|>x\log^{-p}n)\\&\quad+C_3 n\exp\{-C_1\log^p n+C_2\}\\&\leqslant Cn(x\log n)^{-p}+Cn\exp\{-C_1\log^p n\},\end{aligned}$$

where C, C_1, and C_3 are positive constants which may be related to p and C_0 but not to n, $C_2 \equiv 4C_0^{-2}\{EX_1^2 I(|X_1| \leqslant x\log^{-p} n) + EX_1^2 I(|X_1| > x\log^{-p} n)\}$.

As an application of Theorem 2.1, we can deduce the following:

Theorem 2.2. *Let $X_1, X_2, \ldots, X_n$ be an identically distributed* NA *sequence. If, for some $\varepsilon > 0$ and $p \geqslant 2$,*
(B1) $n(\log n)^{p^2} \leqslant (\operatorname{inv}\beta(\varepsilon n))^2$ *for sufficiently large n,*
(B2) $\sum_{n=1}^{\infty} \ell(n)(\operatorname{inv}\beta(\varepsilon n)\log n)^{-p} < \infty$,
(B3) $\sum_{n=1}^{\infty} \ell(n)\exp(-\varepsilon\log^{p} n) < \infty$,
and condition (A1) *is satisfied, then,*

$$\sum_{n=1}^{\infty} \frac{\ell(n)}{n} P\left\{\max_{1\leqslant i\leqslant n} \beta(|S_i|) \geqslant \varepsilon n\right\} < \infty. \tag{2.1}$$

Theorem 2.3. *Let $X_1, X_2, \ldots, X_n$ be an identically distributed* NA *sequence satisfying all conditions of Theorem* 2.2, *then*

$$\sum_{n=1}^{\infty} \frac{\ell(n) - \ell([n/2])}{n} P\left\{\sup_{i\geqslant n} \frac{\beta(|S_i|)}{i} \geqslant \varepsilon\right\} < \infty. \tag{2.2}$$

Taking $x = \operatorname{inv}\beta(\varepsilon n)$ in Theorem 2.1, *it follows from* (B1), (B2), *and* (B3), *we can conclude Theorem* 2.2.

By applying the same technique for the derivation of (8.3.19) in Lin and Lu (1997, p. 190), we can obtain (2.2) directly from (2.1) and this proves Theorem 2.3.

If we take $\ell(n) = n^{p\alpha-1}, \beta(n) = n^{1/\alpha}$, $1/2 < \alpha \leqslant 1$, then we can immediately have the following.

Corollary 2.4. *Let $X_1, X_2, \ldots, X_n$ be an identically distributed* NA *sequence with $E|X_1|^p < \infty$, $p \geqslant 2$. Then, for the given $\varepsilon > 0$ and for $1/2 < \alpha \leqslant 1$, we have*

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left\{\max_{1\leqslant i\leqslant n} |S_i| \geqslant \varepsilon n^{\alpha}\right\} < \infty \tag{2.3}$$

and

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left\{\sup_{i\geqslant n} |S_i|/i^{\alpha} \geqslant \varepsilon\right\} < \infty. \tag{2.4}$$

Remark. The condition of identical distribution can be weakened slightly to be uniformly bounded in probability. If we take $\alpha = 1$, then Corollary 2.4 becomes Theorem 2 in Su and Qin (1997), from which it can be seen that (A1) is a necessary condition for (2.2).

3. Proof of Theorem 2.1

We need a result of Theorem 2 in Su et al. (1996) as a lemma which can be stated as follows:

Lemma 3.1 (Su et al., 1996). *Let* $X_1, X_2, \dots, X_n$ *be an* NA *sequence with* $EX_i = 0$. *For* $p \geqslant 2$, *if* $\beta_p \equiv \sup E|X_n|^p < \infty$, *then there exists some constant* $K_p > 0$ *depending only on* p, *such that for any integer* n,

$$E\left(\max_{1\leqslant k\leqslant n}|S_k|^p\right) \leqslant K_p\{n\beta_p + (n\beta_2)^{p/2}\}.$$

Denote $T_j \equiv X_j - EX_j$, $c_n \equiv x\log^{-p} n$. For $1 \leqslant i \leqslant n$, let

$$S_{1i} = \sum_{j=1}^{i}\{T_jI(|T_j| \leqslant c_n) + c_nI(T_j > c_n) - c_nI(T_j < -c_n)\} \equiv \sum_{j=1}^{i} T_{1j},$$

$$S_{2i} = \sum_{j=1}^{i} T_jI(T_j < -c_n) \equiv \sum_{j=1}^{i} T_{2j},$$

$$S_{3i} = \sum_{j=1}^{i} T_jI(T_j > c_n) \equiv \sum_{j=1}^{i} T_{3j},$$

$$S_{4i} = \sum_{j=1}^{i} c_nI(T_j < -c_n),$$

$$\text{and } S_{5i} = \sum_{j=1}^{i} (-c_n)I(T_j > c_n).$$

Then, we have

$$\begin{aligned}
P\left\{\max_{1\leqslant i\leqslant n}|S_i| \geqslant x\right\} \leqslant{}& P\left\{\max_{1\leqslant i\leqslant n}|S_{1i}| \geqslant x/3\right\} \\
&+ P\left\{\max_{1\leqslant i\leqslant n}|S_{2i}| \geqslant x\log n\right\} + P\left\{x \leqslant \max_{1\leqslant i\leqslant n}|S_{2i}| < x\log n\right\} \\
&+ P\left\{\max_{1\leqslant i\leqslant n}|S_{3i}| \geqslant x\log n\right\} + P\left\{x \leqslant \max_{1\leqslant i\leqslant n}|S_{3i}| < x\log n\right\} \\
&+ P\left\{\max_{1\leqslant i\leqslant n}|S_{4i}| \geqslant x\log^{p^2} n\right\} + P\left\{x/3 \leqslant \max_{1\leqslant i\leqslant n}|S_{4i}| < x\log^{p^2} n\right\} \\
&+ P\left\{\max_{1\leqslant i\leqslant n}|S_{5i}| \geqslant x\log^{p^2} n\right\} + P\left\{x/3 \leqslant \max_{1\leqslant i\leqslant n}|S_{5i}| < x\log^{p^2} n\right\} \\
\equiv{}& I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9. \qquad (3.1)
\end{aligned}$$

In the following, the proof of Theorem 2.1 is divided into five parts.

(a) Estimates of I_2 and I_4

It follows from conditions (A1) and (A2) that

$$\max_{1\leqslant i\leqslant n}|ES_{3i}| \leqslant 2nE|X_1|^p(x\log^{-p} n)^{1-p} \leqslant x/8, \tag{3.2}$$

where we use the fact that $\log n \geqslant C_0^{-1}16^{1/p}E^{1/p}|X_1|^p$.

Denote $U_j = T_{3j} - ET_{3j}$, by (3.2),

$$I_4 = P\{S_{3n} \geqslant x\log n\} \leqslant P\{S_{3n} - ES_{3n} \geqslant x\log n/2\}. \tag{3.3}$$

Conditions (A1) and (A2) yield the following:

$$\begin{aligned}\sup_j EU_j^2 &\leqslant CE|X_1|^pI(|X_1| > c_n)[x\log^{-p} n]^{2-p}\\ &\leqslant CE|X_1|^pI(|X_1| > c_n)Cn^{-(p-2)/2}.\end{aligned}$$

Note that the right-hand side (RHS) of (3.3) equals $P\{\sum_{j=1}^n U_j \geqslant x\log n/2\}$ and $p(4-p) \leqslant 4$ for $p \geqslant 2$, again it follows from Lemma 3.1 and (A1) that

$$\begin{aligned}I_4 &\leqslant C(x\log n)^{-p}\left\{n\sup_j E|U_j|^p + \left(n\sup_j EU_j^2\right)^{p/2}\right\}\\ &\leqslant Cn(x\log n)^{-p}\{Cn + (n^{1-(p-2)/2}E|X_1|^pI(|X_1| > x\log^{-p} n))^{p/2}\}\\ &\leqslant Cn(x\log n)^{-p}\{n + (n^{p(4-p)/4}E^{p/2}|X_1|^pI(|X_1| > x\log^{-p} n))\}\\ &\leqslant Cn(x\log n)^{-p}\{E|X_1|^pI(|X_1| > x\log^{-p} n)\}^{p/2}\\ &\leqslant Cn(x\log n)^{-p}E|X_1|^pI(|X_1| > x\log^{-p} n).\end{aligned} \tag{3.4}$$

Analogously, we can also obtain

$$I_2 \leqslant Cn(x\log n)^{-p}E|X_1|^pI(|X_1| > x\log^{-p} n) \tag{3.5}$$

(b) Estimates of I_3 and I_5

Note that, in I_5, $\max_{1\leqslant i\leqslant n}|S_{3i}| = S_{3n} < x\log n$, and the number of non-zero terms in the summand S_{3n} is given by

$$\begin{aligned}|J| &\equiv |\{j: X_j - EX_j > x\log^{-p} n\}|\\ &\leqslant [\log^{p+1} n] + 1\end{aligned}$$

with probability one, where $|J|$ denotes the cardinality of set J.

By (A2), noting that $n \geqslant \log^{2p^2+p} n$, by the Markov inequality, we have

$$
\begin{aligned}
I_5 &\leqslant P\left\{\sum_{j=1}^{n}(X_j - EX_j)I(X_j - EX_j > x\log^{-p} n)I(j\in J) > x\right\}\\
&\leqslant \sum_{j=1}^{n} E\{|X_j - EX_j|I(|X_j - EX_j| > x\log^{-p} n)I(j\in J)\}x^{-1}\\
&\leqslant E\left\{|X_1 - EX_1|I(|X_1 - EX_1| > x\log^{-p} n)\sum_{j=1}^{n} I(j\in J)\right\}x^{-1}\\
&\leqslant C\log^{p+1} nE|X_1|^pI(|X_1| > x\log^{-p} n)(x\log^{-p} n)^{1-p}x^{-1}\\
&\leqslant Cn(x\log n)^{-p}E|X_1|^pI(|X_1| > x\log^{-p} n). \qquad (3.6)
\end{aligned}
$$

Analogously, we can also conclude that

$$I_3 \leqslant Cn(x\log n)^{-p}E|X_1|^pI(|X_1| > x\log^{-p} n). \qquad (3.7)$$

(c) Estimates of I_6 and I_8

Again it follows from the Markov inequality that

$$
\begin{aligned}
I_6 &\leqslant P\left\{\sum_{j=1}^{n} c_nI(T_j < -c_n) \geqslant x\log^{p^2} n\right\}\\
&\leqslant \sum_{j=1}^{n} c_nP\{|T_j| > c_n\}(x\log^{p^2} n)^{-1}\\
&\leqslant n\log^{-p} nE|X_1|^pI(|X_1| > x\log^{-p} n)(x\log^{-p} n)^{-p}\log^{-p^2} n\\
&\leqslant Cn(x\log n)^{-p}E|X_1|^pI(|X_1| > x\log^{-p} n). \qquad (3.8)
\end{aligned}
$$

Analogously, we can also conclude that

$$I_8 \leqslant Cn(x\log n)^{-p}E|X_1|^pI(|X_1| > x\log^{-p} n). \qquad (3.9)$$

(d) Estimates of I_7 and I_9.

Similarly, in I_7, we note that $\max_{1\leqslant i\leqslant n}|S_{4i}| = S_{4n} < x\log^{p^2} n$ and the number of non-zero terms in the summand S_{4n} is given by the cardinality of the following finite set J_1, i.e.

$$|J_1| = |\{j\colon X_j - EX_j > x\log^{-p} n\}| \leqslant [\log^{p^2+p} n] + 1.$$

Again, by (A2), the fact that $n \geqslant \log^{2p^2+p} n$, and the Markov inequality, one obtains

$$
\begin{aligned}
I_7 &\leqslant Cc_n E\left\{ I(|T_1| > c_n) \sum_{j=1}^{n} I(j \in J_1) \right\} x^{-1} \\
&\leqslant C(\log^{-p} n)(\log^{p(p+1)} n) E|X_1|^p I(|X_1| > x \log^{-p} n)(x \log^{-p} n)^{-p} \\
&\leqslant Cn[x \log n]^{-p} E|X_1|^p I(|X_1| > x \log^{-p} n). \qquad (3.10)
\end{aligned}
$$

Analogously, we have

$$
I_9 \leqslant Cn[x \log n]^{-p} E|X_1|^p I(|X_1| > x \log^{-p} n). \qquad (3.11)
$$

Finally, we give the following:

(e) Estimate of I_1

Note that $ET_{1j} + ET_{2j} + ET_{3j} + Ec_n I(T_j < -c_n) + E(-c_n) I(T_j > c_n) = 0$, by (3.2), we have

$$
\begin{aligned}
\max_{1 \leqslant i \leqslant n} |ES_{1i}| &= \max_{1 \leqslant i \leqslant n} |ES_{2i} + ES_{3i} + ES_{4i} + ES_{5i}| \\
&\leqslant 2 \max_{1 \leqslant i \leqslant n} |ES_{3i}| + 2nc_n E|X_1|^p c_n^{-p} \\
&\leqslant x/6 \quad \text{(by condition (A2))}.
\end{aligned}
$$

Denote $V_j = T_{1j} - ET_{1j}$. We have

$$
\begin{aligned}
I_1 &\leqslant P\left\{ \max_{1 \leqslant i \leqslant n} |S_{1i} - ES_{1i}| \geqslant x/6 \right\} \\
&\leqslant n \max_{1 \leqslant i \leqslant n} P\left\{ \left| \sum_{j=1}^{i} V_j \right| \geqslant x/6 \right\} \\
&= n \max_{1 \leqslant i \leqslant n} P\left\{ \left| \sum_{j=1}^{i} Z_{jn} \right| \geqslant (\log n)^p/12 \right\}, \qquad (3.12)
\end{aligned}
$$

where $Z_{jn} = V_j \log^p n/2x (1 \leqslant j \leqslant n)$. Note also that T_{1j} is a non-decreasing function of T_j, so that $\{Z_{jn}\}(1 \leqslant j \leqslant n)$ is an NA sequence. Since $|V_j| \leqslant 2x \log^{-p} n$, so $|Z_{jn}| \leqslant 1$, and also $\exp(Z_{jn})$ is an NA sequence. Using the fact that $E\exp(T) \leqslant \exp(ET + ET^2)$, when $|T| \leqslant 1$ a.s, and $EZ_{jn} = 0 (1 \leqslant j \leqslant n)$, it follows that

$$
\begin{aligned}
E\left\{ \exp\left(\sum_{j=1}^{i} Z_{jn} \right) \right\} &\leqslant \prod_{j=1}^{i} E\{\exp Z_{jn}\} \\
&\leqslant \prod_{j=1}^{i} \exp\{EZ_{jn} + EZ_{jn}^2\} = \exp\left\{ \sum_{j=1}^{i} EZ_{jn}^2 \right\}.
\end{aligned}
$$

Replacing Z_{jn} by $-Z_{jn}$, we also have

$$E\left\{\exp\left(-\sum_{j=1}^{i} Z_{jn}\right)\right\} \leqslant \exp\left\{\sum_{j=1}^{i} EZ_{jn}^2\right\}.$$

Thus, by (A2), for $1 \leqslant i \leqslant n$, the following is obtained:

$$\begin{aligned}
\sum_{j=1}^{i} \operatorname{var} Z_{jn} &= \sum_{j=1}^{i} EZ_{jn}^2 \\
&= (\log^{2p} n)x^{-2}\sum_{j=1}^{i} EV_j^2/4 \\
&= (\log^{2p} n)x^{-2}\sum_{j=1}^{i} E(T_{1j} - ET_{1j})^2/4 \\
&\leqslant (\log^{2p} n)x^{-2}\sum_{j=1}^{i} ET_{1j}^2 \\
&\leqslant 2(\log^{2p} n)x^{-2}\sum_{j=1}^{i} \{ET_j^2 I(|T_j| \leqslant c_n) + c_n^2 P(|T_j| > c_n)\} \\
&\leqslant 2(\log^{2p} n)x^{-2} n\{ET_1^2 I(|T_1| \leqslant c_n) + c_n^2 ET_1^2 I(|T_1| > c_n)c_n^{-2}\} \\
&\leqslant 4c_0^{-2}\{EX_1^2 I(|X_1| \leqslant c_n) + EX_1^2 I(|X_1| > c_n)\} \\
&\equiv C_2 \quad \text{(say)}.
\end{aligned} \tag{3.13}$$

Therefore, we have

$$\begin{aligned}
P\left\{\left|\sum_{j=1}^{i} Z_{jn}\right| \geqslant \log^p n/12\right\} &\leqslant E\left\{\exp\left(\sum_{j=1}^{i} Z_{jn}\right)\right\}\exp(-C\log^p n) \\
&\leqslant \exp\left\{\sum_{j=1}^{i} EZ_{jn}^2\right\}\exp(-C\log^p n) \\
&\leqslant \exp\{-C\log^p n + C_2\}.
\end{aligned} \tag{3.14}$$

Combining (3.12), we can conclude that

$$I_1 \leqslant Cn\exp\{-C\log^p n + C_2\}. \tag{3.15}$$

Now, combining steps (a)–(e), we complete the proof of Theorem 2.1. □

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