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An optimal replenishment policy for deteriorating items with time-varying demand and partial backlogging

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Abstract

Recently, Papachristos and Skouri developed an inventory model in which unsatisfied demand is partially backlogged at a negative exponential rate with the waiting time. In this article, we complement the shortcoming of their model by adding not only the cost of lost sales but also the non-constant purchase cost.

Keywords: Inventory; Lost sizing; Shortages; Partial backlogging; Deterioration

1. Introduction

In a recent paper [9], Papachristos and Skouri studied a continuous review inventory model with deterministic varying demand and constant deterioration rate. They developed an inventory model that allows for shortages, which are partially backlogged at a negative exponential rate with the waiting time. They then provided an optimal solution to minimize the total cost. However, they did not include the cost of lost sales due to shortages and the purchase cost for a non-constant order quantity into the total cost. If the total cost does not include the cost of lost sales (i.e., the cost of lost sales is zero), then the optimal solution that minimizes the total cost will have a large number of lost sales, which in turn implies a small profit. In addition, if the shortages are partially backlogged, then the total purchase cost is not a constant. Therefore, if we omit the purchase cost from the total cost, it will alter the optimal solution. To correct them, we add both the cost of lost sales and the purchase cost into the total cost suggested by Papachristos and Skouri [9]. Moreover, the demand rate based on their Assumption 5 must be an increasing (or decreasing), continuous, log-concave function of time. From a product life cycle, we know that their demand function is suitable only for the growth stage or the declining stage. For generality, we also relax their Assumption 5 to allow for any positive and log-concave demand pattern. Therefore, the proposed model here is suitable for any given time horizon in a product life cycle. Futhermore, we extend the fraction of unsatisfied demand backordered to any decreasing function of the waiting time

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up to the next replenishment. As a result, the proposed model is in a general framework that includes numerous previous models such as in [1,3–5,7,8,10–15] as special cases. We then prove that the optimal replenishment schedule not only exists but also is unique. In addition, we show that the total relevant cost (i.e. the sum of the holding, backlogging, lost sales, and purchase costs) in the system is a convex function of the number of replenishments. Consequently, the search for the optimal number of replenishments is reduced to finding a local minimum. We further simplify the search process by providing an intuitively good starting value for the optimal number of replenishments. Finally, we characterize the influences of the demand patterns over the replenishment cycles and others.

2. Assumptions and notation

The mathematical model of the inventory replenishment problem is based on the following assumptions:

1. The planning horizon of the inventory problem here is finite and is taken as H time units.
2. Lead time is zero.
3. Shortages are allowed.
4. The initial inventory level is zero.
5. A constant fraction of the on-hand inventory deteriorates per unit of time and there is no repair or replacement of the deteriorated inventory.
6. The fraction of shortages backordered is a decreasing function $\beta(x)$, where x is the waiting time up to the next replenishment, and $0 \leqslant \beta(x) \leqslant 1$ with $\beta(0) = 1$. To guarantee the existence of an optimal solution, we assume that $\beta(x) + H\beta'(x) \geqslant 0$, where $\beta'(x)$ is the first derivative of $\beta(x)$. Note that if $\beta(x) = 1$ (or 0) for all x, then shortages are completely backlogged (or lost).
7. If the objective is minimizing the costs, then we assume that the cost of lost sales is the sum of the cost of lost revenue and the cost of lost goodwill. However, if the objective is maximizing the profits, then the cost of lost sales is the cost of lost goodwill only. In contrast to ours, other researchers may define the cost of lost sales in a maximization problem as gross profit margin plus loss of goodwill to encourage more sales.

For convenience, the following notation is used throughout this paper:

$f(t)$	demand rate at time t, we assume without loss of generality that $f(t)$ is positive and log-concave in the planning horizon $(0, H]$
θ	the deterioration rate
p	the selling price per unit
c_{f}	the fixed purchasing cost per order
c_{v}	the variable purchasing cost per unit
c_{h}	the inventory holding cost per unit per unit time
c_{s}	the backlogging cost per unit per unit time due to shortages
c_{g}	the cost of lost goodwill
c_{l}	the unit cost of lost sales. Note that if the objective is minimizing the costs, then $c_{\mathrm{l}} = p + c_{\mathrm{g}} > c_{\mathrm{v}}$. If the objective is maximizing the profits, then $c_{\mathrm{l}} = c_{\mathrm{g}}$
n	the number of replenishments over $[0, H]$ (a decision variable)
t_i	the ith replenishment time (a decision variable), $i = 1, 2, \ldots, n$, with $0 \leqslant t_1 < t_2 < \cdots < t_n < H$
s_i	the time at which the inventory level reaches zero in the ith replenishment cycle (a decision variable), $i = 1, 2, \ldots, n$

3. Mathematical model

The ith replenishment is made at time t_i. The quantity received at t_i is used partly to meet the accumulated backorders in the previous cycle from time s_{i-1} to t_i ($s_{i-1} < t_i$). The inventory at t_i gradually reduces to zero at s_i ($s_i > t_i$). Consequently, based on whether the inventory is permitted to start and/or end with shortages, we have four possible cases, which were introduced in Teng et al. [13,14]. For an easy comparison, we use the same inventory model as in Papachristos and Skouri [9], which is depicted graphically in Fig. 1. The objective of the inventory problem here is to determine the number of replenishments n, and the timing of the reorder points $\{t_i\}$ and the shortage points $\{s_i\}$ in order to minimize the total relevant cost.

Next, we formulate the level of inventory at time t as $I(t), t_i \leqslant t \leqslant s_i$. Since the inventory is depleted by the combined effect of demand and deterioration, the

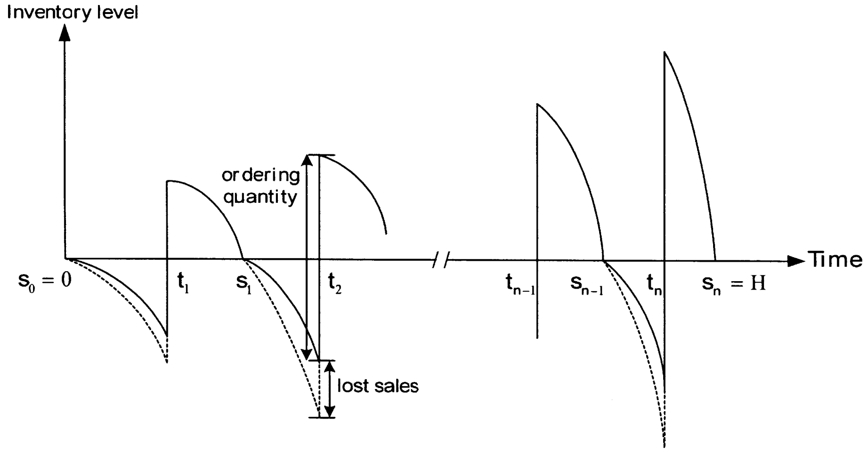

Fig. 1. Graphical representation of inventory model.

inventory level at time t during the ith replenishment cycle is governed by the following differential equation:

$$\frac{dI(t)}{dt} = -f(t) - \theta I(t), \quad t_i \leqslant t \leqslant s_i \tag{1}$$

with the boundary condition $I(s_i) = 0$. Solving the differential equation (1), we have

$$I(t) = e^{-\theta t} \int_t^{s_i} e^{\theta u} f(u)\, du = \int_t^{s_i} e^{\theta(u-t)} f(u)\, du,$$
$$t_i \leqslant t \leqslant s_i. \tag{2}$$

As a result, we obtain the time-weighted inventory during the ith replenishment cycle as

$$I_i = \int_{t_i}^{s_i} I(t)\, dt = \int_{t_i}^{s_i} \left[e^{-\theta t} \int_t^{s_i} e^{\theta u} f(u)\, du \right] dt$$
$$= (1/\theta) \int_{t_i}^{s_i} \left[e^{\theta(t-t_i)} - 1 \right] f(t)\, dt, \quad i = 1, 2, \ldots, n. \tag{3}$$

Similarly, the cumulative number of backorders at time t during $[s_{i-1}, t_i)$ is

$$B(t) = \int_{s_{i-1}}^{t} \beta(t_i - u) f(u)\, du, \quad s_{i-1} \leqslant t \leqslant t_i,$$
$$i = 1, 2, \ldots, n \tag{4}$$

and the cumulative number of lost sales at time t during $[s_{i-1}, t_i)$ is

$$L(t) = \int_{s_{i-1}}^{t} [1 - \beta(t_i - u)] f(u)\, du, \quad s_{i-1} \leqslant t \leqslant t_i,$$
$$i = 1, 2, \ldots, n. \tag{5}$$

Consequently, the time-weighted backorders due to shortages during the ith cycle is

$$S_i = \int_{s_{i-1}}^{t_i} B(t)\, dt = \int_{s_{i-1}}^{t_i} (t_i - t)\beta(t_i - t) f(t)\, dt \tag{6}$$

and the number of lost sales during the ith cycle is

$$L_i = L(t_i) = \int_{s_{i-1}}^{t_i} [1 - \beta(t_i - u)] f(u)\, du. \tag{7}$$

From (2) and (4), we have the order quantity at t_i in the ith replenishment cycle as

$$Q_i = B(t_i) + I(t_i) = \int_{s_{i-1}}^{t_i} \beta(t_i - t) f(t)\, dt$$
$$+ \int_{t_i}^{s_i} e^{\theta(t-t_i)} f(t)\, dt, \quad i = 1, 2, \ldots, n. \tag{8}$$

Therefore, the purchase cost during the ith replenishment cycle is

$$P_i = c_f + c_v Q_i = c_f + c_v$$
$$\times \left[\int_{s_{i-1}}^{t_i} \beta(t_i - t) f(t)\, dt + \int_{t_i}^{s_i} e^{\theta(t-t_i)} f(t)\, dt \right],$$
$$i = 1, 2, \ldots, n \tag{9}$$

and the number of units sold in the ith replenishment cycle is

$$R_i = B(t_i) + \int_{t_i}^{s_i} f(t)\,\mathrm{d}t = \int_{s_{i-1}}^{t_i} \beta(t_i - t) f(t)\,\mathrm{d}t + \int_{t_i}^{s_i} f(t)\,\mathrm{d}t,$$
$$i = 1, 2, \ldots, n. \tag{10}$$

Hence, if n replenishment orders are placed in $[0, H]$, then the total relevant cost of the inventory system during the planning horizon H is as follows:

$$\begin{aligned}
TC(n, \{s_i\}, \{t_i\}) &= \sum_{i=1}^{n} (P_i + c_\mathrm{h} I_i + c_\mathrm{s} S_i + c_\mathrm{l} L_\mathrm{i}) \\
&= nc_\mathrm{f} + c_\mathrm{v} \sum_{i=1}^{n} \left[\int_{s_{i-1}}^{t_i} \beta(t_i - t) f(t)\,\mathrm{d}t \right. \\
&\quad \left. + \int_{t_i}^{s_i} \mathrm{e}^{\theta(t-t_i)} f(t)\,\mathrm{d}t \right] + \frac{c_\mathrm{h}}{\theta} \sum_{i=1}^{n} \int_{t_i}^{s_i} \left[\mathrm{e}^{\theta(t-t_i)} - 1 \right] f(t)\,\mathrm{d}t \\
&\quad + c_\mathrm{s} \sum_{i=1}^{n} \int_{s_{i-1}}^{t_i} (t_i - t) \beta(t_i - t) f(t)\,\mathrm{d}t \\
&\quad + (p + c_\mathrm{g}) \sum_{i=1}^{n} \int_{s_{i-1}}^{t_i} [1 - \beta(t_i - t)] f(t)\,\mathrm{d}t.
\end{aligned} \tag{11}$$

Likewise, the total profit of the inventory system during the planning horizon H is

$$\begin{aligned}
TP(n, \{s_i\}, \{t_i\}) &= \sum_{i=1}^{n} (p^{R_i} - P_i - c_\mathrm{h} I_i - c_\mathrm{s} S_i - c_\mathrm{g} L_i) \\
&= (p - c_\mathrm{v}) \sum_{i=1}^{n} \left[\int_{s_{i-1}}^{t_i} \beta(t_i - t) f(t)\,\mathrm{d}t + \int_{t_i}^{s_i} f(t)\,\mathrm{d}t \right] \\
&\quad - \left(\frac{c_\mathrm{h}}{\theta} + c_\mathrm{v} \right) \sum_{i=1}^{n} \int_{t_i}^{s_i} \left[\mathrm{e}^{\theta(t-t_i)} - 1 \right] f(t)\,\mathrm{d}t \\
&\quad - nc_\mathrm{f} - c_\mathrm{s} \sum_{i=1}^{n} \int_{s_{i-1}}^{t_i} (t_i - t) \beta(t_i - t) f(t)\,\mathrm{d}t \\
&\quad - c_\mathrm{g} \sum_{i=1}^{n} \int_{s_{i-1}}^{t_i} [1 - \beta(t_i - t)] f(t)\,\mathrm{d}t.
\end{aligned} \tag{12}$$

The problem is to determine $n, \{s_i\}$ and $\{t_i\}$ such that $TC(n, \{s_i\}, \{t_i\})$ in (11) is minimized or $TP(n, \{s_i\}, \{t_i\})$ in (12) is maximized. It is obvious from (11) and (12) that

$$TP(n, \{s_i\}, \{t_i\}) + TC(n, \{s_i\}, \{t_i\}) = p \int_0^H f(t)\,\mathrm{d}t, \tag{13}$$

which is a constant. Therefore, we know that the optimal solution that minimizes $TC(n, \{s_i\}, \{t_i\})$ is the same as the optimal solution that maximizes $TP(n, \{s_i\}, \{t_i\})$. The problem now is to determine $n, \{s_i\}$ and $\{t_i\}$ such that $TC(n, \{s_i\}, \{t_i\})$ in (11) is minimized.

4. Theoretical results

For a fixed value of n, the necessary conditions for $TC(n, \{s_i\}, \{t_i\})$ to be minimized are: $\partial TC(n, \{s_i\}, \{t_i\})/\partial s_i = 0$, and $\partial TC(n, \{s_i\}, \{t_i\})/\partial t_i = 0$, for $i = 1, 2, \ldots, n$.

Consequently, we obtain

$$\left(\frac{c_\mathrm{h} + \theta c_\mathrm{v}}{\theta} \right) (\mathrm{e}^{\theta(s_i - t_i)} - 1) = [c_\mathrm{l} - c_\mathrm{l} + c_\mathrm{s}(t_{i+1} - s_i)]\, \beta(t_{i+1} - s_i) + c_\mathrm{l} - c_\mathrm{v} \tag{14}$$

and

$$\begin{aligned}
&\int_{s_{i-1}}^{t_i} \{ c_\mathrm{s} \left[\beta(t_i - t) + (t_i - t) \beta'(t_i - t) \right] \\
&\quad - (c_\mathrm{l} - c_\mathrm{v}) \beta'(t_i - t) \} f(t)\,\mathrm{d}t \\
&\quad = (c_\mathrm{h} + \theta c_\mathrm{v}) \int_{t_i}^{s_i} \mathrm{e}^{\theta(t-t_i)} f(t)\,\mathrm{d}t, \quad i = 1, 2, \ldots, n,
\end{aligned} \tag{15}$$

respectively. Applying (14) and (15), we obtain the following results:

Theorem 1. *For any given n, we have*:

(a) *The solution to Eqs.* (14) *and* (15) *not only exists but is also unique*.

(b) *Eqs.* (14) *and* (15) *are the necessary and sufficient conditions for finding the absolute minimum* $TC(n, \{s_i\}, \{t_i\})$.

Proof. See Appendix A for the proof of Part (a). Now, we show that for any given n, Eqs. (14) and (15) are the necessary and sufficient conditions for finding the absolute minimum $TC(n, \{s_i\}, \{t_i\})$. From (14), we know that the optimal value of s_i (i.e., s_i^*) is the interior point between t_i and t_{i+1} because if $s_i = t_i$ or t_{i+1}, then Eq. (14) does not hold. $TC(n, \{s_i\}, \{t_i\})$ is a continuous (and differentiable) function minimized over the compact set $[0, H]^{2n-1}$, and hence an absolute minimum exists. The optimal value of t_i (i.e., t_i^*) cannot be on the boundary since $TC(n, \{s_i\}, \{t_i\})$ increases when any one of t_is is shifted to the end points 0 or H. In addition, the solution to (14) and (15) is unique as shown in Part (a) of this theorem. Consequently, Eqs. (14) and (15) are the necessary and sufficient conditions for the absolute minimum $TC(n, \{s_i\}, \{t_i\})$. This completes the proof. □

Theorem 1 reduces the $2n$-dimensional problem of finding $\{s_i^*\}$ and $\{t_i^*\}$ to a one-dimensional problem. Since $s_0^* = 0$, we need only to find t_1^* to generate s_1^* by (15), t_2^* by (14), and then the rest of $\{t_i^*\}$ and $\{s_i^*\}$ uniquely by repeatedly using (14) and (15). For any chosen t_1^*, if $s_n^* = H$, then t_1^* is chosen correctly. Otherwise, we can easily find the optimal t_1^* by standard search schemes. The solution procedure for finding $\{t_i^*\}$ and $\{s_i^*\}$ can be obtained by the algorithm in Yang et al. [16] with $L = H/(4n)$ and $U = H/n$.

Next, we show that the total relevant cost $TC(n, \{s_i^*\}, \{t_i^*\})$ is a convex function of the number of replenishments. As a result, the search for the optimal replenishment number, n^*, is reduced to find a local minimum. For simplicity, let

$$TC(n) = TC(n, \{s_i^*\}, \{t_i^*\}). \tag{16}$$

By applying Bellman's principle of optimality [2], we have the following theorem:

Theorem 2. *$TC(n)$ is convex in n.*

Proof. By using the similar technique as in Teng et al. [13] or Friedman [6], the reader can easily prove it. □

To avoid using a brute force enumeration for finding n^* as in [3,4,9], we further simplify the search process by providing an intuitively good starting value for n^*. In fact, the holding cost per unit (including inventory and deterioration costs) is $c_h + \theta c_v$. The unit penalty cost of lost sales (which is the profit per unit) is $c_l - c_v$. For simplicity, we may set the fraction of shortages backordered $\beta(t_i - u)$ to be approximately equal to $\beta(1)$. Therefore, the expected unit cost of stockout approximately is $\beta(1)c_s + [1-\beta(1)](c_l - c_v)$. Substituting the above results into Eq. (15) as in Teng [12], we obtain an estimate of the number of replenishments as

$$\begin{aligned} n = \text{rounded integer of}\{&(c_h + \theta c_v)[\beta(1)c_s \\ &+ [1-\beta(1)](c_l - c_v)]Q(H)H/[2c_f(c_h + \theta c_v \\ &+ \beta(1)c_s + [1-\beta(1)](c_l - c_v))]\}^{1/2}, \end{aligned} \tag{17}$$

where $Q(H)$ is the accumulative demand during the planning horizon H. It is obvious that searching for the optimal number of replenishments by starting with n in (17) instead of $n = 1$ will reduce the computational complexity significantly. The algorithm for determining the optimal replenishment number and schedule is summarized as follows:

Algorithm for finding optimal number and schedule:

Step 0: Choose two initial trial values of n^*, say n as in (17) and $n-1$. Use a standard search method to obtain $\{t_i^*\}$ and $\{s_i^*\}$, and compute the corresponding $TC(n)$ and $TC(n-1)$, respectively.

Step 1: If $TC(n) \geqslant TC(n-1)$, then compute $TC(n-2), TC(n-3), \ldots$, until we find $TC(k) < TC(k-1)$. Set $n^* = k$ and stop.

Step 2: If $TC(n) < TC(n-1)$, then compute $TC(n+1), TC(n+2), \ldots$, until we find $TC(k) < TC(k+1)$. Set $n^* = k$ and stop.

Again, applying (14) and (15), we can characterize the influence of the demand patterns on the length of replenishment cycle and others as follow:

Theorem 3. *If $f(t)$ is increasing with respect to t, then we obtain*:

(a) *The optimal inventory intervals are monotonically decreasing, i.e.*,

$$s_1 - t_1 > s_2 - t_2 > \cdots > s_n - t_n. \tag{18}$$

(b) *The optimal shortage intervals are monotonically decreasing i.e.,*

$$t_2 - s_1 > t_3 - s_2 > \cdots > t_n - s_{n-1}. \tag{19}$$

(c) *The optimal replenishment cycles are monotonically decreasing i.e.,*

$$t_2 - t_1 > t_3 - t_2 > \cdots > t_n - t_{n-1}. \tag{20}$$

Proof. See Appendix B. □

Note that if $f(t)$ is decreasing, then the inequalities in Theorem 3 are reversed. A simple economic interpretation of the above results is as follows. Since demand is increasing with time, we need to shorten the inventory intervals (as well as the shortage intervals, and hence the replenishment cycles) with time in order to lower the holding and deterioration costs (as well as the shortage cost, and hence the total cost), and vice versa.

5. A numerical example

The proposed method is illustrated with the following numerical example:

Example 1. We use the same example as in Papachristos and Skouri [9] to compare the differences. Let $f(t) = 10e^{0.98t}$, $\beta(x) = \exp(-0.2x), H = 4, c_f = 250, c_h = 40, c_s = 200, c_v = 50, c_l = 500$, and $\theta = 0.08$ in appropriate units. Based on $\beta(x) = \exp(-0.2x)$, we set the fraction of shortages backordered to be approximately equal to $1 - 0.2 = 0.8$. By (17), we have $n = 12$. Since $TC(10) = 30842.12, TC(11) = 30777.66$, and $TC(12) = 30782.50$, we know that the optimal number of replenishments is 11 and the optimal time scheduling is shown in Table 1. Comparing the optimal replenishment schedule here to that obtained by Papachristos and Skouri, we have shorter shortage intervals and longer inventory intervals than those in Papachristos and Skouri [9].

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Appendix A

Proof of Part (a) of Theorem 1. Similar to the proof used by Hariga [8], as well as by Papachristos and Skouri [9], the reader can easily prove that if $f(t)$ is positive and log-concave, then there exists a unique solution t_1^* in $(0, H)$ such that $s_n(t_1^*) = H$. Since $s_0 = 0$ and t_1^* is unique, if we can prove that both s_i^* generated by (15) and t_{i+1}^* by (14) are uniquely determined, then we prove (a). For any given s_i and t_i, we set

$$F(x) = [c_v - c_l + c_s(x - s_i)]\,\beta(x - s_i) + c_l - c_v. \tag{A.1}$$

Taking the first derivative of F with respect to x, we obtain

$$F'(x) = c_s\left[\beta(x - s_i) + (x - s_i)\beta'(x - s_i)\right] - (c_l - c_v)\beta'(x - s_i) > 0. \tag{A.2}$$

Since $F(s_i) = 0$, we know that there exists a unique $t_{i+1}^*(> s_i)$ such that

$$F(t_{i+1}^*) = \left(\frac{c_h + \theta c_v}{\theta}\right)(e^{\theta(s_i - t_i)} - 1) > 0. \tag{A.3}$$

Therefore, t_{i+1}^* is uniquely determined by (14). Likewise, for any given s_{i-1} and t_i, let

$$G(x) = (c_h + \theta c_v)\int_{t_i}^{x} e^{\theta(t - t_i)} f(t)\,dt. \tag{A.4}$$

We then have $G(t_i) = 0$, and $G'(x) = (c_h + \theta c_v)e^{\theta(x - t_i)} f(x) > 0$. Therefore, there exists a

Table 1
Optimal time schedule

i	1	2	3	4	5	6	7	8	9	10	11
t_i	0.1719	0.9699	1.5565	2.0187	2.3991	2.7221	3.0023	3.2498	3.4712	3.6715	3.8542
s_i	0.8605	1.4770	1.9564	2.3481	2.6788	2.9649	3.2168	3.4417	3.6448	3.8299	4.0000

unique $s_i^*(>t_i)$ such that

$$G(s_i^*)=\int_{s_{i-1}}^{t_i}\{c_s\left[\beta(t_i-t)+(t_i-t)\beta'(t_i-t)\right] -(c_l-c_v)\beta'(t_i-t)\}f(t)\,\mathrm{d}t>0. \qquad (A.5)$$

This proved Part (a) of Theorem 1. □

Appendix B

Proof of Theorem 3. Applying the Mean Value Theorem to (15), we know that there exist x_1 and x_2 (with $s_{i-1}<x_2<t_i<x_1<s_i$) such that

$$\left(\frac{c_h+\theta c_v}{\theta}\right)(e^{\theta(s_i-t_i)}-1)f(x_1)$$

$$=\{[c_v-c_l+c_s(t_i-s_{i-1})]\,\beta(t_i-s_{i-1})+c_l-c_v\}\,f(x_2).$$

$$=\left(\frac{c_h+\theta c_v}{\theta}\right)(e^{\theta(s_{i-1}-t_{i-1})}-1)f(x_2)$$

(by using (14)). (A.6)

If $f(t)$ is an increasing function, then it is clear that

$$\left(\frac{c_h+\theta c_v}{\theta}\right)(e^{\theta(s_i-t_i)}-1)<$$

$$\left(\frac{c_h+\theta c_v}{\theta}\right)(e^{\theta(s_{i-1}-t_{i-1})}-1). \qquad (A.7)$$

From (A.4), we know that $G(x)$ is a strictly increasing function of x. Thus, $s_i-t_i<s_{i-1}-t_{i-1}$, for $i=1,2,\ldots,n$. This completes the proof of Part (a). Similarly, by using (11) and (A.7), we have

$$[c_v-c_l+c_s(t_{i+1}-s_i)]\,\beta(t_{i+1}-s_i)+c_l-c_v$$

$$<[c_v-c_l+c_s(t_i-s_{i-1})]\,\beta(t_i-s_{i-1})+c_l-c_v. \qquad (A.8)$$

Again, we know from (A.2) that $F(x)$ is a strictly increasing function. Therefore, we prove that $t_{i+1}-s_i<t_i-s_{i-1}$, for $i=1,2,\ldots,n-1$. Finally, the proof of Part (c) immediately follows Parts (a) and (b). □

References

[1] L.C. Barbosa, M. Friedman, Deterministic inventory lot size models—a general root law, Management Sci. 24 (1978) 819–826.

[2] R.E. Bellman, Dynamic Programming, Princeton University Press, Princeton, NJ, 1957.

[3] H.-J. Chang, C.-Y. Dye, An EOQ model for deteriorating items with time varying demand and partial backlogging, J. Oper. Res. Soc. 50 (1999) 1176–1182.

[4] U. Dave, A deterministic lot-size inventory model with shortages and a linear trend in demand, Naval Res. Logist. 36 (1989) 507–514.

[5] W.A. Donaldson, Inventory replenishment policy for a linear trend in demand: an analytical solution Oper. Res. Quart. 28 (1977) 663–670.

[6] M.F. Friedman, Inventory lot-size models with general time-dependent demand and carrying cost function, INFOR 20 (1982) 157–167.

[7] S.K. Goyal, M.A. Hariga, A. Alyan, The trended inventory lot sizing problem with shortages under a new replenishment policy, J. Oper. Res. Soc. 47 (1996) 1286–1295.

[8] M.A. Hariga, Optimal EOQ models for deteriorating items with time-varying demand, J. Oper. Res. Soc. 47 (1996) 1228–1246.

[9] S. Papachristos, K. Skouri, An optimal replenishment policy for deteriorating items with time-varying demand and partial—exponential type—backlogging, Oper. Res. Lett. 27 (2000) 175–184.

[10] M. Resh, M. Friedman, L.C. Barbosa, On a general solution of the deterministic lot size problem with time-proportional demand, Oper. Res. 24 (1976) 718–725.

[11] J.-T. Teng, A note on inventory replenishment policy for increasing demand, J. Oper. Res. Soc. 45 (1994) 1335–1337.

[12] J.-T. Teng, A deterministic replenishment model with linear trend in demand, Oper. Res. Lett. 19 (1996) 33–41.

[13] J.-T. Teng, M.-S. Chern, H.-L. Yang, An optimal recursive method for various inventory replenishment models with increasing demand and shortages, Naval Res. Logist. 44 (1997) 791–806.

[14] J.-T. Teng, M.-S. Chern, H.-L. Yang, Y.J. Wang, Deterministic lot-size inventory models with shortages and deterioration for fluctuating demand, Oper. Res. Lett. 24 (1999) 65–72.

[15] H.-M. Wee, A deterministic lot-size inventory model for deteriorating items with shortages and a declining market, Comput. Oper. Res. 22 (1995) 345–356.

[16] H.-L. Yang, J.-T. Teng, M.-S. Chern, Deterministic inventory lot-size models under inflation with shortages and deterioration for fluctuating demand, Naval Res. Logist. 48 (2001) 144–158.