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# On an EOQ model for deteriorating items with time-varying demand and partial backlogging

J-T Teng[1]*, H-L Yang[2] and L-Y Ouyang[3]

[1]*The William Paterson University of New Jersey, Wayne, NJ, USA;* [2]*Hung Kuang Institute of Technology, Taichung, Taiwan, ROC; and* [3]*Tamkang University, Tamsui, Taipei, Taiwan, ROC*

For seasonal products, fashionable commodities and high-tech products with a short product life cycle, the willingness of a customer to wait for backlogging during a shortage period is diminishing with the length of waiting time. Recently, Chang and Dye developed an inventory model in which the backlogging rate declines as the waiting time increases. In this paper, we complement the shortcoming of their model by adding the non-constant purchase cost into the model. In addition, we show that the total cost is a convex function of the number of replenishments. We further simplify the search process by providing an intuitively good starting value, which reduces the computational complexity significantly. Finally, we characterize the influences of the demand patterns over the replenishment cycles and others.
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**Keywords:** inventory; lot sizing; shortages; partial backlogging; deterioration

## Introduction

For fashionable commodities and other products with a short life cycle, the willingness of a customer to wait for backlogging during a shortage period is declining with the length of waiting time. Hence, the longer the waiting time, the smaller the backlogging rate. To reflect this phenomenon, Chang and Dye[1] developed an inventory model in which the proportion of customers who would like to accept backlogging is the reciprocal of a linear function of the waiting time. Concurrently, Papachristos and Skouri[2] established a partially backlogged inventory model in which the backlogging rate decreases exponentially as the waiting time increases.

In Chang and Dye's paper, they neither included the purchase cost for a non-constant order quantity into the total cost nor defined the opportunity cost due to lost sales clearly. If the shortages are partially backlogged, then the total purchase cost is not a constant. Therefore, if we omit the purchase cost from the total cost, it will alter the optimal solution. In addition, if the opportunity cost is defined to be less than the purchase cost per unit, then the optimal solution that minimizes the total cost will have a large number of lost sales, which in turn implies a small profit. Lately, Goyal and Giri[3] noted that the opportunity cost was taken to be too small as \$30 (which is only 15% of the unit purchase cost of \$200) in Chang and Dye,[1] and suggested the opportunity cost to be \$60 or \$90, which would be more appropriate and meaningful. Unfortunately, the opportunity cost based on their suggestion is still cheaper than the unit purchase cost. Consequently, the solution will tend to have a large number of lost sales. To correct this, we amend the opportunity cost and add the purchase cost into the total cost suggested by Chang and Dye.[1] We then show that not only the optimal replenishment schedule exists uniquely, but also the total cost in the system is a convex function of the number of replenishments. Consequently, the search for the optimal number of replenishments is reduced to finding a local minimum. We further simplify the search process by providing an intuitively good starting value, which reduces the computational complexity significantly. Finally, we characterize the influences of the demand patterns over the replenishment cycles and others.

## Mathematical model

To shorten the paper, we adopt the same notation and assumptions as in Chang and Dye.[1] However, for correctness, we assume that not only the opportunity cost per unit $C_3$ is greater than the unit purchase cost $C_4$ but also $f(t)$ is positive and log-concave in (0, $H$]. As a result, we obtain the total number of backorders during the $i$th cycle as

$$\int_{s_{i-1}}^{t_i} \frac{f(u)}{1+\alpha(t_i-u)}\mathrm{d}u, \quad s_{i-1} \leqslant t \leqslant t_i, \; i = 1, 2, \ldots, n, \tag{1}$$

and the order quantity at $t_i$ in the $i$th replenishment cycle as

$$Q_i = \int_{s_{i-1}}^{t_i} \frac{f(t)}{1+\alpha(t_i-t)}\mathrm{d}t + \int_{t_i}^{s_i} \mathrm{e}^{\theta(t-t_i)} f(t)\mathrm{d}t, \quad i = 1, 2, \ldots, n \tag{2}$$

*Correspondence: J-T Teng, Department of Marketing and Management Sciences, The William Paterson University of New Jersey, Wayne, NJ 07470-2103, USA.
E-mail: tengj@wpunj.edu

Therefore, the purchase cost during the $i$th replenishment cycle is

$$P_i = A + C_4 Q_i = A + C_4\left[\int_{s_{i-1}}^{t_i} \frac{f(t)}{1+\alpha(t_i - t)}\mathrm{d}t + \int_{t_i}^{s_i} \mathrm{e}^{\theta(t-t_i)} f(t)\mathrm{d}t\right] \quad (3)$$

Hence, if $n$ replenishment orders are placed in [0, $H$], then we can formulate the total relevant cost as the sum of the purchase cost, inventory holding cost, shortage cost and opportunity cost due to lost sales as follows:

$$\begin{aligned} TC(n, \{s_i\}, \{t_i\}) &= nA + C_4\sum_{i=1}^{n}\left[\int_{s_{i-1}}^{t_i} \frac{f(t)}{1+\alpha(t_i - t)}\mathrm{d}t + \int_{t_i}^{s_i} \mathrm{e}^{\theta(t-t_i)} f(t)\mathrm{d}t\right] \\ &+ \frac{C_i}{\theta}\sum_{t=1}^{n}\int_{t_i}^{s_i}[\mathrm{e}^{\theta(t-t_i)} - 1]f(t)\mathrm{d}t + (C_2 + \alpha C_3) \\ &\times \sum_{i=1}^{n}\int_{s_{i-1}}^{t_i}\frac{(t_i - t)}{1+\alpha(t_i - t)}f(t)\mathrm{d}t \end{aligned} \quad (4)$$

The problem is to determine $n$, $\{s_i\}$ and $\{t_i\}$ such that $TC(n,\{s_i\},\{t_i\})$ is minimized.

### *Theoretical results*

For a fixed value of $n$, the necessary conditions for $TC(n,\{s_i\},\{t_i\})$ to be minimized are $\partial TC(n,\{s_i\},\{t_i\})/\partial s_i = 0$ and $\partial TC(n,\{s_i\},\{t_i\})/\partial t_i = 0$ for $i = 1, 2,\ldots, n$. Consequently, we obtain

$$\begin{aligned} &[C_2 + \alpha(C_3 - C_4)](t_{i+1} - s_i)/[1 + \alpha(t_{i+1} - s_i)] \\ &\quad = (C_1/\theta + C_4)(\mathrm{e}^{\theta(s_i - t_i)} - 1) \end{aligned} \quad (5)$$

and

$$(C_1 + \theta C_4)\int_{t_i}^{s_i}\mathrm{e}^{\theta(t-t_i)}f(t)\mathrm{d}t = [C_2 + \alpha(C_3 - C_4)]\int_{s_{i-1}}^{t_i}\frac{f(t)}{[1+\alpha(t_i - t)]^2}\mathrm{d}t \quad (6)$$

respectively. Applying (5) and (6), we obtain the following result.

**Theorem 1** *For any given n, we have*:

(a) *The solution to* (5) *and* (6) *not only exists but is also unique*.

(b) *The solution to* (5) *and* (6) *obtains the unique global minimum*.

**Proof** By using a similar proof as in Hariga,[4] as well as in Chang and Dye,[1] we can easily prove (a). See Appendix A for the detailed proof of (a). Next, we show that for any given $n$. Equations (5) and (6) are the necessary and sufficient conditions for finding the global minimum $TC(n,\{s_i\},\{t_i\})$. From (5), we know that the optimal value of $s_i$ (ie, $s_i^*$) is the interior point between $t_i$ and $t_{i+1}$ because if $s_i = t_i$ or $t_{i+1}$, then Equation (5) does not hold. $TC(n,\{s_i\},\{t_i\})$ is a continuous (and differentiable) function minimized over the compact set $[0, H]^{2n-1}$, and hence a global minimum exists. The optimal value of $t_i$ (ie, $t_i^*$) cannot be on the boundary since $TC(n,\{s_i\},\{t_i\})$ increases when any one of the $t_i$'s is shifted to the end points 0 or $H$. Therefore, there must exist at least an inner global minimum solution that satisfies (5) and (6). In addition, the solution to (5) and (6) is unique as shown in part (a) of this theorem. Consequently, Equations (5) and (6) are the necessary and sufficient conditions for the global minimum $TC(n,\{s_i\},\{t_i\})$. This completes the proof. □

Note that our proof of (b) here is much simpler than that in Hariga,[4] and in Chang and Dye,[1] in which they proved that the associated Hessian matrix of the solution to (5) and (6) has positive principal minors.

Theorem 1 reduces the 2$n$-dimensional problem of finding $\{s_i^*\}$ and $\{t_i^*\}$ to a one-dimensional problem. Since $s_0^* = 0$, we need only to find $t_1^*$ to generate $s_1^*$ by (6), $t_2^*$ by (5), and then the rest of $\{t_i^*\}$ and $\{s_i^*\}$ uniquely by repeatedly using (5) and (6). For any chosen $t_1^*$, if $s_n^* = H$, then $t_1^*$ is chosen correctly. Otherwise, we can easily find the optimal $t_1^*$ by standard search schemes. For any given value of $n$, the values of $\{t_i^*\}$ and $\{s_i^*\}$ can be obtained by the algorithm in Yang *et al*[5] with $L = H/(4n)$ and $U = H/n$ as initial trial values of $t_1^*$.

Next, we show that the total relevant cost $TC(n, \{s_i^*\}, \{t_i^*\})$ is a convex function of the number of replenishments. As a result, the search for the optimal replenishment number, $n$*, is reduced to finding a local minimum. For simplicity, let

$$TC(n) = TC(n, \{s_i^*\}, \{t_i^*\}) \quad (7)$$

By applying Bellman's principal of optimality,[6] we have the following theorem.

**Theorem 2** *TC*($n$) *is convex* in $n$.

**Proof** By using a similar technique as in Friedman,[7] Papachristos and Skouri[2] or Teng *et al*,[8,9] the reader can easily prove it. □

Note that Theorem 2 simplifies the search for $n$* to a local minimum of $TC(n)$. To avoid using a brute force enumeration as in Chang and Dye,[1] Dave[10] or Papachristos and Skouri,[2] we further simplify the search process by providing an intuitively good starting value for $n$*. In fact, the holding cost per unit (including inventory and deterioration costs) is $C_1 + \theta C_4$. The unit penalty cost of lost sales is $C_3 - C_4$. We know from (1) that the backlogging rate is approximately equal to $1/(1+\alpha)$. Therefore, the unit cost of stockout is approximately equal to $C_2/(1+\alpha)+\alpha(C_3-C_4)/(1+\alpha)$. Substituting the above results into Equation (15) as in Teng,[11] we obtain an estimate of the number of replenishments as

$$n = \text{rounded integer of } \{(C_1+\theta C_4)[C_2/(1+\alpha) + \alpha(C_3-C_4)/(1+\alpha)]Q(H)H/[2A(C_1+\theta C_4 + C_2/(1+\alpha)+\alpha(C_3-C_4)/(1+\alpha))]\}^{1/2} \tag{8}$$

where $Q(H)$ is the accumulative demand during the planning horizon $H$. It is obvious that searching for $n^*$ by starting with $n$ in (8) will reduce the computational complexity significantly, compared to starting with $n=1$ (such as in Chang and Dye,[1] Dave[10] or Papachristos and Skouri[2]). The algorithm for determining the optimal number of replenishments $n^*$ is similar to the algorithm in Teng *et al*[8] with two initial trial values of $n^*$, say $n$ as in (8) and $n-1$.

Again, applying (5) and (6), we can characterize the influence of the demand patterns on the length of the replenishment cycle and others as follows:

**Theorem 3** *If $f(t)$ is increasing with respect to $t$, then we obtain*:

(a) *The optimal inventory intervals are monotonically decreasing, that is,*

$$s_1 - t_1 > s_2 - t_2 > \cdots > s_n - t_n$$

(b) *The optimal shortage intervals are monotonically decreasing, that is,*

$$t_2 - s_1 > t_3 - s_2 > \cdots > t_n - s_{n-1}$$

(c) *The optimal replenishment cycles are monotonically decreasing, that is,*

$$t_2 - t_1 > t_3 - t_2 > \cdots > t_n - t_{n-1}$$

**Proof** See Appendix B.

Note that if $f(t)$ is decreasing, then the inequalities in Theorem 3 are reversed. A simple economic interpretation of the above results is as follows. Since demand is increasing with time, we need to shorten the inventory intervals (as well as the shortage intervals, and hence the replenishment cycles) with time in order to lower the holding and deterioration costs (as well as the shortage cost, and hence the total cost), and vice versa.

## Numerical examples

We provide the following numerical examples to illustrate the proposed method.

**Example 1** Let $f(t)=40+3t$, $H=4$, $A=250$, $C_1=80$, $C_2=120$, $C_3=300$, $C_4=150$, $\theta=0.08$, $\alpha=20$ in appropriate units. By (8), we have $n=9$, and $TC(8)=33747.52$, $TC(9)=33533.37$, $TC(10)=33412.46$, $TC(11)=33359.32$, $TC(12)=33356.95$, $TC(13)=33393.59$. Thus, the replenishment number during the planning horizon $H$ is 12 and the optimal replenishment schedule is shown in Table 1.

**Example 2** Let $f(t)=50-3t$, $H=4$, $A=250$, $C_1=80$, $C_2=120$, $C_3=300$, $C_4=150$, $\theta=0.08$, $\alpha=20$ in appropriate units. By (8), we have $n=8$, and $TC(7)=32636.26$, $TC(8)=32326.68$, $TC(9)=32140.96$, $TC(10)=32042.15$, $TC(11)=32006.65$, $TC(12)=32018.66$. Thus, the replenishment number during the planning horizon $H$ is 11 and the optimal replenishment schedule is shown in Table 2.

It is clear from Table 1 (or 2) that the inventory intervals (as well as the shortage intervals, and hence the replenishment cycles) are decreasing (or increasing) sequentially, which are coincident with the results in Theorem 3.

## Conclusions

In this paper, we provide an appropriate way to minimize the costs in a partially backlogged inventory model. We then show that the optimal replenishment schedule not only exists but is also unique. In addition, we point out that the total relevant cost is a convex function of the number of replenishments. Consequently, the search for the optimal number of replenishments is reduced to find a local minimum. We further simplify the search process by providing an intuitively good starting value, which reduces the computational complexity significantly by comparing with a brute force enumeration as in Chang and Dye,[1]

**Table 1** Optimal replenishment schedule for Example 1

| *i* | *1* | *2* | *3* | *4* | *5* | *6* | *7* | *8* | *9* | *10* | *11* | *12* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t_i$ | 0.0130 | 0.3674 | 0.7173 | 1.0630 | 1.4046 | 1.7423 | 2.0764 | 2.4069 | 2.7340 | 3.0578 | 3.3786 | 3.6963 |
| $s_i$ | 0.3546 | 0.7047 | 1.0505 | 1.3923 | 1.7302 | 2.0644 | 2.3951 | 2.7224 | 3.0464 | 3.3672 | 3.6851 | 4.0000 |

**Table 2** Optimal replenishment schedule for Example 2

| *i* | *1* | *2* | *3* | *4* | *5* | *6* | *7* | *8* | *9* | *10* | *11* |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $t_i$ | 0.0121 | 0.3547 | 0.7010 | 1.0511 | 1.4052 | 1.7635 | 2.1262 | 2.4936 | 2.8658 | 3.2431 | 3.6259 |
| $s_i$ | 0.3425 | 0.6886 | 1.0385 | 1.3924 | 1.7505 | 2.1130 | 2.4801 | 2.8521 | 3.2292 | 3.6117 | 4.0000 |

Dave[10] or Papachristos and Skouri.[2] Finally, we provide numerical examples for illustration.

The proposed model can be extended in several ways. Firstly, we can easily extend the backlogging rate of unsatisfied demand to any decreasing function $\beta(x)$, where $x$ is the waiting time up to the next replenishment, and $0 \leqslant \beta(x) \leqslant 1$ with $\beta(0)=1$. Secondly, we can generalize the positive log-concave demand function to any positive continuous function (eg, Teng *et al*[9]). Thirdly, we can consider that the demand is a function of time as well as price (eg, Abad[12]). Finally, we can also incorporate the quantity discount and the learning curve phenomenon into the model.

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## Appendix .A Proof of part (a) of Theorem 1

From Hariga[4] (1239), we know that if $g(t)$ is positive and log-concave, then

$$g(y) - g(x) \leqslant [g'(x)/g(x)] \int_x^y g(t)\mathrm{d}t \qquad (A1)$$

Applying (A1) to $g(t) = \mathrm{e}^{\theta(t-t_i)} f(t)$ (which is also positive and log-concave), we have

$$\mathrm{e}^{\theta(s_i-t_i)} f(s_i) - f(t_i) \leqslant (f'(t_i)/f(t_i) + \theta) \int_{t_i}^{s_i} \mathrm{e}^{\theta(t-t_i)} f(t)\mathrm{d}t \quad (A2)$$

Multiplying both sides of (A2) by $(C_1 + \theta C_4)$ and using (6), we obtain

$$\begin{aligned}
&(C_1+\theta C_4)\left[\mathrm{e}^{\theta(s_i-t_i)} f(s_i) - f(t_i) - \theta \int_{t_i}^{s_i} \mathrm{e}^{\theta(t-t_i)} f(t)\mathrm{d}t\right] \\
&\leqslant [C_2 + \alpha(C_3 - C_4)][f'(t_i)/f(t_i)] \int_{s_{i-1}}^{t_i} \frac{f(t)}{[1+\alpha(t_i-t)]^2}\mathrm{d}t \\
&\leqslant [C_2 + \alpha(C_3 - C_4)] \int_{s_{i-1}}^{t_i} \frac{f'(t)}{[1+\alpha(t_i-t)]^2}\mathrm{d}t
\end{aligned} \qquad (A3)$$

since $f'(t_i)/f(t_i) \leq f'(t)/f(t)$, for $s_{t-1} \leqslant t \leqslant t_I$. Using the method of integrating by parts to the right-hand-side integral of (A3), we obtain

$$\int_{s_{i-1}}^{t_i} \frac{f'(t)}{[1+\alpha(t_i-t)]^2}\mathrm{d}t = \left[f(t_i) - \frac{f(s_{i-1})}{[1+\alpha(t_i-s_{i-1})]^2} - \int_{s_{i-1}}^{t_i} \frac{2\alpha f(t)}{[1+\alpha(t_i-t)]^3}\mathrm{d}t\right] \qquad (A4)$$

Thus,

$$\begin{aligned}
&(C_1+\theta C_4)\left[\mathrm{e}^{\theta(s_i-t_i)} f(s_i) - f(t_i) - \theta \int_{t_i}^{s_i} \mathrm{e}^{\theta(t-t_i)} f(t)\mathrm{d}t\right] \\
&\leqslant [C_2 + \alpha(C_3 - C_4)]\left[f(t_i) - \frac{f(s_{i-1})}{[1+\alpha(t_i-s_{i-1})]^2} - \int_{s_i}^{t_i} \frac{2\alpha f(t)}{[1+\alpha(t_i-t)]^3}\mathrm{d}t\right]
\end{aligned} \qquad (A5)$$

Since $s_0 = 0$, if $t_1$ is known, then other $s_i$'s and $t_i$'s can be obtained from (5) and (6). Thus, the other $s_i$'s and $t_i$'s can be regarded as functions of $t_1$. By taking implicit differentiation on (5) and (6) with respect to $t_1$, we have

$$\begin{aligned}
&\frac{C_2+\alpha(C_3-C_4)}{[1+\alpha(t_{i+1}-s_i)]^2}\left(\frac{\mathrm{d}t_{i+1}}{\mathrm{d}t_1} - \frac{\mathrm{d}s_i}{\mathrm{d}t_1}\right) \\
&= (C_1+\theta C_4)\mathrm{e}^{\theta(s_i-t_i)}\left(\frac{\mathrm{d}s_i}{\mathrm{d}t_1} - \frac{\mathrm{d}t_i}{\mathrm{d}t_1}\right)
\end{aligned} \qquad (A6)$$

and

$$\begin{aligned}
&(C_1+\theta C_4)\mathrm{e}^{\theta(s_i-t_i)} f(s_i)\left(\frac{\mathrm{d}s_i}{\mathrm{d}t_1} - \frac{\mathrm{d}t_i}{\mathrm{d}t_1}\right) \\
&\quad + (C_1+\theta C_4)\left[\mathrm{e}^{\theta(s_i-t_i)} f(s_i) - f(t_i) - \theta\left(\int_{t_i}^{s_i} \mathrm{e}^{\theta(t-t_i)} f(t)\mathrm{d}t\right)\right]\frac{\mathrm{d}t_i}{\mathrm{d}t_1} \\
&= [C_2+\alpha(C_3-C_4)]\left[f(t_i)\frac{\mathrm{d}t_i}{\mathrm{d}t_1} - \frac{f(s_{i-1})}{[1+\alpha(t_i-s_{i-1})]^2}\frac{\mathrm{d}s_{i-1}}{\mathrm{d}t_1}\right. \\
&\quad \left. - \left(\int_{s_{i-1}}^{t_i} \frac{2\alpha f(t)\mathrm{d}t}{[1+\alpha(t_i-t)]^3}\right)\frac{\mathrm{d}t_i}{\mathrm{d}t_1}\right]
\end{aligned} \qquad (A7)$$

respectively. Applying (A5) (A7) with $i=1$, and using the fact that $\mathrm{d}s_0/\mathrm{d}t_1 = 0$ and $\mathrm{d}t_1/\mathrm{d}t_1 = 1$, we have

$$\begin{aligned}
&(C_1+\theta C_4)\mathrm{e}^{\theta(s_1-t_1)} f(s_1)\left(\frac{\mathrm{d}s_1}{\mathrm{d}t_1} - \frac{\mathrm{d}t_1}{\mathrm{d}t_1}\right) \\
&\geqslant [C_2+\alpha(C_3-C_4)]\frac{f(s_0)}{[1+\alpha(t_1-s_0)]^2} > 0
\end{aligned} \qquad (A8)$$

This implies that $\mathrm{d}s_1/\mathrm{d}t_1 > \mathrm{d}t_1/\mathrm{d}t_1 = 1 > 0$. It is obvious from (A6) that

$$\begin{aligned}&\mathrm{d}t_{i+1}/\mathrm{d}t_1 - \mathrm{d}s_i/\mathrm{d}t_1 > 0 \text{ if and only if}\\ &\mathrm{d}s_i/\mathrm{d}t_1 - \mathrm{d}t_i/\mathrm{d}t_1 > 0,\\ &\text{for } i = 1, 2, \ldots, n-1\end{aligned} \qquad (A9)$$

Consequently, we know $\mathrm{d}t_2/\mathrm{d}t_1 > \mathrm{d}s_1/\mathrm{d}t_1 > \mathrm{d}t_1/\mathrm{d}t_1 = 1 > 0$. Similarly, multiplying (A5) by $\mathrm{d}t_i/\mathrm{d}t_1$ (which is greater than 0 by (A9)) and subtracting it from (A7), we obtain

$$\begin{aligned}&(C_1 + \theta C_4)\mathrm{e}^{\theta(s_i - t_i)} f(s_i)\left(\frac{\mathrm{d}s_i}{\mathrm{d}t_1} - \frac{\mathrm{d}t_i}{\mathrm{d}t_1}\right)\\ &\geqslant [C_2 + \alpha(C_3 - C_4)]\frac{f(s_{i-1})}{[1 + \alpha(t_i - s_{i-1})]^2} \qquad (A10)\\ &\left(\frac{\mathrm{d}t_i}{\mathrm{d}t_1} - \frac{\mathrm{d}s_{i-1}}{\mathrm{d}t_1}\right) > 0 \quad \text{for } i = 2, 3, \ldots, n\end{aligned}$$

Hence, we obtain $\mathrm{d}s_2/\mathrm{d}t_1 > \mathrm{d}t_2/\mathrm{d}t_1 > \mathrm{d}s_1/\mathrm{d}t_1 > \mathrm{d}t_1/\mathrm{d}t_1 = 1 > 0$. Using (A9) and (A10) repeatedly, we get

$$\begin{aligned}&\frac{\mathrm{d}s_n}{\mathrm{d}t_1} > \frac{\mathrm{d}t_n}{\mathrm{d}t_1} > \frac{\mathrm{d}s_{n-1}}{\mathrm{d}t_1} > \frac{\mathrm{d}t_{n-1}}{\mathrm{d}t_1} > \cdots > \frac{\mathrm{d}s_2}{\mathrm{d}t_1} > \frac{\mathrm{d}t_2}{\mathrm{d}t_1} > \frac{\mathrm{d}s_1}{\mathrm{d}t_1} > \frac{\mathrm{d}t_1}{\mathrm{d}t_1}\\ &= 1 > 0\end{aligned} \qquad (A11)$$

Therefore, $s_i$ and $t_{i+1}$ (with $i > 1$) obtained by (5) and (6) are monotonically increasing with $t_1$. It is obvious from (5) and (6) that $s_n(t_1) < H$ if $t_1 = 0$ and $s_n(t_1) > H$ if $t_1 = H$. Therefore, there exists a unique $t_1$ in (0, $H$) such that $s_n(t_1) = H$. This proves part (a) of Theorem 1. □

## Appendix .B Proof of Theorem 3

Applying the mean value theorem to (6), we know that there exist $x_1$ and $x_2$ (with $s_{i-1} < x_2 < t_1 < x_1 < s_i$) such that

$$\begin{aligned}&(C_1/\theta + C_4)(\mathrm{e}^{\theta(s_i - t_i)} - 1)f(x_1)\\ &= [C_2 + \alpha(C_3 - C_4)](t_i - s_{i-1})f(x_2)/[1 + \alpha(t_i - s_{i-1})]\\ &= (C_1/\theta + C_4)(\mathrm{e}^{\theta(s_{i-1} - t_{i-1})} - 1)f(x_2) \quad (\text{by using (5)})\end{aligned} \qquad (A12)$$

If $f(t)$ is an increasing function, then it is clear that

$$(C_1/\theta + C_4)(\mathrm{e}^{\theta(s_i - t_i)} - 1) < (C_1/\theta + C_4)(\mathrm{e}^{\theta(s_{i-1} - t_{i-1})} - 1) \qquad (A13)$$

Thus, $s_i - t_i < s_{i-1} - t_{i-1}$ for $i = 2, 3, \ldots, n$. This completes the proof of part (a).

Similarly, using (5) again, we have

$$(t_{i+1} - s_i)/[1 + \alpha(t_{i+1} - s_1)] < (t_i - s_{i-1})/[1 + \alpha(t_i - s_{i-1})] \qquad (A14)$$

Let $g(x) = x/(1 + \alpha x)$, $x \geqslant 0$. We have $g(0) = 0$ and $g'(x) = 1/(1 + \alpha x)^2 > 0$. This implies that $g(x)$ is a strictly increasing function. Thus, we have $t_{i+1} - s_i < t_i - s_{i-1}$, for $i = 1, 2, \ldots, n-1$. Finally, the proof of part (c) immediately follows parts (a) and (b). □

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