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Fuzzy Mixture Inventory Model with Variable Lead-Time Based on Probabilistic Fuzzy Set and Triangular Fuzzy Number

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Abstract—This article considers the fuzzy problems for the mixture inventory model involving variable lead-time with backorders and lost sales. We first use the probabilistic fuzzy set to construct a new random variable for lead-time demand, and derive the total expected annual cost in the fuzzy sense. Then, the average demand per year is fuzzified as the triangular fuzzy number. For this case, two methods of defuzzification, namely signed distance and centroid, are employed to find the value of total expected annual cost in the fuzzy sense. Next, the backorder rate of the demand during the stock-out period is also fuzzified as the triangular fuzzy number, and the value of total expected annual cost in the fuzzy sense is derived using the signed distance. For the proposed models, we provide a solution procedure to find the optimal lead-time and the optimal order quantity such that the total expected annual cost in the fuzzy sense has a minimum value. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Fuzzy inventory, Lead-time, Signed distance, Fuzzy total cost.

1. INTRODUCTION

Lead-time plays an important role in production/inventory control systems. As stated in [1], lead-time usually consists of the following components: order preparation, order transit, supplier lead-time, delivery time, and setup time. The classical inventory models often assume lead-time as a given parameter or a random variable, which is not subject to control. However, this assumption may not always be true. In some practical situations, the above components of lead-

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time could be accomplished earlier than the regular time if one is willing to pay extra costs. For example, one may adopt the special delivery (by air) instead of ordinary delivery (by water) to shorten the delivery time, while the air freight rate is higher than the water freight rate. In this sense, lead-time is controllable (reducible). Furthermore, through the Japanese successful experiences of using just-in-time (JIT) production, the benefits associated with efforts to reduce lead-time, such as lowering the safety stock, reducing the loss caused by stock-out, increasing the service level to the customer, and gaining the competitive advantages in business, have been evidenced. Inventory models considering lead-time as a decision variable have been developed by several researchers. Liao and Shyu [2] first presented a continuous review inventory model in which the order quantity is predetermined and lead-time is a unique variable. They formulated the crashing cost function for reduced lead-time by a piecewise linear function. Ben-Daya and Raouf [3] extended [2] to include both lead-time, and order quantity as decision variables. Later, Ouyang *et al.* [4] provided a more meaningful mathematical model for the decision-maker; they extended [3] by allowing shortages with a mixture of backorders and lost sales.

The above lead-time reduction models are based on the continuous review inventory systems, where the uncertain lead-time demand is formulated by a random variable using the approach from the traditional probability theory. In addition, the other inputs such as the average demand per year and the backorder rate of the demand during the stock-out period in the model proposed in [4] are assumed with crisp values. However, in real inventory systems, various types of uncertainties and imprecision including randomness and fuzziness often exist. In this article, we shall consider a possible situation where fuzziness and randomness appear simultaneously in lead-time demand, and adopt the probabilistic fuzzy set proposed by Hirota [5] to deal with this situation. Furthermore, to assess the annual average demand or the backorder rate by a crisp value is not an easy task, since they may have a little fluctuation in unstable environments. For this scenario, it is more suitable to describe these factors by linguistic terms such as approximately equal to some certain amounts or consider their values to be located in some intervals. This study will also apply the fuzzy sets concept initially introduced by Zadeh [6] to formulate those uncertain factors.

In recent years, several researchers have developed various types of inventory problems in fuzzy environments. For example, Petrović and Sweeney [7] fuzzified the demand, lead-time, and inventory level into triangular fuzzy numbers in an inventory control model. Vujošević *et al.* [8] extended the classical EOQ model by introducing the fuzziness of ordering cost and holding cost. Chen and Wang [9] fuzzified the demand, ordering cost, inventory cost, and backorder cost into trapezoidal fuzzy numbers in the EOQ model with backorder. Roy and Maiti [10] presented a fuzzy EOQ model with demand-dependent unit cost under limited storage capacity. Gen *et al.* [11] considered the fuzzy input data expressed by fuzzy numbers, where the interval mean value concept is used to help solve the problem. Ishii and Konno [12] fuzzified the shortage cost into an L-shape fuzzy number in a classical newsboy problem aimed to find an optimal ordering quantity in the sense of fuzzy ordering. Chang *et al.* [13] presented a fuzzy model for inventory with backorder, where the backorder quantity was fuzzified as the triangular fuzzy number. Lee and Yao [14] and Lin and Yao [15] discussed the production inventory problems, where [14] fuzzified the demand quantity and production quantity per day, and [15] fuzzified the production quantity per cycle, all to be the triangular fuzzy numbers. Yao *et al.* [16] proposed the EOQ model in the fuzzy sense, where both order quantity and total demand were fuzzified as the triangular fuzzy numbers. Ouyang and Yao [17] presented a mixture inventory model involving variable lead-time, where the annual average demand was fuzzified as the triangular fuzzy number and statistic-fuzzy number. Specifically, in [13–17], they used the extension principle and centroid method to find the total cost in the fuzzy sense, and showed that it is close to the crisp total cost when fuzzy is small.

From literature review, we note that although several fuzzy inventory models have been presented, little has been done on addressing the issue of lead-time reduction in fuzzy environments.

The purpose of this article is to recast Ouyang *et al.*'s mixture inventory model [4] involving variable lead-time with backorders and lost sales by introducing the fuzziness of lead-time demand, the average demand per year, and the backorder rate of the demand during the stock-out period. We aim at providing an alternative approach of modeling uncertainty that may appear in real situations, while we do not attempt to establish the superiority of proposing new models to reduce more inventory cost than before. Moreover, in addition to the centroid method that is often used for defuzzification, a new ranking method for fuzzy numbers, namely the signed distance, introduced by Yao and Wu [18] will be employed to solve our problems.

This article is organized as follows. In Section 2, some definitions and propositions related to this study are introduced, and a brief review of Ouyang *et al.*'s model [4] is provided. In Section 3, three fuzzy inventory models involving variable lead-time are presented. We first use the probabilistic fuzzy set to construct a new random variable for lead-time demand, and obtain the total expected annual cost in the fuzzy sense. Then, the average demand per year is fuzzified as the triangular fuzzy number. The signed distance and centroid methods are employed to find the value of total expected annual cost in the fuzzy sense. Next, the backorder rate is also fuzzified as the triangular fuzzy number, and the value of total expected annual cost in the fuzzy sense is derived using the signed distance. These results are summarized in Theorems 1–3. In Section 4, we derive the optimal order quantity and the optimal lead-time by minimizing the total expected annual cost in the fuzzy sense. Numerical examples are carried out to illustrate the results. In Section 5, we discuss some problems for the proposed models. Section 6 summarizes the work done in this article.

2. PRELIMINARIES

In order to consider the fuzziness for Ouyang *et al.*'s mixture inventory model [4], some definitions and propositions relative to this study are introduced first.

2.1. Formulas

DEFINITION 1. *(See [19].) A fuzzy set $\tilde{a}$ defined on $R = (-\infty, \infty)$ is called a fuzzy point, if the membership function of $\tilde{a}$ is given by*

$$\mu_{\tilde{a}}(x) = \begin{cases} 1, & x = a, \\ 0, & x \neq a. \end{cases}$$

DEFINITION 2. *(See [5].) Let $(\Omega, \mathcal{A}, P)$ be the probability space, and T be the total space. The probabilistic fuzzy set A on T is defined as the following function:*

$$\mu_A : T \times \Omega \to \Omega_c = [0, 1], \qquad (t, x) \to \mu_A(t, x).$$

For each fixed $t \in T$, $\mu_A(t, \cdot)$ is a measurable function on $(\mathcal{A}, \mathcal{A}_c)$, where $\mathcal{A}_c$ is a Borel field of Ω_c. For each fixed x, $\mu_A(t, x)$ is the membership function of the fuzzy set A.

DEFINITION 3. *The fuzzy set $\tilde{A} = (a, b, c)$, where $a < b < c$ and defined on R, is called the triangular fuzzy number, if the membership function of $\tilde{A}$ is given by*

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{(x-a)}{(b-a)}, & a \leq x \leq b, \\ \dfrac{(c-x)}{(c-b)}, & b \leq x \leq c, \\ 0, & \text{otherwise}. \end{cases}$$

Then, the centroid of $\tilde{A}$ can be derived as

$$C\left(\tilde{A}\right) = \frac{\int_{-\infty}^{\infty} x\mu_{\tilde{A}}(x)\,dx}{\int_{-\infty}^{\infty} \mu_{\tilde{A}}(x)\,dx} = \frac{1}{3}(a + b + c). \tag{1}$$

Furthermore, when $c = b = a$, then $\tilde{A} = (a, a, a) = \tilde{a}$ becomes a fuzzy point.

For any $a, b, p, q, k \in R$, $a < b$, and $p < q$, the operations of crisp interval are as follows (see, e.g., [20,21]):

(i)
$$[a, b] + [p, q] = [a + p, b + q];$$

(ii)
$$[a, b] - [p, q] = [a - q, b - p];$$

(iii)
$$k[a, b] = \begin{cases} [ka, kb], & k > 0, \\ [kb, ka], & k < 0. \end{cases} \tag{2}$$

Further, if $a > 0$ and $p > 0$, then

(iv)
$$[a, b] \cdot [p, q] = [ap, bq].$$

DEFINITION 4. *A fuzzy set $[a, b; \alpha]$, where $0 \le \alpha \le 1$, $a < b$ and defined on R, is called a level α fuzzy interval, if the membership function of $[a, b; \alpha]$ is given by*

$$\mu_{[a,b;\alpha]}(x) = \begin{cases} \alpha, & a \le x \le b, \\ 0, & \textit{otherwise}. \end{cases}$$

Let $\mathbf{F}$ be the family of all fuzzy sets defined on R that satisfy the following two conditions:

(i) $\tilde{B} \in \mathbf{F}$, the α-cut $B(\alpha) = [B_l(\alpha), B_u(\alpha)]$ of $\tilde{B}$ exists for every $\alpha \in [0, 1]$;

(ii) $B_l(\alpha)$ and $B_u(\alpha)$ are continuous functions on $0 \le \alpha \le 1$.

Then, for any $\tilde{B} \in \mathbf{F}$, from the decomposition theorem, $\tilde{B}$ can be expressed as

$$\tilde{B} = \bigcup_{0 \le \alpha \le 1} \alpha I_{B(\alpha)}(x), \tag{3}$$

where $I_{B(\alpha)}(x)$ is the characteristic function of $B(\alpha)$, and

$$\mu_{\alpha I_{B(\alpha)}(x)}(z) = \begin{cases} \alpha, & z \in B(\alpha), \\ 0, & z \notin B(\alpha). \end{cases}$$

From Definition 4 and equation (3), it is clear for $x \in R$, $\alpha I_{B(\alpha)}(x) = \mu_{[B_l(\alpha), B_u(\alpha); \alpha]}(x)$, $\forall \alpha \in [0, 1]$. Hence, $\tilde{B}$ can further be expressed as

$$\tilde{B} = \bigcup_{0 \le \alpha \le 1} [B_l(\alpha), B_u(\alpha); \alpha]. \tag{4}$$

Next, as in [18], we introduce the concept of signed distance of fuzzy set on $\mathbf{F}$, which will be needed later. We first consider the signed distance for a point defined on R.

DEFINITION 5. *For any a and $0 \in R$, define the signed distance from a to 0 as $d_0(a, 0) = a$. If $a > 0$, the distance from a to 0 is $a = d_0(a, 0)$; if $a < 0$, the distance from a to 0 is $-a = -d_0(a, 0)$. Hence, $d_0(a, 0) = a$ is called the signed distance from a to 0.*

For $\tilde{B} \in \mathbf{F}$ with the α-cut $B(\alpha) = [B_l(\alpha), B_u(\alpha)]$, $\alpha \in [0, 1]$, the signed distance of two endpoints $B_l(\alpha)$ and $B_u(\alpha)$ of this α-cut to the origin 0 is $d_0(B_l(\alpha), 0) = B_l(\alpha)$ and $d_0(B_u(\alpha), 0) = B_u(\alpha)$, respectively (according to Definition 5). Their average, $[B_l(\alpha) + B_u(\alpha)]/2$ is taken as the signed distance of α-cut $[B_l(\alpha), B_u(\alpha)]$ to 0, that is, the signed distance of interval $[B_l(\alpha), B_u(\alpha)]$ to 0 is defined as: $d_0([B_l(\alpha), B_u(\alpha)], 0) = [B_l(\alpha) + B_u(\alpha)]/2$. In addition, for every $\alpha \in [0, 1]$,

there is a one-to-one mapping between the *level α fuzzy interval* $[B_l(\alpha), B_u(\alpha); \alpha]$ and real interval $[B_l(\alpha), B_u(\alpha)]$, i.e., the following corresponding is one-to-one mapping:

$$[B_l(\alpha), B_u(\alpha); \alpha] \leftrightarrow [B_l(\alpha), B_u(\alpha)]. \tag{5}$$

Also, the fuzzy point $\tilde{0}$ is a mapping to the real number 0. Hence, the signed distance of $[B_l(\alpha), B_u(\alpha); \alpha]$ to $\tilde{0}$ can be defined as

$$d\left([B_l(\alpha), B_u(\alpha); \alpha], \tilde{0}\right) = d_0\left([B_l(\alpha), B_u(\alpha)], 0\right) = \frac{[B_l(\alpha) + B_u(\alpha)]}{2}. \tag{6}$$

Furthermore, for $\tilde{B} \in \mathbf{F}$, since the above function is continuous on $0 \leq \alpha \leq 1$, we can use the integration to obtain the mean value of the signed distance. That is,

$$\int_0^1 d\left([B_l(\alpha), B_u(\alpha); \alpha], \tilde{0}\right)\, d\alpha = \frac{1}{2}\int_0^1 (B_l(\alpha) + B_u(\alpha))\, d\alpha. \tag{7}$$

Thus, from equations (4) and (7), we can define the signed distance of a fuzzy set $\tilde{B} \in \mathbf{F}$ to $\tilde{0}$ as follows.

DEFINITION 6. *For $\tilde{B} \in \mathbf{F}$, define the signed distance of $\tilde{B}$ to $\tilde{0}$ as*

$$d\left(\tilde{B}, \tilde{0}\right) = \int_0^1 d\left([B_l(\alpha), B_u(\alpha); \alpha], \tilde{0}\right)\, d\alpha = \frac{1}{2}\int_0^1 (B_l(\alpha) + B_u(\alpha))\, d\alpha.$$

For the triangular fuzzy number $\tilde{A} = (a, b, c)$, the α-cut of $\tilde{A}$ is $A(\alpha) = [A_l(\alpha), A_u(\alpha)]$, $\alpha \in [0, 1]$, where $A_l(\alpha) = a + (b - a)\alpha$ and $A_u(\alpha) = c - (c - b)\alpha$. From Definition 6, we get

$$d\left(\tilde{A}, \tilde{0}\right) = \frac{1}{4}(2b + a + c). \tag{8}$$

Moreover, for $\tilde{A} = (a, b, c)$, the relationship between centroid $C(\tilde{A})$ (in equation (1)) and signed distance $d(\tilde{A}, \tilde{0})$ (in equation (8)) is as follows. Let $M = (a + c)/2$. By the results that $b - d(\tilde{A}, \tilde{0}) = (2b - a - c)/4 = (b - M)/2$, $d(\tilde{A}, \tilde{0}) - C(\tilde{A}) = (b - M)/6$, and $C(\tilde{A}) - M = (b - M)/3$, we have the following.

(i) If $b < M$, then $b < d(\tilde{A}, \tilde{0}) < C(\tilde{A}) < M$.
(ii) If $b > M$, then $M < C(\tilde{A}) < d(\tilde{A}, \tilde{0}) < b$.
(iii) If $b = M$, then $b = d(\tilde{A}, \tilde{0}) = C(\tilde{A}) = M$.

The results of (i) and (ii) are shown in Figures 1 and 2, respectively.

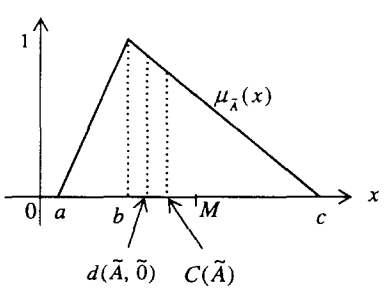

Figure 1. The case of $b < M$.

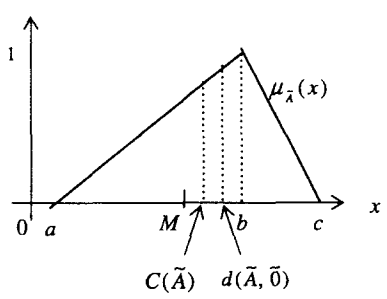

Figure 2. The case of $b > M$.

NOTE 1. From Figures 1 and 2, it can be found that $\mu_{\tilde{A}}(C(\tilde{A})) < \mu_{\tilde{A}}(d(\tilde{A},\tilde{0})) < \mu_{\tilde{A}}(b) = 1$, i.e., the membership grade at $d(\tilde{A},\tilde{0})$ is greater than that at $C(\tilde{A})$. Furthermore, from equations (1) and (8), $C(\tilde{A}) = (a+b+c)/3$ and $d(\tilde{A},\tilde{0}) = (a+2b+c)/4$, we note that both of them are weighted mean of a set of numbers (a,b,c), whereas $d(\tilde{A},\tilde{0})$ is more meaningful than $C(\tilde{A})$, since the maximum membership grade of $\tilde{A} = (a,b,c)$ occurs at point b, which has a larger weight in $d(\tilde{A},\tilde{0})$ than in $C(\tilde{A})$. Hence, when we defuzzify the triangular fuzzy number, it is better to use signed distance than centroid.

In addition, for two fuzzy sets $\tilde{B},\tilde{G} \in \mathbf{F}$, where $\tilde{B} = \bigcup_{0\le\alpha\le1}[B_l(\alpha), B_u(\alpha);\alpha]$, and $\tilde{G} = \bigcup_{0\le\alpha\le1}[G_l(\alpha), G_u(\alpha);\alpha]$, from equations (2) and (5), we have

$$\tilde{B}(+)\tilde{G} = \bigcup_{0\le\alpha\le1}[B_l(\alpha)+G_l(\alpha), B_u(\alpha)+G_u(\alpha);\alpha], \tag{9}$$

$$k(\cdot)\tilde{B} = \begin{cases} \bigcup_{0\le\alpha\le1}[kB_l(\alpha), kB_u(\alpha);\alpha], & \text{if } k>0, \\ \bigcup_{0\le\alpha\le1}[kB_u(\alpha), kB_l(\alpha);\alpha], & \text{if } k<0. \end{cases} \tag{10}$$

Then, from equations (9), (10), and Definition 6, we obtain the following proposition.

PROPOSITION 1. *For $\tilde{B},\tilde{G} \in \mathbf{F}$, and $k \in R$,*

(i) $d(\tilde{B}(+)\tilde{G},\tilde{0}) = d(\tilde{B},\tilde{0}) + d(\tilde{G},\tilde{0})$;
(ii) $d(k(\cdot)\tilde{B},\tilde{0}) = kd(\tilde{B},\tilde{0})$;
(iii) $d(\tilde{B}(+)\tilde{k},\tilde{0}) = d(\tilde{B},\tilde{0}) + k$.

Finally, for two triangular fuzzy numbers, we have the following proposition.

PROPOSITION 2. *For $\tilde{A} = (a,b,c)$ and $\tilde{B} = (p,q,r)$, and $k \in R$,*

(i) $C(\tilde{A}(+)\tilde{B}) = C(\tilde{A}) + C(\tilde{B})$;
(ii) $C(k(\cdot)\tilde{A}) = kC(\tilde{A})$;
(iii) $C(\tilde{A}(+)\tilde{k}) = C(\tilde{A}) + k$.

PROOF. Since $\tilde{A}(+)\tilde{B} = (a+p, b+q, c+r)$, $k(\cdot)\tilde{A} = (ka, kb, kc)$ if $k>0$, $k(\cdot)\tilde{A} = (kc, kb, ka)$ if $k<0$, $\tilde{k} = (k,k,k)$, and from equation (1), we can get the above results.

2.2. Review of Ouyang *et al.*'s Model

To develop the proposed models, we adopt the following notation and assumptions used in [4].

Notation

$D =$ average demand per year,
$Q =$ order quantity,
$A =$ fixed ordering cost per order,
$h =$ inventory holding cost per item per year,
$\pi =$ fixed penalty cost per unit short,
$\pi_0 =$ marginal profit per unit,
$L =$ length of lead-time,
$r =$ reorder point,
$X =$ lead-time demand, which is normally distributed with finite mean μL and standard deviation $\sigma\sqrt{L}$, where μ and σ denote the mean and standard deviation of the demand per unit time,
$x^+ =$ maximum value of x and 0, i.e., $x^+ = \max\{x, 0\}$,
$E(\cdot) =$ mathematical expectation.

Assumptions

(1) The reorder point, r = expected demand during lead-time + safety stock (SS), and SS = $k \cdot$ (standard deviation of lead-time demand), i.e., $r = \mu L + k\sigma\sqrt{L}$, where k is the safety factor and satisfies $P(X > r) = P(Z > k) = q$, Z represents the standard normal random variable, and q represents the allowable stock-out probability during lead-time L, and q is given.

(2) Inventory is continuously reviewed. Replenishments are made whenever the inventory level falls to the reorder point r.

(3) The lead-time L has n mutually independent components each having a different crashing cost for reducing lead-time. The ith component has a minimum duration a_i and normal duration b_i, and a crashing cost per unit time c_i. Furthermore, we assume that $c_1 \leq c_2 \leq \cdots \leq c_n$.

(4) The components of lead-time are crashed one at a time starting with the component of least c_i and so on.

(5) If we let $L_0 = \sum_{j=1}^{n} b_j$ and L_i be the length of lead-time with components $1, 2, \ldots, i$ crashed to their minimum duration, then L_i can be expressed as $L_i = \sum_{j=1}^{n} b_j - \sum_{j=1}^{i}(b_j - a_j)$, $i = 1, 2, \ldots, n$; and the lead-time crashing cost per cycle $U(L)$ for a given $L \in [L_i, L_{i-1}]$ is given by

$$U(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j) \qquad \text{and} \qquad U(L_0) = 0. \tag{11}$$

By the above assumptions and considering that only a fraction β $(0 \leq \beta \leq 1)$ of the demand during the stock-out period can be backordered, Ouyang *et al.* [4] established the total expected annual cost as follows:

$$\begin{aligned}
\text{EAC}(Q, L) &= \text{setup cost} + \text{holding cost} + \text{stock-out cost} + \text{lead-time crashing cost} \\
&= A\frac{D}{Q} + h\left[\frac{Q}{2} + r - \mu L + (1-\beta)E(X-r)^+\right] \\
&\quad + \frac{D}{Q}[\pi + \pi_0(1-\beta)]E(X-r)^+ + \frac{D}{Q}U(L) \\
&= \frac{D}{Q}\left\{A + [\pi + \pi_0(1-\beta)]E(X-r)^+ + U(L)\right\} \\
&\quad + h\left[\frac{Q}{2} + k\sigma\sqrt{L} + (1-\beta)E(X-r)^+\right],
\end{aligned} \tag{12}$$

for $Q > 0$, $L \in [L_i, L_{i-1}]$, $i = 1, 2, \ldots, n$, where $E(X-r)^+$ is the expected demand shortage at the end of the cycle.

3. FUZZY MIXTURE INVENTORY MODEL INVOLVING VARIABLE LEAD-TIME

3.1. Fuzzy Mixture Inventory Model Involving Variable Lead-Time with Probabilistic Fuzzy Set

In contrast to Ouyang *et al.*'s model [4], here we consider the fuzzy mixture inventory model with the probabilistic fuzzy set as the following. Let $(R, \mathcal{B}, P)$ be the probability space, where R is the set of real numbers, $\mathcal{B}$ is the *Borel field* on R, and P is a probability measure. The lead-time demand, X, in Section 2.2 is a crisp random variable on $(R, \mathcal{B}, P)$, which is assumed to be normally distributed with mean μL and standard deviation $\sigma\sqrt{L}$, i.e., $X \sim N(\mu L, \sigma^2 L)$. For

notational convenience, from now on, we denote $\mu_L = \mu L$ and $\sigma_L = \sigma\sqrt{L}$, then the probability density function (pdf) of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma_L} \exp\left[-\frac{(x-\mu_L)^2}{2\sigma_L^2}\right], \qquad -\infty < x < \infty. \tag{13}$$

From the definition of L_i (Assumption 5), we have $\min_{0\le i\le n} L_i = L_n$ and $\max_{0\le i\le n} L_i = L_0$, hence, $L_n \le L \le L_0$. For any fixed $L \in [L_i, L_{i-1}]$, $i = 1, 2, \ldots, n$, the value μ_L of lead-time demand (LTD) may have a little fluctuation in the uncertain and/or unstable environments. For this scenario, it is more suitable to describe the value of LTD by an interval $[\mu_L - \Delta_1, \mu_L + \Delta_2]$, where Δ_1, Δ_2 are determined by the decision-maker and should satisfy the conditions $0 < \Delta_1 < \mu_{L_n}$ and $0 < k\sigma_{L_0} < \Delta_2$. In order to find the corresponding fuzzy set with this interval $[\mu_L - \Delta_1, \mu_L + \Delta_2]$, we take a value μ_L^* from the interval $[\mu_L - \Delta_1, \mu_L + \Delta_2]$ and then compare it with μ_L (LTD of crisp case). If $\mu_L^* = \mu_L$, then we define the *error* $|\mu_L^* - \mu_L| = 0$. In the fuzzy sense, we can use the term *confidence level* instead of *error*. When the error is zero, the confidence level will be the largest, and we set it to be 1. If μ_L^* is located in $[\mu_L - \Delta_1, \mu_L)$ or $(\mu_L, \mu_L + \Delta_2]$, the farther the value μ_L^* deviates from μ_L, the larger the error $|\mu_L^* - \mu_L|$, and hence, the smaller the confidence level. When $\mu_L^* = \mu_L - \Delta_1$ and $\mu_L^* = \mu_L + \Delta_2$, the errors $|\mu_L^* - \mu_L|$ will attain to the largest, and the confidence level will be the smallest and we set it be zero.

Therefore, corresponding to the interval $[\mu_L - \Delta_1, \mu_L + \Delta_2]$, we set the following triangular fuzzy number:

$$\tilde{\mu}_L = (\mu_L - \Delta_1, \mu_L, \mu_L + \Delta_2), \tag{14}$$

where $0 < \Delta_1 < \mu_{L_n}$ and $0 < k\sigma_{L_0} < \Delta_2$. Note that the membership grade of $\tilde{\mu}_L$ is 1 at point μ_L, decreases as the point deviates from μ_L, and reaches zero at the endpoints $\mu_L - \Delta_1$ and $\mu_L + \Delta_2$. Since the properties of membership grade and confidence level are the same, consequently, when the confidence level is treated as the membership grade, corresponding to the interval $[\mu_L - \Delta_1, \mu_L + \Delta_2]$, it is reasonable to set the above triangular fuzzy number $\tilde{\mu}_L$.

By Note 1, utilizing the signed distance to defuzzify $\tilde{\mu}_L$, we obtain

$$\mu_L^* \equiv d\left(\tilde{\mu}_L, \tilde{0}\right) = \mu_L + \frac{1}{4}(\Delta_2 - \Delta_1) = \frac{3}{4}\mu_L + \frac{1}{4}\Delta_2 + \frac{1}{4}(\mu_L - \Delta_1) > 0. \tag{15}$$

$\mu_L^*$is regarded as the value of LTD in the fuzzy sense and $\mu_L^* \in [\mu_L - \Delta_1, \mu_L + \Delta_2]$. If $\Delta_1 < \Delta_2$, then $\mu_L < \mu_L^*$; if $\Delta_2 < \Delta_1$, then $\mu_L^* < \mu_L$; and if $\Delta_1 = \Delta_2$, then $\mu_L = \mu_L^*$.

Furthermore, by the assumption that the reorder point $r = \mu_L + k\sigma_L$, and from an observation x of the crisp random variable X, we obtain a fuzzy point $\tilde{x} = (x, x, x)$. Corresponding to the crisp random variable $X - \mu_L$, the observation is $x - \mu_L$. Then, from equation (14), we get $\tilde{P} \equiv \tilde{x} - \tilde{\mu}_L = (x - \mu_L - \Delta_2, x - \mu_L, x - \mu_L + \Delta_1)$, and obtain the following membership function of the probabilistic fuzzy set $\tilde{P}$:

$$\mu_{\tilde{P}}(t, x) = \begin{cases} \dfrac{t - (x - \mu_L - \Delta_2)}{\Delta_2}, & x - \mu_L - \Delta_2 \le t \le x - \mu_L, \\ \dfrac{(x - \mu_L + \Delta_1) - t}{\Delta_1}, & x - \mu_L \le t \le x - \mu_L + \Delta_1, \\ 0, & \text{otherwise}. \end{cases} \tag{16}$$

Note that from Definition 2, x can be viewed as the random part, and t can be viewed as the fuzzy part.

From the above probabilistic fuzzy set, and by Note 1 using the signed distance to defuzzify, the relationship of random part can be derived as follows:

$$w \equiv d\left(\tilde{P}, \tilde{0}\right) = x - \mu_L + \frac{1}{4}(\Delta_1 - \Delta_2). \tag{17}$$

Hence, from equation (17), we obtain the corresponding crisp random variable

$$W = X - \mu_L + \frac{1}{4}(\Delta_1 - \Delta_2). \tag{18}$$

By the probability theorem, the pdf of W can be derived from the pdf of X, i.e., the pdf of W is

$$\begin{aligned} g(w) &= f\left(w + \mu_L - \frac{1}{4}(\Delta_1 - \Delta_2)\right)\frac{dx}{dw} \\ &= \frac{1}{\sqrt{2\pi}\,\sigma_L}\exp\left[-\frac{[w - (\Delta_1 - \Delta_2)/4]^2}{2\sigma_L^2}\right], \qquad -\infty < w < \infty. \end{aligned} \tag{19}$$

Note that instead of using the random variable X in [4], here we consider a new random variable W for the problem; therefore, the term $E(X-r)^+$ in equation (12) is replaced by $E(W-r)^+$.

Now, we derive $E(W-r)^+$. From equation (19), let $s = (1/\sigma_L)[w - (\Delta_1 - \Delta_2)/4]$. Then, $w - r = \sigma_L s - r + (\Delta_1 - \Delta_2)/4 \geq 0 \Leftrightarrow s \geq (1/\sigma_L)[r - (\Delta_1 - \Delta_2)/4]$. Therefore, we have

$$\begin{aligned} E(W-r)^+ &= \int_{[r-(\Delta_1-\Delta_2)/4]/\sigma_L}^{\infty}\left[\sigma_L s - r + \frac{1}{4}(\Delta_1 - \Delta_2)\right]\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{s^2}{2}\right)ds \\ &= \sigma_L\phi\left(\frac{r - (\Delta_1 - \Delta_2)/4}{\sigma_L}\right) \\ &\quad - \left[r - \frac{1}{4}(\Delta_1 - \Delta_2)\right]\left[1 - \Phi\left(\frac{r - (\Delta_1 - \Delta_2)/4}{\sigma_L}\right)\right], \end{aligned} \tag{20}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and the cumulative distribution function (cdf) of the standard normal distribution, respectively.

From the above, we obtain the following theorem.

THEOREM 1. *In equation (12), using the probabilistic fuzzy set of equation (16), and equation (18), the term $E(X-r)^+$ is changed to be $E(W-r)^+$ (as shown in equation (20)). In this case, the total expected annual cost in the fuzzy sense is given by*

$$\begin{aligned} \mathrm{EACP}(Q, L; \Delta_1, \Delta_2) &= \frac{D}{Q}\left\{A + [\pi + \pi_0(1-\beta)]E(W-r)^+ + U(L)\right\} \\ &\quad + h\left[\frac{Q}{2} + k\sigma\sqrt{L} + (1-\beta)E(W-r)^+\right]. \end{aligned} \tag{21}$$

Note that we will derive the corresponding optimal solution of equation (21) in Section 4, and discuss the relationship between $E(X-r)^+$ and $E(W-r)^+$ in Section 5.B later.

3.2. Fuzzy Mixture Inventory Model Involving Variable Lead-Time with Probabilistic Fuzzy Set and Triangular Fuzzy Number

In the real situation, due to various uncertainties, the annual average demand may have a little fluctuation, especially, in a perfect competitive market. Therefore, it is difficult for the decision-maker to assess the annual average demand by a crisp value D, but easier to determine it by an interval $[D - \Delta_3, D + \Delta_4]$. Similar to the previous approach, corresponding to the interval $[D - \Delta_3, D + \Delta_4]$, we can set the following triangular fuzzy number:

$$\tilde{D} = (D - \Delta_3, D, D + \Delta_4), \tag{22}$$

where Δ_3 and Δ_4 are determined by the decision-maker and should satisfy the conditions $0 < \Delta_3 < D$ and $0 < \Delta_4$.

Now, we employ the methods of signed distance and centroid to defuzzify $\tilde{D}$. By the signed distance method, we get

$$D^* \equiv d\left(\tilde{D}, \tilde{0}\right) = D + \frac{1}{4}(\Delta_4 - \Delta_3) > 0. \tag{23}$$

By the centroid method, we get

$$D^{(0)} \equiv C\left(\tilde{D}\right) = D + \frac{1}{3}(\Delta_4 - \Delta_3) > 0. \tag{24}$$

Both D^* and $D^{(0)}$ are regarded as the values of annual demand in the fuzzy sense.

From equation (21) in Theorem 1, for each $Q > 0$, and $L \in [L_i, L_{i-1}]$, $i = 1, 2, \ldots, n$, let

$$\begin{aligned} H_{(Q,L)}(D) &\equiv \mathrm{EACP}(Q, L; \Delta_1, \Delta_2) \\ &= \frac{D}{Q}\left\{A + [\pi + \pi_0(1-\beta)]E(W-r)^+ + U(L)\right\} \\ &\quad + h\left[\frac{Q}{2} + k\sigma\sqrt{L} + (1-\beta)E(W-r)^+\right]. \end{aligned} \tag{25}$$

Next, by incorporating the fuzziness of annual demand into equation (25), i.e., replacing D by $\tilde{D}$ as described in equation (22), we obtain the fuzzy total cost as follows:

$$\begin{aligned} H_{(Q,L)}\left(\tilde{D}\right) &= \frac{1}{Q}\left\{A + [\pi + \pi_0(1-\beta)]E(W-r)^+ + U(L)\right\}\tilde{D} \\ &\quad + h\left[\frac{Q}{2} + k\sigma\sqrt{L} + (1-\beta)E(W-r)^+\right]. \end{aligned} \tag{26}$$

Note that here the operations of fuzzy sets, $(+)$, $(-)$, $(\cdot)$, for simplification, are expressed as $+$, $-$, $\cdot$, and the fuzzy point $\tilde{a}$ is expressed as a.

Then, we get the following theorem.

THEOREM 2. *The values of total expected annual cost in the fuzzy sense $H_{(Q,L)}(\tilde{D})$ are as follows.*

(i) *Using the signed distance method to defuzzify equation (26) results in*

$$\begin{aligned} &\mathrm{EACPS}(Q, L; \Delta_j, j = 1,2,3,4) = \mathrm{EACP}(Q, L; \Delta_1, \Delta_2) \\ &+ \frac{(\Delta_4 - \Delta_3)}{4Q}\left\{A + [\pi + \pi_0(1-\beta)]E(W-r)^+ + U(L)\right\}. \end{aligned} \tag{27}$$

(ii) *Using the centroid method to defuzzify equation (26) results in*

$$\begin{aligned} &\mathrm{EACPC}(Q, L; \Delta_j, j = 1,2,3,4) = \mathrm{EACP}(Q, L; \Delta_1, \Delta_2) \\ &+ \frac{(\Delta_4 - \Delta_3)}{3Q}\left\{A + [\pi + \pi_0(1-\beta)]E(W-r)^+ + U(L)\right\}. \end{aligned} \tag{28}$$

PROOF.

(i) From Proposition 1 with equations (23) and (26), we have

$$\begin{aligned} d\left(H_{(Q,L)}\left(\tilde{D}\right), \tilde{0}\right) &= \frac{1}{Q}\left\{A + [\pi + \pi_0(1-\beta)]E(W-r)^+ + U(L)\right\}\left[D + \frac{1}{4}(\Delta_4 - \Delta_3)\right] \\ &\quad + h\left[\frac{Q}{2} + k\sigma\sqrt{L} + (1-\beta)E(W-r)^+\right] \\ &= \mathrm{EACP}(Q, L; \Delta_1, \Delta_2) \\ &\quad + \frac{(\Delta_4 - \Delta_3)}{4Q}\left\{A + [\pi + \pi_0(1-\beta)]E(W-r)^+ + U(L)\right\}. \end{aligned}$$

(ii) Similar to Proof (i), from Proposition 2 with equations (24) and (26), we can get the result.

The corresponding optimal solutions of equations (27) and (28) will be derived later in Section 4. Also, we will discuss which value, obtained by equations (27) or (28), is better in Section 5.C.

3.3. Fuzzy Mixture Inventory Model Involving Variable Lead-Time with Probabilistic Fuzzy Set and Two Triangular Fuzzy Numbers

In the real market, it is difficult for the decision-maker to know explicitly how many customers will accept backorders when the stock-out occurs. Therefore, in this section, we further incorporate the fuzziness of backorder rate into the model. Following the approach in Section 3.2, from Theorem 1 (equation (21)), for each $Q > 0$, and $L \in [L_i, L_{i-1}]$, $i = 1, 2, \ldots, n$, let

$$\begin{aligned} G_{(Q,L)}(D,\beta) \equiv \mathrm{EACP}(Q,L;\Delta_1,\Delta_2) &= \frac{D}{Q}\left\{A + (\pi+\pi_0)E(W-r)^+ + U(L)\right\} \\ &- \frac{\pi_0 D\beta}{Q}E(W-r)^+ + h\left[\frac{Q}{2} + k\sigma\sqrt{L} + E(W-r)^+\right] - h\beta E(W-r)^+. \end{aligned} \tag{29}$$

Similar to equation (22), we fuzzify the backorder rate β of the demand during the stock-out period as the following triangular fuzzy number:

$$\tilde{\beta} = (\beta - \Delta_5, \beta, \beta + \Delta_6), \tag{30}$$

where Δ_5 and Δ_6 are determined by the decision-maker and should satisfy the conditions $0 < \Delta_5 < \beta$ and $0 < \Delta_6$.

Then, by Note 1 using the signed distance method to defuzzify $\tilde{\beta}$, we get

$$\beta^* \equiv d\left(\tilde{\beta}, \tilde{0}\right) = \beta + \frac{1}{4}(\Delta_6 - \Delta_5) > 0, \tag{31}$$

which is the value of backorder rate of the demand during the stock-out period in the fuzzy sense.

When D and β in equation (29) are fuzzified to be $\tilde{D}$ and $\tilde{\beta}$ as described in equations (22) and (30), respectively, we obtain the following fuzzy total expected annual cost:

$$\begin{aligned} G_{(Q,L)}\left(\tilde{D},\tilde{\beta}\right) &= \frac{1}{Q}\left\{A + (\pi+\pi_0)E(W-r)^+ + U(L)\right\}\tilde{D} - \frac{\pi_0}{Q}E(W-r)^+\tilde{D}\cdot\tilde{\beta} \\ &+ h\left[\frac{Q}{2} + k\sigma\sqrt{L} + E(W-r)^+\right] - hE(W-r)^+\tilde{\beta}. \end{aligned} \tag{32}$$

From equations (22) and (30), and for every $\alpha \in [0,1]$, we obtain the left endpoint and right endpoint of α-cut of $\tilde{D}$ and $\tilde{\beta}$, respectively, as follows:

$$\begin{aligned} D_l(\alpha) &= D - (1-\alpha)\Delta_3 > 0, & D_u(\alpha) &= D + (1-\alpha)\Delta_4 > 0, \\ \beta_l(\alpha) &= \beta - (1-\alpha)\Delta_5 > 0, & \beta_u(\alpha) &= \beta + (1-\alpha)\Delta_6 > 0. \end{aligned} \tag{33}$$

Also, the α-cut of $\tilde{D}\cdot\tilde{\beta}$ is $(D\beta)(\alpha) = [(D\beta)_l(\alpha), (D\beta)_u(\alpha)]$, where

$$(D\beta)_l(\alpha) = D_l(\alpha)\beta_l(\alpha) \qquad \text{and} \qquad (D\beta)_u(\alpha) = D_u(\alpha)\beta_u(\alpha). \tag{34}$$

THEOREM 3. *The value of total expected annual cost in the fuzzy sense $G_{(Q,L)}(\tilde{D},\tilde{\beta})$ obtained by the signed distance method is as follows:*

$$\begin{aligned} &\mathrm{EACP}^*(Q,L;\Delta_j, j = 1,2,\ldots,6) \\ &= \mathrm{EACP}(Q,L;\Delta_1,\Delta_2) + \mathrm{FTER}(Q,L;\Delta_j, j = 3,4,5,6), \end{aligned} \tag{35}$$

where

$$\begin{aligned} \mathrm{FTER}(Q,L;\Delta_j, j = 3,4,5,6) &= \frac{(\Delta_4 - \Delta_3)}{4Q}\left\{A + (\pi+\pi_0)E(W-r)^+ + U(L)\right\} \\ &- \frac{\pi_0}{Q}E(W-r)^+\left[\frac{1}{4}D(\Delta_6 - \Delta_5) + \frac{1}{4}\beta(\Delta_4 - \Delta_3) + \frac{1}{6}(\Delta_3\Delta_5 + \Delta_4\Delta_6)\right] \\ &- \frac{h}{4}E(W-r)^+(\Delta_6 - \Delta_5). \end{aligned}$$

PROOF. From equations (32)–(34), the signed distance of $\tilde{D}\cdot\tilde{\beta}$ to $\tilde{0}$ is

$$\begin{aligned} d\left(\tilde{D}\cdot\tilde{\beta},\tilde{0}\right) &= \frac{1}{2}\int_0^1 \left[(D\beta)_l(\alpha)+(D\beta)_u(\alpha)\right]d\alpha \\ &= \frac{1}{2}\int_0^1 \left\{[D-(1-\alpha)\Delta_3][\beta-(1-\alpha)\Delta_5]\right. \\ &\quad \left.+[D+(1-\alpha)\Delta_4][\beta+(1-\alpha)\Delta_6]\right\}d\alpha \\ &= D\beta+\frac{1}{4}D(\Delta_6-\Delta_5)+\frac{1}{4}\beta(\Delta_4-\Delta_3)+\frac{1}{6}(\Delta_3\Delta_5+\Delta_4\Delta_6). \end{aligned} \tag{36}$$

Thus, from Proposition 1 and equations (23), (31), (32), and (36), we obtain

$$\begin{aligned} d\left(G_{(Q,L)}\left(\tilde{D},\tilde{\beta}\right),\tilde{0}\right) &= \frac{1}{Q}\left\{A+(\pi+\pi_0)E(W-r)^+ + U(L)\right\}\left[D+\frac{1}{4}(\Delta_4-\Delta_3)\right] \\ &\quad -\frac{\pi_0}{Q}E(W-r)^+\left[D\beta+\frac{1}{4}D(\Delta_6-\Delta_5)+\frac{1}{4}\beta(\Delta_4-\Delta_3)\right. \\ &\quad \left.+\frac{1}{6}(\Delta_3\Delta_5+\Delta_4\Delta_6)\right]+h\left[\frac{Q}{2}+k\sigma\sqrt{L}+E(W-r)^+\right] \\ &\quad -hE(W-r)^+\left[\beta+\frac{1}{4}(\Delta_6-\Delta_5)\right] \\ &= \mathrm{EACP}(Q,L;\Delta_1,\Delta_2)+\mathrm{FTER}(Q,L;\Delta_j,j=3,4,5,6). \end{aligned}$$

REMARK 1. For the fuzzy total expected annual cost $G_{(Q,L)}(\tilde{D},\tilde{\beta})$ of equation (32), if we want to defuzzify it using the centroid method, we need to find the membership function of $G_{(Q,L)}(\tilde{D},\tilde{\beta})$, which can be expressed as $\mu_{G_{(Q,L)}(\tilde{D},\tilde{\beta})}(z)=\sup_{(x,y)\in f^{-1}(z)}\mu_{\tilde{D}}(x)\wedge\mu_{\tilde{\beta}}(y)$, using the extension principle, and from equation (29) we have

$$\begin{aligned} f(x,y) = \frac{x}{Q}\left\{A+(\pi+\pi_0)E(W-r)^+ + U(L)\right\} - \frac{\pi_0 xy}{Q}E(W-r)^+ \\ +h\left[\frac{Q}{2}+k\sigma\sqrt{L}+E(W-r)^+\right]-hyE(W-r)^+. \end{aligned}$$

In this sense, it is difficult to derive $\mu_{G_{(Q,L)}(\tilde{D},\tilde{\beta})}(z)$. Therefore, in this case, we adopt the signed distance to defuzzify the fuzzy total cost rather than centroid.

4. THE OPTIMAL SOLUTION

This section provides the solution procedure for the models proposed in Section 3. In order to consider the optimal solutions of Theorems 1–3 at the same time, we define the following terms. Let

$$\begin{aligned} M_1(L) &= D\left\{A+[\pi+\pi_0(1-\beta)]E(W-r)^+ + U(L)\right\}, \\ M_2(L) &= M_1(L)+\frac{(\Delta_4-\Delta_3)}{4}\left\{A+[\pi+\pi_0(1-\beta)]E(W-r)^+ + U(L)\right\}, \\ M_3(L) &= M_1(L)+\frac{(\Delta_4-\Delta_3)}{3}\left\{A+[\pi+\pi_0(1-\beta)]E(W-r)^+ + U(L)\right\}, \\ M_4(L) &= M_1(L)+\frac{(\Delta_4-\Delta_3)}{4}\left\{A+(\pi+\pi_0)E(W-r)^+ + U(L)\right\} \\ &\quad -\pi_0 E(W-r)^+\left[\frac{1}{4}D(\Delta_6-\Delta_5)+\frac{1}{4}\beta(\Delta_4-\Delta_3)+\frac{1}{6}(\Delta_3\Delta_5+\Delta_4\Delta_6)\right], \\ N_1(L) &= N_2(L)=N_3(L)=h\left[k\sigma\sqrt{L}+(1-\beta)E(W-r)^+\right], \\ N_4(L) &= N_1(L)-\frac{(\Delta_6-\Delta_5)}{4}hE(W-r)^+. \end{aligned}$$

Then, equation (21) in Theorem 1, equations (27) and (28) in Theorem 2, and equation (35) in Theorem 3 can be expressed as

$$K_1(Q,L) \equiv \text{EACP}(Q,L;\Delta_1,\Delta_2) = \frac{M_1(L)}{Q} + \frac{hQ}{2} + N_1(L), \tag{37}$$

$$K_2(Q,L) \equiv \text{EACPS}(Q,L;\Delta_j, j=1,2,3,4) = \frac{M_2(L)}{Q} + \frac{hQ}{2} + N_2(L), \tag{38}$$

$$K_3(Q,L) \equiv \text{EACPC}(Q,L;\Delta_j, j=1,2,3,4) = \frac{M_3(L)}{Q} + \frac{hQ}{2} + N_3(L), \tag{39}$$

and

$$K_4(Q,L) \equiv \text{EACP}^*(Q,L;\Delta_j, j=1,2,\ldots,6) = \frac{M_4(L)}{Q} + \frac{hQ}{2} + N_4(L), \tag{40}$$

respectively.

The problem is to determine the optimal order quantity and the optimal lead-time such that the total expected annual cost in the fuzzy sense has a minimum value. That is, for each $j \in \{1,2,3,4\}$, minimize

$$K_j(Q,L) = \frac{M_j(L)}{Q} + \frac{hQ}{2} + N_j(L), \qquad \text{for } Q > 0 \quad \text{and} \quad L \in [L_i, L_{i-1}].$$

To this end, for fixed $L \in [L_i, L_{i-1}]$, we take the first and second partial derivatives of $K_j(Q,L)$, $j = 1,2,3,4$, with respect to Q, which lead to

$$\frac{\partial}{\partial Q}K_j(Q,L) = -\frac{M_j(L)}{Q^2} + \frac{h}{2} \quad \text{and} \quad \frac{\partial^2}{\partial Q^2}K_j(Q,L) = \frac{2M_j(L)}{Q^3}, \qquad j = 1,2,3,4.$$

It is clear if $M_j(L) > 0$, then $\frac{\partial^2 K_j(Q,L)}{\partial Q^2} > 0$, and the minimum value of $K_j(Q,L)$ will occur at the point Q that satisfies $\frac{\partial K_j(Q,L)}{\partial Q} = 0$, $j = 1,2,3,4$. Solving this equation for Q, we obtain $Q = \sqrt{2M_j(L)/h} \equiv Q_j^*(L)$, $j = 1,2,3,4$. Hence, for fixed $L \in [L_i, L_{i-1}]$, the minimum total expected annual cost in the fuzzy sense is $K_j(Q_j^*(L), L)$, $j = 1,2,3,4$.

Next, let $S = \{L \mid L \in [L_i, L_{i-1}],\ i = 1,2,\ldots,n\}$. And from equation (11), let

$$U_i(L) \equiv U(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j), \qquad i = 1,2,\ldots,n,$$

and

$$U_0(L) \equiv U(L_0) = 0.$$

Thus, the mathematical expression of the problem is given by

$$\min_{Q>0, L\in S} K_j(Q,L) = \min_{L\in S}\min_{Q>0} K_j(Q,L) = \min_{L\in S} K_j(Q_j^*(L), L) \equiv K_j(Q_j^*(L^*), L^*). \tag{41}$$

In this case, for each $j \in \{1,2,3,4\}$, the optimal lead-time L^*, the optimal order quantity $Q_j^*(L^*)$, and the minimum total expected annual cost in the fuzzy sense $K_j(Q_j^*(L^*), L^*)$ can be determined easily by any numerical analysis methods.

Numerical Examples

To illustrate the results of proposed models, we consider an inventory system with the data used in [4]: $D = 600$ units/year, $A = \$200$ per order, $h = \$20$ per unit per year, $\pi = \$50$ per unit short, $\pi_0 = \$150$ per unit, and $\sigma = 7$ units/week. But instead of taking $q = 0.2$ (the allowable

Table 1. Lead-time data.

Lead-Time Component i	Normal Duration b_i (days)	Minimum Duration a_i (days)	Unit Crashing Cost c_i (\$/day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

Table 2. Lead-time crashing cost.

i	$U_i(L)$	
1	$0.4(56-L) = 22.4 - 0.4L,$	for $42 \le L \le 56$ (L in days)
2	$1.2(42-L) + 0.4 \times 14 = 56 - 1.2L,$	for $28 \le L \le 42$ (L in days)
3	$5(28-L) + 0.4 \times 14 + 1.2 \times 14 = 162.4 - 5L,$	for $21 \le L \le 28$ (L in days)

stock-out probability during lead-time), here we take a more reasonable value $q = 0.05$. Hence, when the lead-time demand follows normal distribution, we have the safety factor $k = 1.645$. Besides, the lead-time has three components with data shown in Table 1.

In this case, we have $L_0 = 56$ days, $L_1 = 56 - 14 = 42$ days, $L_2 = 42 - 14 = 28$ days, $L_3 = 28 - 7 = 21$ days, hence, $L_3 = \min L_i = 21$ days (= 3 weeks), $L_0 = \max L_i = 56$ days (= 8 weeks), $0 < \Delta_1 < \mu L_3 = 34.62$, $\Delta_2 > k\sigma\sqrt{L_0} = 32.57$, $0 < \Delta_3 < D = 600$, $0 < \Delta_4$. And the lead-time crashing cost is as shown in Table 2.

EXAMPLE 1. In this example, we determine the optimal solutions for Theorem 2 (i.e., $K_2(Q, L)$ and $K_3(Q, L)$ in equations (38) and (39), respectively). For $\beta = 0.5$ and various sets of $(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$, using the numerical analysis method, we obtain the computed results as shown in Table 3.

The above results show that when $\Delta_3 < \Delta_4$, $K_2(Q_2^*(L^*), L^*) < K_3(Q_3^*(L^*), L^*)$, i.e., the minimum total expected annual cost in the fuzzy sense obtained by SDM is less than that obtained by CM; for these cases, we prefer choosing SDM. Conversely, when $\Delta_4 < \Delta_3$, $K_3(Q_3^*(L^*), L^*) < K_2(Q_2^*(L^*), L^*)$; for these cases, we prefer choosing CM. Section 5.C will discuss this problem.

Table 3. The optimal solutions of Theorem 2 (L^* in weeks).

Given Parameters				Theorem 2						Choose
				Signed Distance Method (SDM)			Centroid Method (CM)			
Δ_1	Δ_2	Δ_3	Δ_4	L^*	$Q_2^*(L^*)$	$K_2(Q_2^*(L^*), L^*)$	L^*	$Q_3^*(L^*)$	$K_3(Q_3^*(L^*), L^*)$	
5	50	25	50	4	116.1	2782.9	4	116.3	2786.9	SDM
5	50	30	100	4	117.2	2804.4	4	117.7	2815.4	SDM
5	50	35	150	4	118.3	2825.6	4	119.1	2843.6	SDM
10	35	25	50	4	116.1	2782.9	4	116.3	2786.9	SDM
10	35	30	100	4	117.2	2804.4	4	117.7	2815.4	SDM
10	35	35	150	4	118.3	2825.6	4	119.1	2843.6	SDM
5	50	50	25	4	114.9	2758.9	4	114.7	2754.8	CM
5	50	100	30	4	113.8	2737.0	4	113.2	2725.6	CM
5	50	150	35	4	112.7	2714.9	4	111.8	2695.9	CM
10	35	50	25	4	114.9	2758.9	4	114.7	2754.8	CM
10	35	100	30	4	113.8	2737.0	4	113.2	2725.6	CM
10	35	150	35	4	112.7	2714.9	4	111.8	2695.9	CM

Furthermore, in order to compare the results with those obtained from the crisp model [4], we denote the chosen optimal solutions by L^{**}, Q^{**}, and MK^{**}. Also, for $\beta = 0.5$ and $q = 0.05$, using the procedure proposed in [4], we find that the optimal lead-time $L_s = 4$, the optimal order quantity $Q_s = 124.7$, and the minimum total expected annual cost $\mathrm{EAC}_s = 2956.5$. Thus, the result of Theorem 2 with that of crisp case can be compared as follows: $\mathrm{Rel}\,L = [(L^{**} - L_s)/L_s] \times 100\%$, $\mathrm{Rel}\,Q = [(Q^{**} - Q_s)/Q_s] \times 100\%$, and $\mathrm{Rel}\,TC = [(MK^{**} - \mathrm{EAC}_s)/\mathrm{EAC}_s] \times 100\%$. Using the values in Table 3 and these formulas, we obtain the results listed in Table 4.

It can be observed for each case of $(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$, the solution L^{**} is the same as L_s, while Q^{**} and MK^{**} are different from Q_s and EAC_s, respectively ($Q^{**} < Q_s$ and $MK^{**} < \mathrm{EAC}_s$). We note that the numerical results depend on the given values of problem parameters, which therefore, for other cases, it may get different results. In Sections 5.B and 5.D, we will discuss this problem.

Table 4. Compare the results of Theorem 2 with that obtained from the crisp model [4].

Given Parameters				Theorem 2			Related Errors (%)		
Δ_1	Δ_2	Δ_3	Δ_4	L^{**}	Q^{**}	MK^{**}	Rel L	Rel Q	Rel TC
5	50	25	50	4	116.1	2782.9	0	−6.90	−5.87
5	50	30	100	4	117.2	2804.4	0	−6.01	−5.14
5	50	35	150	4	118.3	2825.6	0	−5.13	−4.43
10	35	25	50	4	116.1	2782.9	0	−6.90	−5.87
10	35	30	100	4	117.2	2804.4	0	−6.01	−5.14
10	35	35	150	4	118.3	2825.6	0	−5.13	−4.43
5	50	50	25	4	114.7	2754.8	0	−8.02	−6.82
5	50	100	30	4	113.2	2725.6	0	−9.22	−7.81
5	50	150	35	4	111.8	2695.9	0	−10.34	−8.81
10	35	50	25	4	114.7	2754.8	0	−8.02	−6.82
10	35	100	30	4	113.2	2725.6	0	−9.22	−7.81
10	35	150	35	4	111.8	2695.9	0	−10.34	−8.81

Table 5. The optimal solutions of Theorems 1 and 3 (L^* in weeks).

Given Parameters		Theorem 1			Given Parameters				Theorem 3		
Δ_1	Δ_2	L^*	$Q_1^*(L^*)$	$K_1(Q_1^*(L^*), L^*)$	Δ_3	Δ_4	Δ_5	Δ_6	L^*	$Q_4^*(L^*)$	$K_4(Q_4^*(L^*), L^*)$
10	35	4	115.5	2770.9	25	50	0	0	4	116.1	2782.9
10	35	4	115.5	2770.9	30	100	0	0	4	117.2	2804.4
10	35	4	115.5	2770.9	35	150	0	0	4	118.3	2825.6
5	50	4	115.5	2770.9	25	25	0	0	4	115.5	2770.9
15	45	4	115.5	2770.9	50	50	0	0	4	115.5	2770.9
20	40	4	115.5	2770.9	100	100	0	0	4	115.5	2770.9
10	35	4	115.5	2770.9	25	50	0.1	0.3	4	116.1	2782.9
10	35	4	115.5	2770.9	25	50	0.2	0.2	4	116.1	2782.9
10	35	4	115.5	2770.9	25	50	0.3	0.1	4	116.1	2782.9
10	35	4	115.5	2770.9	50	100	0.1	0.3	4	116.7	2794.9
10	35	4	115.5	2770.9	50	100	0.2	0.2	4	115.5	2794.9
10	35	4	115.5	2770.9	50	100	0.3	0.1	4	115.5	2794.9

EXAMPLE 2. In this example, we determine the optimal solution for Theorem 1 (i.e., $K_1(Q,L)$ in equation (37)) and Theorem 3 (i.e., $K_4(Q,L)$ in equation (40)). For $\beta = 0.5$, we first calculate the results of Theorem 1 for given (Δ_1, Δ_2), and then Theorem 3 by further given $(\Delta_3, \Delta_4, \Delta_5, \Delta_6)$. Using the numerical analysis method, we obtain the computed results as shown in Table 5.

From these results, the following can be observed.

(i) The solutions of Theorem 1 are insensitive to the parameters (Δ_1, Δ_2). The reason is that in this example for the various given sets of (Δ_1, Δ_2), the values of $E(W-r)^+$ (in equation (20)) are all close to zero, hence, the solutions obtained from equation (21) in Theorem 1 are identical.

(ii) When $\Delta_3 = \Delta_4$ and $\Delta_5 = \Delta_6 = 0$, the results of Theorem 3 are the same as Theorem 1.

Furthermore, by comparing the results with that shown in Table 3, it can be found when $\Delta_5 = \Delta_6 = 0$, the results of Theorem 3 are identical to those obtained in Theorem 2 using SDM.

5. DISCUSSIONS

5.A. The Relationships Between Theorems 1, 2, and 3

(A.1) In Theorem 3 (i.e., equation (40)), letting $\Delta_5 = \Delta_6 = 0$, we obtain

$$K_4(Q,L) = \frac{M_4(L)}{Q} + \frac{hQ}{2} + N_4(L) = \frac{M_2(L)}{Q} + \frac{hQ}{2} + N_2(L) = K_2(Q,L),$$

which means Theorem 3 is reduced to Theorem 2(i) (i.e., equation (38)). Hence, it can be concluded that Theorem 2(i) is a special case of Theorem 3.

(A.2) In Theorem 2(i) (i.e., equation (38)) and (ii) (i.e., equation (39)), letting $\Delta_3 = \Delta_4$, we obtain $K_2(Q,L) = K_3(Q,L) = K_1(Q,L)$, which means Theorem 2(i) and (ii) are reduced to Theorem 1. That is, Theorem 1 is a special case of Theorem 2.

(A.3) In Theorem 3 (i.e., equation (40)), letting $\Delta_3 = \Delta_4$ with $\Delta_5 = \Delta_6 = 0$, or letting $\Delta_3 = \Delta_4 = 0$ with $\Delta_5 = \Delta_6$, we get $K_4(Q,L) = K_1(Q,L)$, which means Theorem 3 is reduced to Theorem 1. That is, Theorem 1 is a special case of Theorem 3.

5.B. Comparing $E(W-r)^+$ with $E(X-r)^+$

Equations (12) and (21) can be rewritten as

$$\begin{aligned} \mathrm{EAC}(Q,L) &= \frac{D}{Q}[A + U(L)] + h\left(\frac{Q}{2} + k\sigma\sqrt{L}\right) \\ &+ \left\{\frac{D}{Q}[\pi + \pi_0(1-\beta)] + h(1-\beta)\right\} E(X-r)^+ \end{aligned} \tag{42}$$

and

$$\begin{aligned} \mathrm{EACP}(Q,L;\Delta_1,\Delta_2) &= \frac{D}{Q}[A + U(L)] + h\left(\frac{Q}{2} + k\sigma\sqrt{L}\right) \\ &+ \left\{\frac{D}{Q}[\pi + \pi_0(1-\beta)] + h(1-\beta)\right\} E(W-r)^+, \end{aligned} \tag{43}$$

respectively. Thus, the difference of them is

$$\begin{aligned} \mathrm{DCP} &\equiv \mathrm{EAC}(Q,L) - \mathrm{EACP}(Q,L;\Delta_1,\Delta_2) \\ &= \left\{\frac{D}{Q}[\pi + \pi_0(1-\beta)] + h(1-\beta)\right\} [E(X-r)^+ - E(W-r)^+]. \end{aligned}$$

Because $(D/Q)[\pi+\pi_0(1-\beta)]+h(1-\beta) > 0$, and from equation (17) that $w = x-\mu_L+(\Delta_1-\Delta_2)/4$, we can obtain the following results.

CASE 1. If $\mu_L > (\Delta_1 - \Delta_2)/4$, then $w < x$, or equivalently $w - r < x - r$, and hence, $E(W-r)^+ < E(X-r)^+$, which implies DCP > 0, i.e., $\mathrm{EACP}(Q, L; \Delta_1, \Delta_2) < \mathrm{EAC}(Q, L)$, $\forall Q > 0$, $L \in S$. Therefore, $\min_{Q>0, L\in S} \mathrm{EACP}(Q, L; \Delta_1, \Delta_2) < \min_{Q>0, L\in S} \mathrm{EAC}(Q, L)$.

CASE 2. If $\mu_L < (\Delta_1 - \Delta_2)/4$, following the same approach as in Case 1, we obtain $E(W - r)^+ > E(X-r)^+$, and hence, $\mathrm{EACP}(Q, L; \Delta_1, \Delta_2) > \mathrm{EAC}(Q, L)$, $\forall Q > 0$, $L \in S$, therefore, $\min_{Q>0, L\in S} \mathrm{EACP}(Q, L; \Delta_1, \Delta_2) > \min_{Q>0, L\in S} \mathrm{EAC}(Q, L)$.

CASE 3. If $\mu_L = (\Delta_1 - \Delta_2)/4$, then $E(W-r)^+ = E(X-r)^+$, and hence, $\mathrm{EACP}(Q, L; \Delta_1, \Delta_2) = \mathrm{EAC}(Q, L)$, $\forall Q > 0$, $L \in S$, therefore,

$$\min_{Q>0, L\in S} \mathrm{EACP}(Q, L; \Delta_1, \Delta_2) = \min_{Q>0, L\in S} \mathrm{EAC}(Q, L).$$

That is, in this case, the fuzzy mixture inventory model with probabilistic fuzzy set (proposed in Theorem 1) is equivalent to Ouyang *et al.*'s model [4].

5.C. Signed Distance and Centroid Methods

In Theorem 2, we employed the methods of signed distance and centroid to defuzzify the fuzzy total cost. Now, we further discuss which method is better. From equations (27) and (28), we get

$$\begin{aligned}\mathrm{DCS} &\equiv \mathrm{EACPC}(Q, L; \Delta_j, j = 1,2,3,4) - \mathrm{EACPS}(Q, L; \Delta_j, j = 1,2,3,4)\\ &= \frac{(\Delta_4 - \Delta_3)}{12Q}\left\{A + [\pi + \pi_0(1-\beta)]E(W-r)^+ + U(L)\right\}.\end{aligned}$$

Because $A + [\pi + \pi_0(1-\beta)]E(W-r)^+ + U(L) > 0$ and $Q > 0$, we have the following results.

CASE 1. If $\Delta_3 < \Delta_4$, then DCS > 0, i.e., $\mathrm{EACPS}(Q, L; \Delta_j, j = 1,2,3,4) < \mathrm{EACPC}(Q, L; \Delta_j, j = 1,2,3,4)$. This means the total expected annual cost in the fuzzy sense obtained using the signed distance is less than that of using centroid. In this situation, it is better for the decision-maker to adopt the total cost $\mathrm{EACPS}(Q, L; \Delta_j, j = 1,2,3,4)$ obtained by signed distance to determine the optimal solution.

CASE 2. If $\Delta_3 > \Delta_4$, then DCS < 0, i.e., $\mathrm{EACPC}(Q, L; \Delta_j, j = 1,2,3,4) < \mathrm{EACPS}(Q, L; \Delta_j, j = 1,2,3,4)$. In this situation, it is better to adopt the total cost $\mathrm{EACPC}(Q, L; \Delta_j, j = 1,2,3,4)$ obtained by centroid to determine the optimal solution.

CASE 3. If $\Delta_3 = \Delta_4$, then DCS $= 0$, i.e., $\mathrm{EACPS}(Q, L; \Delta_j, j = 1,2,3,4) = \mathrm{EACPC}(Q, L; \Delta_j, j = 1,2,3,4)$. In this situation, either the total cost obtained by signed distance or centroid could be adopted to determine the optimal solution.

5.D. The Relationship Between Theorem 2 and Crisp Case

In Section 5.B, Case 3, we know that when $\mu_L = (\Delta_1 - \Delta_2)/4$, Theorem 1 is equivalent to the crisp case, and from Section 5.A, (A.2), we know that when $\Delta_3 = \Delta_4$, Theorem 2 reduces to Theorem 1. Thus, it can be concluded that when $\mu_L = (\Delta_1 - \Delta_2)/4$ and $\Delta_3 = \Delta_4$, Theorem 2 reduces to the crisp case.

6. CONCLUSIONS

In this article, we apply the fuzzy sets theory to reformulate the mixture inventory model involving variable lead-time with partial backorders. Three fuzzy models are proposed. First, we use the probabilistic fuzzy set to construct a new random variable for lead-time demand and obtain the total expected annual cost in the fuzzy sense. Second, we use the triangular fuzzy number to represent the imprecise annual demand and obtain the model with fuzzy total cost. For this fuzzy model, we employ two methods of defuzzification, the signed distance and centroid,

to derive the value of total cost in the fuzzy sense. Third, we further fuzzify the backorder rate of the demand during the stock-out period as the triangular fuzzy number, and derive the value of total cost in the fuzzy sense by signed distance. The main results of the above cases are summarized in Theorems 1–3. Furthermore, through the mathematical analysis, we show the relationship between these theorems, and illustrate in what situation adopting the total cost obtained by signed distance or centroid is better.

Finally, we would like to point out that the advantage of fuzzy models themselves are they keep the uncertainties, which can capture the real situations better than the crisp model does. In this paper, since we do not attempt to establish the superiority of proposing new models to reduce more inventory cost than before, but providing an alternative approach of modeling uncertainties for the decision-maker, therefore, the results of numerical examples did not show a significant reduction in inventory cost comparing with that obtained from the crisp model [4].

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