Asia-Pacific Journal of Operational Research
Vol. 21, No. 4 (2004) 435–446

INVENTORY MODELS WITH STOCK-DEPENDENT DEMAND AND NONLINEAR HOLDING COSTS FOR DETERIORATING ITEMS

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Received February 2003
Revised July 2003

In this paper, we discuss why it is appropriate maximize the profits, instead of minimizing the costs, in an inventory system with an inventory-level-dependent demand rate. In addition, we restate Urban's viewpoint that the restriction of zero ending-inventory is not necessary in an inventory-level-dependent demand model. Consequently, we amend Giri and Chaudhuri's inventory model for deteriorating items by changing the objective to maximize the profits and relaxing the restriction of zero ending-inventory. Finally, we provide a couple of examples to show that both the order quantity and the profit obtained from our proposed model are significantly larger than those in Giri and Chaudhuri's model, in which the objective is to minimize the costs.

Keywords: Inventory; lot sizing; stock-dependent demand; deterioration.

1. Introduction

In recent years, marketing researchers and practitioners have recognized the phenomenon that the demand for some items can depend on the inventory level on display. Levin *et al.* (1972) observed that large piles of consumer goods displayed in a supermarket will lead the customer to buy more. Lately, Silver and Peterson (1985) noted that sales at the retail level tend to be proportional to stock displayed. Consequently, Baker and Urban (1988) first established an EOQ model for a power-form inventory-level-dependent demand pattern (i.e., the demand rate at time t is $D(t) = \alpha[I(t)]^{\beta}$, where $I(t)$ is the inventory level at time t, $\alpha > 0$, and $0 < \beta < 1$). Then, Mandal and Phaujdar (1989) developed a production inventory model for deteriorating items with uniform rate of production and linearly stock-dependent demand (i.e., $D(t) = \alpha + \beta I(t)$, where both α and $\beta > 0$). Later on, Datta and Pal (1990) presented an inventory model in which the demand rate is dependent on the instantaneous inventory level until a given inventory level S is achieved, after which the demand rate becomes constant (i.e., $D(t) = \alpha[I(t)]^{\beta}$, if $I(t) > S$ and $D(t) = \alpha S^{\beta}$, if $0 \le I(t) \le S$). Urban (1992) then relaxed the unnecessary zero ending-inventory at the end of each order cycle as imposed in Datta and Pal (1990). Pal *et al.* (1993) again extended the model of Baker and Urban (1988) for perishable products that deteriorate at a constant rate. Similarly, Giri *et al.* (1996) generalized Urban's model for constant deteriorating items. Recently, Giri and Chaudhuri (1998) extended the EOQ model to allow for a nonlinear holding cost. Other papers related to this area are Bar-Lev *et al.* (1994), Gerchak and Wang (1994), Hwang and Hahn (2000), Mandal and Maiti (1999, 2000), Ray and Chaudhuri (1997), and others.

First, in this paper, we explain why it is appropriate to maximize the profits instead of minimizing the costs in an inventory system with an inventory-level-dependent demand rate. As stated by Urban (1992), "It may be desirable to order larger quantities, resulting in stock remaining at the end of the cycle, due to the potential profits resulting from the increased demand." Therefore, it is quite clear that the objective for a large inventory in an inventory system with an inventory-level-dependent demand rate is to increase the profits. Otherwise, in the inventory system that possesses an inventory-level-dependent demand rate, a higher level of inventory causes not only higher inventory and deterioration costs but also higher purchasing costs due to demand increase. Consequently, if the goal is to minimize the costs, then an inventory-level-dependent demand rate causes a lower level of inventory than the traditional EOQ model with a constant demand rate. It defeats the purpose of an inventory-level-dependent demand rate that is desirable to maintain large inventory for potential profits obtained from the increased demand. Therefore, the objective here must be to maximize the profits, not to minimize the costs. Secondly, we then re-establish the objective in Giri and Chaudhuri's model (1998) by changing it to maximize the profits in this paper. Again, in an inventory system that possesses an inventory-level-dependent demand rate, it may be desirable to order larger quantities and have positive ending-inventory. Therefore, the zero ending-inventory may not be the optimal inventory policy. So, we also relax the terminal condition of zero ending-inventory imposed in Giri and Chaudhuri's model (1998) to develop an appropriate model that allows for a positive ending-inventory. Finally, numerical examples are presented to illustrate our results. Our computational results show that both the order quantity and the average net profit obtained by our model are significantly larger than those obtained by Giri and Chaudhuri's inventory model.

2. Assumptions and Notation

A single-item deterministic inventory model for deteriorating items with stock-dependent demand rate and the nonlinear holding cost is presented under the following assumptions and notations:

1. Lead time is zero.
2. Replenishments are instantaneous.
3. The fixed purchasing cost K per order is known and constant.

4. Both the selling price p and the variable purchase cost c per unit are known and constant. In other words, neither price discounts nor the effects of inflation are taken into consideration.
5. The constant deterioration rate θ $(0 < \theta \ll 1)$ is only applied to on-hand inventory.
6. All replenishment cycles are identical. Consequently, only a typical planning cycle with T length is considered (i.e., the planning horizon is $[0, T]$).
7. The demand rate $R(I(t))$ is deterministic and given by the following expression:

$$R(I(t)) = \alpha I(t)^{\beta}, \qquad \alpha > 0, \quad \text{and} \quad 0 < \beta < 1,$$

   where $I(t)$ is the inventory level at time t, and both α and β are constant.
8. The initial and ending inventory are assumed to be S, which is greater than or equal to zero. The order quantity Q enters into inventory at time $t = 0$. Consequently, $I(0) = Q + S$. During the time interval $[0, T]$, the inventory is depleted by the combination of demand and deterioration. At time T, the inventory level falls to S, i.e., $I(T) = S$. The graphical representation of the inventory system is depicted in Figure 1.

The mathematical problem here is to determine the optimal values of Q and S such that the average net profit in a replenishment cycle is maximized. From Assumption 7, we know that the demand is zero if the inventory level reaches zero. Consequently, the inventory level will never fall below zero. In other words, shortages will not occur under Assumption 7.

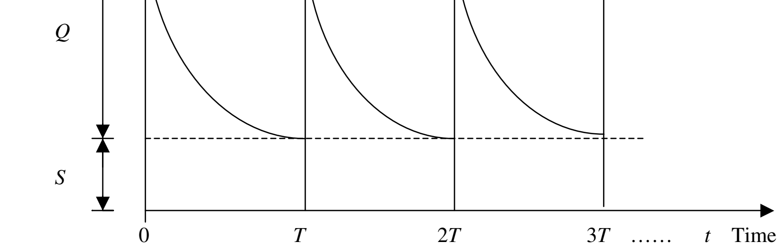

Fig. 1. The inventory system.

3. Mathematical Model

The differential equation describing the instantaneous states of $I(t)$ in the interval $[0, T]$ is given by

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} + \theta I(t) = -\alpha(I(t))^{\beta}, \quad 0 \le t \le T, \tag{3.1}$$

with the initial and terminal conditions being $I(0) = Q + S$ and $I(T) = S$, respectively. Solving (3.1), we obtain

$$(1-\beta)\theta t = \ln\left(1 + \frac{\theta}{\alpha}(Q+S)^{1-\beta}\right) - \ln\left(1 + \frac{\theta}{\alpha}I(t)^{1-\beta}\right).$$

On expanding the right-hand side, the first-order approximation of θ gives

$$t \approx \frac{(Q+S)^{1-\beta} - I^{1-\beta}}{\alpha(1-\beta)}\left[1 - \frac{\theta}{2\alpha}\left((Q+S)^{1-\beta} + I^{1-\beta}\right)\right]. \tag{3.2}$$

Taking the first derivative of t with respect to I, we have

$$\mathrm{d}t \approx \frac{1}{\alpha(1-\beta)}\left[-(1-\beta)I^{-\beta} + \frac{\theta(1-\beta)}{\alpha}I^{1-2\beta}\right]\mathrm{d}I. \tag{3.3}$$

Since $I(T) = S$, we obtain from (3.2) that the cycle time T is given by

$$T \approx \frac{(Q+S)^{1-\beta} - S^{1-\beta}}{\alpha(1-\beta)}\left[1 - \frac{\theta}{2\alpha}\left((Q+S)^{1-\beta} + S^{1-\beta}\right)\right]. \tag{3.4}$$

The total purchase cost PC in $[0, T]$ is given by

$$\mathrm{PC} = K + cQ. \tag{3.5}$$

Note that we know from Assumption 8 that the initial inventory is S, and the order quantities are Q units at the beginning of the period. Therefore, the starting inventory level of the period is $S + Q$. At time T, the inventory level is back to S. Thus, the order quantities Q are depleted to zero at time T by the combined effect of demand and deterioration. Consequently, the total variable purchasing cost cQ is the sum of the deterioration cost for deteriorated items and the procurement cost for items in demand.

The total revenue TR over the period $[0, T]$ is given by

$$\mathrm{TR} = p \int_0^T \alpha I(t)^\beta \, \mathrm{d}t.$$

Substituting (3.3) into TR, we get

$$\mathrm{TR} \approx p \int_0^T \alpha I^\beta \frac{1}{\alpha(1-\beta)} \left[-(1-\beta) I^{-\beta} + \frac{\theta(1-\beta)}{\alpha} I^{1-2\beta} \right] \mathrm{d}I,$$

with the conditions $I(0) = Q + S$, and $I(T) = S$. Consequently, after integrating, we obtain

$$\mathrm{TR} \approx p \left[Q + \frac{\theta}{\alpha(2-\beta)} (S^{2-\beta} - (Q+S)^{2-\beta}) \right]. \tag{3.6}$$

Next, we will discuss two types of inventory holding costs just as in Giri and Chaudhuri (1998).

3.1. *Nonlinear time-dependent holding cost*

For this model, we assume that the cost of holding an inventory $\mathrm{d}I$ up to time t is $ht^n \, \mathrm{d}I$, where n and $h > 0$. Therefore, the total inventory holding cost HC in $[0, T]$ is given by

$$\mathrm{HC} = \int_S^{Q+S} ht^n \, \mathrm{d}I. \tag{3.7}$$

Substituting (3.2) into (3.7), we have

$$\mathrm{HC} \approx \frac{h}{\alpha^n (1-\beta)^n} \int_S^{Q+S} ((Q+S)^{1-\beta} - I^{1-\beta})^n \left[1 - \frac{\theta}{2\alpha} ((Q+S)^{1-\beta} + I^{1-\beta}) \right]^n \mathrm{d}I.$$

Since

$$\left[1 - \frac{\theta}{2\alpha} ((Q+S)^{1-\beta} + I^{1-\beta}) \right]^n \approx 1 - \frac{n\theta}{2\alpha} [(Q+S)^{1-\beta} + I^{1-\beta}],$$

we obtain

$$\begin{aligned} \mathrm{HC} \approx \Bigg\{ & \left(1 - \frac{n\theta}{2\alpha} (Q+S)^{1-\beta} \right) \int_S^{Q+S} [(Q+S)^{1-\beta} - I^{1-\beta}]^n \, \mathrm{d}I \\ & - \frac{n\theta}{2\alpha} \int_S^{Q+S} I^{1-\beta} [(Q+S)^{1-\beta} - I^{1-\beta}]^n \, \mathrm{d}I \Bigg\} \times \frac{h}{\alpha^n (1-\beta)^n}. \end{aligned}$$

For convenience, let $(Q+S)^{1-\beta}-I^{1-\beta}=(Q+S)^{1-\beta}Z$. Hence, HC can be rewritten as follows:

$$\begin{aligned}\mathrm{HC} \approx{}& \frac{h}{\alpha^n(1-\beta)^n}\left\{\left(1-\frac{n\theta}{2\alpha}(Q+S)^{1-\beta}\right)\left(\frac{(Q+S)^{(1-\beta)n+1}}{1-\beta}\right)\right.\\ &\left.\times\int_0^{1-(S/(Q+S))^{1-\beta}} Z^n(1-Z)^{\beta/(1-\beta)}\,\mathrm{d}Z\right\}-\frac{h}{\alpha^n(1-\beta)^n}\\ &\times\left\{\left(\frac{n\theta}{2\alpha}\right)\left(\frac{(Q+S)^{(1-\beta)n+(1-\beta)+1}}{1-\beta}\right)\int_0^{1-(S/(Q+S))^{1-\beta}} Z^n(1-Z)^{1/(1-\beta)}\,\mathrm{d}Z\right\}.\end{aligned} \tag{3.8}$$

Since the total profit TP over the period $[0,T]$ is given by TP = TR − HC − PC, the average net profit per unit of time is

$$\mathrm{AP}=\frac{\mathrm{TP}}{T}=\frac{\mathrm{TR}-\mathrm{HC}-\mathrm{PC}}{T}. \tag{3.9}$$

Our problem here is to determine the order quantity Q^* and the reorder point S^* in order to maximize AP as shown in (3.9). For fixed n, the necessary conditions for AP to be maximum are

$$\frac{\partial \mathrm{AP}}{\partial Q}=0 \quad\text{and}\quad \frac{\partial \mathrm{AP}}{\partial S}=0,$$

which in turn imply (see the Appendix for details)

$$\begin{aligned}&\left\{p-\frac{p\theta}{\alpha}(Q+S)^{1-\beta}-\frac{\partial \mathrm{HC}}{\partial Q}-c\right\}\left[\frac{\theta}{\alpha^2}S^{1-2\beta}-\frac{S^{-\beta}}{\alpha}\right]\\ &\quad=\left[-\frac{\theta}{\alpha^2}(Q+S)^{1-2\beta}+\frac{(Q+S)^{-\beta}}{\alpha}\right]\times\left\{-p+\frac{p\theta}{\alpha}S^{1-\beta}+\frac{h}{\alpha^n(1-\beta)^n}\right.\\ &\qquad\left.\times\left(1-\left(\frac{S}{Q+S}\right)^{1-\beta}\right)^n(Q+S)^{(1-\beta)n}\left[1-\frac{n\theta}{2\alpha}((Q+S)^{1-\beta}+S^{1-\beta})\right]+c\right\}\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}&\left\{p-\frac{p\theta}{\alpha}(Q+S)^{1-\beta}-\frac{\partial \mathrm{HC}}{\partial Q}-c\right\}\times\frac{(Q+S)^{1-\beta}-S^{1-\beta}}{\alpha(1-\beta)}\\ &\quad\times\left[1-\frac{\theta}{2\alpha}\left((Q+S)^{1-\beta}+S^{1-\beta}\right)\right]\\ &=(\mathrm{TR}-\mathrm{HC}-\mathrm{PC})\times\left[-\frac{\theta}{\alpha^2}(Q+S)^{1-2\beta}+\frac{(Q+S)^{-\beta}}{\alpha}\right],\end{aligned} \tag{3.11}$$

respectively. From (3.8), we have

$$\begin{aligned}\frac{\partial \mathrm{HC}}{\partial Q} = {} & \frac{h}{\alpha^n(1-\beta)^n}\left[-\frac{n\theta((1-\beta)(1+n)+1)}{2\alpha(1-\beta)}(Q+S)^{(1-\beta)(1+n)}\right. \\ & \left. + \frac{(1-\beta)n+1}{1-\beta}(Q+S)^{(1-\beta)n}\right] \times \int_0^{1-(S/(Q+S))^{1-\beta}} Z^n(1-Z)^{\beta/(1-\beta)}\,\mathrm{d}Z \\ & + \frac{h}{\alpha^n(1-\beta)^n}\left(1-\left(\frac{S}{(Q+S)}\right)^{1-\beta}\right)^n S(Q+S)^{(1-\beta)n-1} \\ & \times \left(1-\frac{n\theta}{2\alpha}((Q+S)^{1-\beta}+S^{1-\beta})\right) + \frac{h}{\alpha^n(1-\beta)^n}\left[-\frac{n\theta((1-\beta)(1+n)+1)}{2\alpha(1-\beta)}\right. \\ & \left. \times (Q+S)^{(1-\beta)(1+n)}\right] \times \int_0^{1-(S/(Q+S))^{1-\beta}} Z^n(1-Z)^{1/(1-\beta)}\,\mathrm{d}Z.\end{aligned}$$

Note that the optimal values of Q and S cannot be on the boundary since AP decreases when any one of Q and S is shifted to the end points 0. In other words, the optimal values of Q and S must be the solution to (3.10) and (3.11). Solving (3.10) and (3.11) simultaneously, if we obtain a unique solution for Q and S, then they are the optimal solution, say Q^* and S^*. Substituting the values of Q^* and S^* into (3.4), we will obtain the optimum value of T, say T^*. We then substitute the values of Q^*, S^*, and T^* into (3.9), and get the maximum value of the average net profit per unit of time AP, say AP*. Due to the complexity of the problem, we are unable to prove that the solution to (3.10) and (3.11) is unique. Therefore, there may be more than one solution to (3.10) and (3.11). If the solution to (3.10) and (3.11) is indeed more than one, then the reader can simply calculate the corresponding AP for each solution, and choose the optimal solution that will provide the maximum AP.

3.2. *Nonlinear stock-dependent holding cost*

In this case, we assume that the rate of change of holding cost is proportional to a power function of the on-hand inventory given by

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{HC}) = hI^n, \quad I = I(t) \tag{3.12}$$

where both n and $h > 0$, n being an integer. To find the holding cost per cycle, we integrate (3.12) over time t between two limits $t = 0$ and $t = T$. Hence

$$\mathrm{HC} = \int_0^T hI^n\,\mathrm{d}t. \tag{3.13}$$

Substituting (3.3) into (3.13), we get

$$\begin{aligned}\mathrm{HC} &\approx \int_0^T hI^n \frac{1}{\alpha(1-\beta)}\left[-(1-\beta)I^{-\beta} + \frac{\theta(1-\beta)}{\alpha}I^{1-2\beta}\right]\mathrm{d}I \\ &= \frac{h}{\alpha}\left[\frac{1}{n-\beta+1}\left[(Q+S)^{n-\beta+1} - S^{n-\beta+1}\right] + \frac{\theta}{\alpha(n-2\beta+2)}\right. \\ &\quad \left.\times\left[S^{n-2\beta+2} - (Q+S)^{n-2\beta+2}\right]\right]. \end{aligned} \tag{3.14}$$

Similarly, we can obtain (3.15) and (3.16) in the same way as (3.10) and (3.11) in Section 3.1. Again, solving (3.15) and (3.16) simultaneously, we get the optimal values Q^* and S^*. We then use (3.4) and (3.9) to obtain the corresponding T^* and AP^*, respectively.

$$\begin{aligned}&\left\{p - \frac{p\theta}{\alpha}(Q+S)^{1-\beta} - \frac{h}{\alpha}\left[(Q+S)^{n-\beta} - \frac{\theta}{\alpha}(Q+S)^{n-2\beta+1}\right] - c\right\}\left[\frac{\theta}{\alpha^2}S^{1-2\beta} - \frac{S^{-\beta}}{\alpha}\right] \\ &= \left\{-p + \frac{p\theta}{\alpha}S^{1-\beta} + \frac{h}{\alpha}\left[S^{n-\beta} - \frac{\theta}{\alpha}S^{n-2\beta+1}\right] + c\right\} \\ &\quad \times\left[-\frac{\theta}{\alpha^2}(Q+S)^{1-2\beta} + \frac{(Q+S)^{-\beta}}{\alpha}\right], \end{aligned} \tag{3.15}$$

and

$$\begin{aligned}&\left\{p - \frac{p\theta}{\alpha}(Q+S)^{1-\beta} - \frac{h}{\alpha}\left[(Q+S)^{n-\beta} - \frac{\theta}{\alpha}(Q+S)^{n-2\beta+1}\right] - c\right\} \\ &\quad \times \frac{(Q+S)^{1-\beta} - S^{1-\beta}}{\alpha(1-\beta)} \times \left[1 - \frac{\theta}{2\alpha}\left((Q+S)^{1-\beta} + S^{1-\beta}\right)\right] \\ &= (\mathrm{TR} - \mathrm{HC} - \mathrm{PC})\left[-\frac{\theta}{\alpha^2}(Q+S)^{1-2\beta} + \frac{(Q+S)^{-\beta}}{\alpha}\right]. \end{aligned} \tag{3.16}$$

4. Numerical Examples

Here, we give numerical examples to illustrate the differences between our proposed model and the model by Giri and Chaudhuri (1998).

Example 1. Let $\alpha = 5$, $c = \$5$ per unit, $h = \$0.02$ per unit, $K = \$10$ per order, $\theta = 0.01$, $p = \$50$ per unit, $n = 2$, and $\beta = 0.1$. Since both (3.10) and (3.11) are nonlinear and extremely difficult to solve, we solve them by using *Maple V* software. The computational result shows the following optimal values: $Q^* = 62.3738$, $S^* = 305.2291$, $T^* = 4.3519$ and $\mathrm{AP}^* = 372.1537$. By using Giri and Chaudhuri's inventory model, we get the optimal values as follows: $Q^* = 27.7320$, $T^* = 4.3326$ and $\mathrm{AP}^* = 278.3049$. Comparing the two computational results, our model provides an average net profit that is 33.72% higher than theirs.

Example 2. The parameter values are taken as follows: $\alpha = 0.5$, $c = \$10$ per unit, $h = \$0.3$ per unit, $K = \$10$ per order, $\theta = 0.03$, $p = \$50$ per unit, $n = 2$, and $\beta = 0.5$.

Solving (3.15) and (3.16) by using *Maple V* software, we obtain the optimal values as $Q^* = 5.6024$ and $S^* = 3.5796$. We then get $T^* = 4.1046$ and $\mathrm{AP}^* = 33.2212$ from (3.4) and (3.9), respectively. Similarly, the optimal values for Giri and Chaudhuri's inventory model are as follows: $Q^* = 2.3567$, $T^* = 5.9521$, and $\mathrm{AP}^* = 13.0211$. Comparing the two optimal average net profits, we obtain the average net profit, which is 155.13% higher than theirs.

5. Conclusion

In this paper, we explain why it is appropriate to maximize the profits in an inventory system with stock-dependent demand. We then correctly develop the inventory models with stock-dependent demand and nonlinear holding costs for deteriorating items. We not only reestablish Giri and Chaudhuri's inventory model by changing the objective to maximize the profits instead of minimizing the costs, but also relax the unnecessary terminal condition of zero ending-inventory in order to get the ideal inventory policy. Finally, we use two examples to show that our proposed model generates both an order quantity and an average net profit significantly higher than those in Giri and Chaudhuri's inventory model.

The proposed model can be extended in several ways. For instance, we may extend the constant selling price and purchasing cost to vary with time. Also, we could consider a linearly stock-dependent demand pattern (e.g., Mandal and Phaujdar (1989)), instead of a power-form demand function here. Finally, we could study the effects of inflation as well as discount rates.

Acknowledgments

I would like to thank the editor and the referees for their constructive suggestions. I am grateful to Dr. Jinn-Tsair Teng for inspiring me to do this research.

Appendix

Using $\frac{\partial \mathrm{AP}}{\partial Q} = 0$ and $\frac{\partial \mathrm{AP}}{\partial S} = 0$, we get

$$\left(\frac{\partial \mathrm{TR}}{\partial Q} - \frac{\partial \mathrm{HC}}{\partial Q} - \frac{\partial \mathrm{PC}}{\partial Q}\right) T = (\mathrm{TR} - \mathrm{HC} - \mathrm{PC}) \frac{\partial T}{\partial Q} \tag{A.1}$$

and

$$\left(\frac{\partial \mathrm{TR}}{\partial S} - \frac{\partial \mathrm{HC}}{\partial S} - \frac{\partial \mathrm{PC}}{\partial S}\right) T = (\mathrm{TR} - \mathrm{HC} - \mathrm{PC}) \frac{\partial T}{\partial S}, \tag{A.2}$$

respectively. Consequently, dividing (A.1) by (A.2), we obtain

$$\left(\frac{\partial \mathrm{TR}}{\partial Q} - \frac{\partial \mathrm{HC}}{\partial Q} - \frac{\partial \mathrm{PC}}{\partial Q}\right) \Big/ \left(\frac{\partial \mathrm{TR}}{\partial S} - \frac{\partial \mathrm{HC}}{\partial S} - \frac{\partial \mathrm{PC}}{\partial S}\right) = \frac{\partial T}{\partial Q} \Big/ \frac{\partial T}{\partial S}, \tag{A.3}$$

which implies

$$\left(\frac{\partial \mathrm{TR}}{\partial Q} - \frac{\partial \mathrm{HC}}{\partial Q} - \frac{\partial \mathrm{PC}}{\partial Q}\right)\frac{\partial T}{\partial S} = \left(\frac{\partial \mathrm{TR}}{\partial S} - \frac{\partial \mathrm{HC}}{\partial S} - \frac{\partial \mathrm{PC}}{\partial S}\right)\frac{\partial T}{\partial Q}. \tag{A.4}$$

From Eqs. (3.4)–(3.6) and (3.8), we obtain

$$\frac{\partial T}{\partial Q} = \left[-\frac{\theta}{\alpha^2}(Q+S)^{1-2\beta} + \frac{(Q+S)^{-\beta}}{\alpha}\right], \tag{A.5}$$

$$\frac{\partial T}{\partial S} = \left[-\frac{\theta}{\alpha^2}(Q+S)^{1-2\beta} + \frac{(Q+S)^{-\beta}}{\alpha}\right] + \left[\frac{\theta}{\alpha^2}S^{1-2\beta} - \frac{S^{-\beta}}{\alpha}\right], \tag{A.6}$$

$$\frac{\partial \mathrm{PC}}{\partial Q} = c, \quad \frac{\partial \mathrm{PC}}{\partial S} = 0, \tag{A.7}$$

$$\frac{\partial \mathrm{TR}}{\partial Q} = p - \frac{p\theta}{\alpha}(Q+S)^{1-\beta}, \quad \frac{\partial \mathrm{TR}}{\partial S} = \frac{p\theta}{\alpha}\left(S^{1-\beta} - (Q+S)^{1-\beta}\right), \tag{A.8}$$

$$\begin{aligned}\frac{\partial \mathrm{HC}}{\partial Q} &= \frac{h}{\alpha^n(1-\beta)^n}\left[-\frac{n\theta((1-\beta)(1+n)+1)}{2\alpha(1-\beta)}(Q+S)^{(1-\beta)(1+n)}\right.\\ &\quad \left. + \frac{(1-\beta)n+1}{1-\beta}(Q+S)^{(1-\beta)n}\right] \times \int_0^{1-(S/(Q+S))^{1-\beta}} Z^n(1-Z)^{\beta/(1-\beta)}\,\mathrm{d}Z\\ &\quad + \frac{h}{\alpha^n(1-\beta)^n}\left(1-\left(\frac{S}{Q+S}\right)^{1-\beta}\right)^n S(Q+S)^{(1-\beta)n-1}\\ &\quad \times\left(1-\frac{n\theta}{2\alpha}((Q+S)^{1-\beta}+S^{1-\beta})\right)\\ &\quad + \frac{h}{\alpha^n(1-\beta)^n}\left[-\frac{n\theta((1-\beta)(1+n)+1)}{2\alpha(1-\beta)}(Q+S)^{(1-\beta)(1+n)}\right]\\ &\quad \times \int_0^{1-(S/(Q+S))^{1-\beta}} Z^n(1-Z)^{1/(1-\beta)}\,\mathrm{d}Z,\end{aligned} \tag{A.9}$$

and

$$\begin{aligned}\frac{\partial \mathrm{HC}}{\partial S} &= \frac{h}{\alpha^n(1-\beta)^n}\left[-\frac{n\theta((1-\beta)(1+n)+1)}{2\alpha(1-\beta)}(Q+S)^{(1-\beta)(1+n)}\right.\\ &\quad \left. + \frac{(1-\beta)n+1}{1-\beta}(Q+S)^{(1-\beta)n}\right] \times \int_0^{1-(S/(Q+S))^{1-\beta}} Z^n(1-Z)^{\beta/(1-\beta)}\,\mathrm{d}Z\\ &\quad + \frac{h}{\alpha^n(1-\beta)^n}\left(1-\left(\frac{S}{Q+S}\right)^{1-\beta}\right)^n Q(Q+S)^{(1-\beta)n-1}\\ &\quad \times\left(-1+\frac{n\theta}{2\alpha}((Q+S)^{1-\beta}+S^{1-\beta})\right)\end{aligned}$$

$$+\frac{h}{\alpha^n(1-\beta)^n}\left[-\frac{n\theta((1-\beta)(1+n)+1)}{2\alpha(1-\beta)}(Q+S)^{(1-\beta)(1+n)}\right]$$
$$\times\int_0^{1-(S/(Q+S))^{1-\beta}} Z^n(1-Z)^{1/(1-\beta)}\,\mathrm{d}Z. \tag{A.10}$$

Substituting Eqs. (A.5)–(A.10) into (A.4) and simplifying terms, we have

$$\left\{p-\frac{p\theta}{\alpha}(Q+S)^{1-\beta}-\frac{\partial\mathrm{HC}}{\partial Q}-c\right\}\left[\frac{\theta}{\alpha^2}S^{1-2\beta}-\frac{S^{-\beta}}{\alpha}\right]$$
$$=\left[-\frac{\theta}{\alpha^2}(Q+S)^{1-2\beta}+\frac{(Q+S)^{-\beta}}{\alpha}\right]$$
$$\times\left\{-p+\frac{p\theta}{\alpha}S^{1-\beta}+\frac{h}{\alpha^n(1-\beta)^n}\left(1-\left(\frac{S}{Q+S}\right)^{1-\beta}\right)^n(Q+S)^{(1-\beta)n}\right.$$
$$\left.\times\left[1-\frac{n\theta}{2\alpha}((Q+S)^{1-\beta}+S^{1-\beta})\right]+c\right\}$$

So, we obtain Eq. (3.10). To obtain Eq. (3.11), we substitute Eqs. (3.4), (A.5), (A.7)–(A.9) into (A.1), and get

$$\left\{p-\frac{p\theta}{\alpha}(Q+S)^{1-\beta}-\frac{\partial\mathrm{HC}}{\partial Q}-c\right\}\times\frac{(Q+S)^{1-\beta}-S^{1-\beta}}{\alpha(1-\beta)}$$
$$\times\left[1-\frac{\theta}{2\alpha}\left((Q+S)^{1-\beta}+S^{1-\beta}\right)\right]$$
$$=(\mathrm{TR}-\mathrm{HC}-\mathrm{PC})\times\left[-\frac{\theta}{\alpha^2}(Q+S)^{1-2\beta}+\frac{(Q+S)^{-\beta}}{\alpha}\right],$$

which is Eq. (11).

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