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Fuzzy mixture inventory model involving fuzzy random variable lead time demand and fuzzy total demand

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Abstract

This article considers the mixture inventory model involving variable lead time with backorders and lost sales. We first fuzzify the random lead-time demand to be a fuzzy random variable and obtain the total cost in the fuzzy sense. Then, we further fuzzify the total demand to be the triangular fuzzy number and derive the fuzzy total cost. By the centroid method of defuzzification, we derive the estimate of total cost in the fuzzy sense. Also, we find the optimal solution for order quantity and lead time in the fuzzy sense such that the total cost has a minimum value. A numerical example is provided to illustrate the results of proposed model.

Keywords: Inventory; Lead time; Fuzzy random variable; Fuzzy total cost

1. Introduction

The issue of lead-time reduction has received a great deal of attention in the field of production/inventory management. As stated in Tersine [15], lead time usually consists of the following components: order preparation, order transit, supplier lead time, delivery time and setup time. Although most of the literature dealing with inventory problems viewed lead time as an uncontrollable variable, however, in some practical situations, lead time can be reduced by controlling some or all of its components. The benefits associated with efforts to reduce lead time, such as lower the safety stock, reduce the loss caused by stock

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out, increase the service level to the customer, and gain the competitive advantages in business, can be clearly perceived through the Japanese successful experiences of using Just-In-Time (JIT) production. Concerning the inventory systems with lead-time reduction, Liao and Shyu [8] first presented a continuous review inventory model in which the order quantity is predetermined and lead time is a unique variable that can be controlled by paying extra crashing cost. This model has been extended by Ben-Daya and Raouf [1] to include both lead time and order quantity as decision variables. Later, Ouyang et al. [10] developed a more general model, where they extended [1] by allowing shortages and considered that only a fraction of the demand during the stockout period can be backordered.

The above lead-time reduction models [1,8,10] are based on the continuous review inventory systems in which the uncertainty of demand during lead time is tackled from the traditional probability theory and the annual average demand is assessed by a crisp value. However, various types of uncertainties and imprecision including randomness and fuzziness are inherent in real inventory environments, In this article, we shall then address the issue of lead-time reduction under such circumstances. Specifically, we attempt to develop a fuzzy inventory model by considering the fuzziness and randomness for lead-time demand, which will be represented by a fuzzy random variable, based on the concept proposed by Puri and Ralescu [13]. Also, for the annual average demand, due to the fact that it may fluctuate a little in an unstable environment and is difficult to assess by a crisp value, we shall consider it as the fuzzy number.

It should be pointed out that the fuzzy sets concept was initially introduced by Zadeh [19] in 1965. Besides, in the literature, there are several researchers presented various types of fuzzy inventory models. For example, Petrovic and Sweeney [12] fuzzified the demand, lead time and inventory level into triangular fuzzy numbers in an inventory control model. Vujosevic et al. [16] extended the EOQ model by introducing the fuzziness of ordering cost and holding cost. Chen and Wang [3] fuzzified the demand, ordering cost, inventory cost, and backorder cost into trapezoidal fuzzy numbers in EOQ model with backorder. Roy and Maiti [14] presented a fuzzy EOQ model with demand-dependent unit cost under limited storage capacity. Gen et al. [4] considered the fuzzy input data expressed by fuzzy numbers, where the interval mean value concept is used to help solving the problem. Ishii and Konno [5] fuzzified the shortage cost into L shape fuzzy number in a classical newsboy problem aimed to find an optimal ordering quantity in the sense of fuzzy ordering. Chang et al. [2] presented a fuzzy model for inventory with backorder, where the backorder quantity was fuzzified as the triangular fuzzy number. Lee and Yao [7] and Lin and Yao [9] discussed the production inventory problems, where [7] fuzzified the demand quantity and production quantity per day, and [9] fuzzified the production quantity per cycle, all to be the triangular fuzzy numbers. Yao et al. [17] proposed the EOQ model in the fuzzy sense, where both order quantity and total demand were fuzzified as the triangular fuzzy numbers. Yao and Su [18] presented the fuzzy EOQ model for the inventory with backorders, where the total demand was fuzzified to be the interval-value fuzzy set. Ouyang and Yao [11] presented a mixture inventory model involving variable lead time, where the annual average demand was fuzzified as the triangular fuzzy number and as the statistic-fuzzy number.

From literature survey, we note that although several fuzzy inventory models have been presented, little has been done on addressing the issue of lead-time reduction. The purpose of this article is to recast Ouyang et al.'s [10] mixture inventory model involving variable lead time with backorders and lost sales by further considering the fuzziness of lead-time demand and annual average demand. We aim at providing an alternative approach of modeling uncertainty that may appear in real situations; whereas we do not attempt to establish the superiority of proposing a new model to reduce more inventory cost than previous one.

This article is organized as follows. In Section 2, a brief review of Ouyang et al.'s [10] model is given. In Section 3, we develop the fuzzy mixture inventory model involving variable lead time. Using the centroid method of defuzzification, we derive the estimate of total cost in the fuzzy sense. Two theorems are obtained. In Section 4, we derive the optimal order quantity and the optimal lead time by minimizing the estimate of total cost in the fuzzy sense. A numerical example is provided to illustrate the results. In Section 5, we discuss some problems for the proposed model. Section 6 summarizes the work done in this article.

2. Review of Ouyang et al.'s model

To develop the proposed model, we adopt the following notation and assumptions used in Ouyang et al. [10].

Notation

- D average demand per year
- Q order quantity
- A fixed ordering cost per order
- h inventory holding cost per item per year
- π fixed penalty cost per unit short
- π_0 marginal profit per unit
- L length of lead time
- r reorder point
- X lead-time demand, which is normally distributed with finite mean μL and standard deviation $\sigma\sqrt{L}$, where μ and σ denote the mean and standard deviation of the demand per unit time
- x^+ maximum value of x and 0, i.e., $x^+ = \max\{x, 0\}$
- $E(\cdot)$ mathematical expectation.

2.1. Assumptions

(1) The reorder point, r=expected demand during lead time + safety stock (SS), and $SS = k\cdot$(standard deviation of lead-time demand), i.e., $r = \mu L + k\sigma\sqrt{L}$ where k is the safety factor and satisfies $P(X>r) = P(Z>k) = q$, Z represents the standard normal random variable, and q represents the allowable stockout probability during lead time L, and q is given.

(2) Inventory is continuously reviewed. Replenishments are made whenever the inventory level falls to the reorder point r.

(3) The lead time L has n mutually independent components each having a different crashing cost for reducing lead time. The ith component has a minimum duration a_i and normal duration b_i and a crashing cost per unit time c_i. Furthermore, we assume that $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_n$.

(4) The components of lead time are crashed one at a time starting with the component of least c_i and so on.

(5) If we let $L_0 = \sum_{j=1}^{n} b_j$ and L_i be the length of lead time with components $1, 2, \ldots, i$ crashed to their minimum duration, then L_i can be expressed as $L_i = \sum_{j=1}^{n} b_j - \sum_{j=1}^{i}(b_j - a_j),\ i = 1, 2, \ldots, n$; and the lead time crashing cost per cycle $U(L)$ for a given $L \in [L_i, L_{i-1}]$, is given by

$$U(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j) \quad \text{and} \quad U(L_0) = 0. \tag{1}$$

By the above assumptions and considering that only a fraction β $(0 \leqslant \beta \leqslant 1)$ of the demand during the stockout period can be backordered, Ouyang et al. [10] established the total expected annual cost as follows.

$$
\begin{aligned}
EAC(Q,L) &= \text{setup cost} + \text{holding cost} + \text{stockout cost} + \text{lead time crashing cost} \\
&= A\frac{D}{Q} + h\left[\frac{Q}{2} + r - \mu L + (1-\beta)E(X-r)^+\right] + \frac{D}{Q}[\pi + \pi_0(1-\beta)]E(X-r)^+ + \frac{D}{Q}U(L) \\
&= \frac{D}{Q}\{A + [\pi + \pi_0(1-\beta)]E(X-r)^+ + U(L)\} + h\left[\frac{Q}{2} + k\sigma\sqrt{L} + (1-\beta)E(X-r)^+\right], \qquad (2)
\end{aligned}
$$

for $Q>0$, $L\in[L_i, L_{i-1}]$, $i=1,2,\ldots,n$, where $E(X-r)^+$ is the expected demand shortage at the end of the cycle.

3. Fuzzy mixture inventory model involving variable lead time

In contrast to Ouyang et al.'s [10] model, we will consider the fuzzy mixture inventory model in this article. Let $(R,\mathscr{B},P)$ be the probability space, where R is the set of real number, $\mathscr{B}$ is the *Borel field* on R, and P is a probability measure. The lead-time demand, X, in Section 2 is a random variable on $(R,\mathscr{B},P)$, which is assumed to be normally distributed with mean μL and standard deviation $\sigma\sqrt{L}$, i.e., $X\sim N(\mu L,\sigma\sqrt{L})$. For notational convenience, from now on, we denote $\mu_L=\mu L$ and $\sigma_L=\sigma\sqrt{L}$, where μ is an estimate (a known value), i.e., μ_L is an estimate. Then, the probability density function (pdf) of X is

$$
f(x) = \frac{1}{\sqrt{2\pi}\sigma_L}\mathrm{e}^{-\frac{(x-\mu_L)^2}{2\sigma_L^2}}, \quad -\infty < x < \infty. \qquad (3)
$$

X is called the crisp random variable. Corresponding to X, we consider the fuzzy random variable $\widetilde{X}$ here.

By definition of L_i (Assumption 5), we have $\min_{0\leqslant i\leqslant n}L_i=L_n$ and $\max_{0\leqslant i\leqslant n}L_i=L_0$ hence $L_n\leqslant L\leqslant L_0$. In the uncertain and/or unstable environments, for any $L\in[L_i,L_{i-1}]$, $i=1,2,\ldots,n$, it is difficult for the decision-maker to determine the lead-time demand (LTD) with a single value $E(X)=\mu_L$, rather it may easier to determine LTD by an interval $[\mu_L-\Delta_1,\ \mu_L+\Delta_2]$ where Δ_1, Δ_2 are determined by the decision-maker and should satisfy the conditions: $0<\Delta_1<\mu_{L_n}$ and $0<k\sigma_{L_0}<\Delta_2$ (see Eqs. (6) and (7) in the following). Since $[\mu_L-\Delta_1,\mu_L+\Delta_2]$ is an interval, so the decision-maker must take an appropriate value (we denote it by $\hat{\mu}_L$) from the inside of this interval as the estimate of LTD. If the chosen value is μ_L, it is the same as $E(X)=\mu_L$ of the crisp case, then the error of estimation $|\hat{\mu}_L-\mu_L|$ is 0. Moreover, if the chosen value is located in the left-hand side (LHS) or right-hand side (RHS) of μ_L, then the further the chosen value $\hat{\mu}_L$ is away from μ_L, the larger the error of estimation $|\hat{\mu}_L-\mu_L|$, and the largest errors will occur at the end points of interval $[\mu_L-\Delta_1,\ \mu_L+\Delta_2]$.

In the fuzzy viewpoint, we may employ the confidence level instead of error. For the case of $\hat{\mu}_L=\mu_L$ the error is 0, and the confidence level is in the largest and we let it be 1. In contrast, the further the value $\hat{\mu}_L$ is away from μ_L, the smaller the confidence level to be. At the end points of the interval, i.e., $\hat{\mu}_L=\mu_L-\Delta_1$ and $\hat{\mu}_L=\mu_L+\Delta_2$, the confidence level is in the smallest and we let it be 0.

Next, let us consider the following triangular fuzzy number,

$$
\tilde{\mu}_L = (\mu_L-\Delta_1, \mu_L, \mu_L+\Delta_2), \qquad (4)
$$

where $0<\Delta_1<\mu_{L_n}$ and $0<k\sigma_{L_0}<\Delta_2$ (see Eqs. (6) and (7)). The membership grade of $\tilde{\mu}_L$ is 1 at point μ_L, decreases as the point away from μ_L, and reaches 0 at the end points $\mu_L-\Delta_1$ and $\mu_L+\Delta_2$. Since the properties of membership grade and confidence level are the same, consequently, when the membership grade is treated as the confidence level, corresponding to the interval $[\mu_L - \Delta_1, \mu_L + \Delta_2]$ it is reasonable to set the triangular fuzzy number $\tilde{\mu}_L$ as Eq. (4).

Utilizing the centroid method to defuzzify $\tilde{\mu}_L$, we obtain

$$C(\tilde{\mu}_L) = \mu_L + \tfrac{1}{3}(\Delta_2 - \Delta_1) = \tfrac{2}{3}\mu_L + \tfrac{1}{3}\Delta_2 + \tfrac{1}{3}(\mu_L - \Delta_1) > 0. \quad (5)$$

$C(\tilde{\mu}_L)$ is regarded as the estimate of LTD in the fuzzy sense and $C(\tilde{\mu}_L) \in [\mu_L - \Delta_1, \mu_L + \Delta_2]$. For the special case $\Delta_1 = \Delta_2$, we have $C(\tilde{\mu}_L) = \mu_L$.

Furthermore, from the assumption that the reorder point $r = \mu_L + k\sigma_L$, we set $\tilde{r}$ is a fuzzy point with membership function $m_{\tilde{r}}(x) = 1$ if $x = r$, and $m_{\tilde{r}}(x) = 0$ if $x \neq r$. We then obtain the following triangular fuzzy number:

$$\tilde{\mu}_L(-)\tilde{r} = (\mu_L - r - \Delta_1, \mu_L - r, \mu_L - r + \Delta_2). \quad (6)$$

Note that $\tilde{r}$ is identical with the fuzzy number $\tilde{r} = (r, r, r)$, and the arithmetic of fuzzy numbers can be found in several textbooks, e.g. [6].

From above, $0 < \Delta_1 < \mu_{L_n}$ and $0 < r - \mu_L = k\sigma_L < k\sigma_{L_0} < \Delta_2$, we then have

$$\mu_L - r - \Delta_1 < \mu_L - r0; \quad 0 < \mu_L - r + \Delta_2. \quad (7)$$

From [13], the fuzzy random variable can be defined as a mapping from R of probability space $(R, \mathscr{B}, P)$ to a family of membership functions. Corresponding to the crisp random variable X, we set the fuzzy random variable $\widetilde{X}$ as

$$\widetilde{X} : s(\in R) \rightarrow \widetilde{X}(s), \quad (8)$$

where $\widetilde{X}(s)$ is the membership function. Let the fuzzy set that has $\widetilde{X}(\mu_L)$ as membership function be $\widetilde{X}^*(\mu_L) = \tilde{\mu}_L$ (as defined in Eq. (4)).

Next, let $Y = X - r$, then we can obtain the pdf of random variable Y as:

$$g(y) = \frac{1}{\sqrt{2\pi}\sigma_L} e^{-\frac{(y+r-\mu_L)^2}{2\sigma_L^2}}, \quad -\infty < y < \infty. \quad (9)$$

Corresponding to the crisp random variable Y, the fuzzy random variable $\widetilde{Y}$ is defined as:

$$\widetilde{Y} : t(\in R) \rightarrow \widetilde{Y}(t), \quad (10)$$

where $\widetilde{Y}(t)$ is the membership function. Also, let the fuzzy set that has $\widetilde{Y}(\mu_L)$ as membership function be $\widetilde{Y}^*(\mu_L) = \widetilde{X}^*(\mu_L)(-)\tilde{r} = \tilde{\mu}_L(-)\tilde{r}$ (as defined in Eq. (6)). Then, we have

$$\widetilde{Y}(\mu_L)(y) = \begin{cases} \frac{y-(\mu_L - r - \Delta_1)}{\Delta_1}, & \mu_L - r - \Delta_1 \leqslant y \leqslant \mu_L - r, \\ \frac{(\mu_L - r + \Delta_2) - y}{\Delta_2}, & \mu_L - r \leqslant y \leqslant \mu_L - r + \Delta_2, \\ 0, & \text{otherwise}. \end{cases} \quad (11)$$

The picture is shown in Fig. 1.

$\widetilde{Y}(\mu_L)(y)$ is a continuous function on $-\infty < y < \infty$. From the crisp probability theory, we note that $\widetilde{Y}(\mu_L)(Y)$ is a crisp random variable. From Fig. 1 and $y \geqslant 0$, we can derive the expectation $E(\widetilde{Y}(\mu_L)(Y))^+$ as follows:

$$E(\widetilde{Y}(\mu_L)(Y))^+ = \int_0^{\mu_L - r + \Delta_2} \left(\frac{(\mu_L - r + \Delta_2) - y}{\Delta_2} \times \frac{1}{\sqrt{2\pi}\sigma_L} e^{-\frac{(y+r-\mu_L)^2}{2\sigma_L^2}} \right) dy(> 0). \quad (12)$$

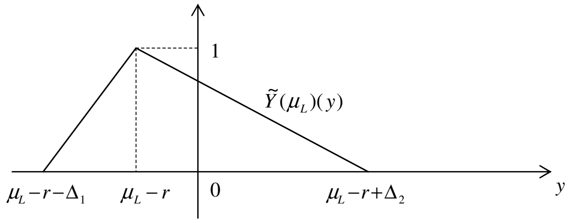

Fig. 1. Triangular fuzzy number $\widetilde{Y}^*(\mu_L)$.

Let $w = \frac{y+r-\mu_L}{\sigma_L}$, $\phi(a) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{a^2}{2}}$, and $\Phi(a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a}\mathrm{e}^{-\frac{w^2}{2}}\mathrm{d}w$, then from $r=\mu_L+k\sigma_L$, we obtain

$$E(\widetilde{Y}(\mu_L)(Y))^+ = \frac{1}{\sqrt{2\pi}\Delta_2}\int_k^{\frac{\Delta_2}{\sigma_L}}(\Delta_2 - \sigma_L w)\mathrm{e}^{-\frac{w^2}{2}}\mathrm{d}w = \Phi\left(\frac{\Delta_2}{\sigma_L}\right) - \Phi(k) + \frac{\sigma_L}{\Delta_2}\left[\phi\left(\frac{\Delta_2}{\sigma_L}\right) - \phi(k)\right], \tag{13}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ is the pdf and the cumulative distribution function (cdf) of the standard normal distribution, respectively.

Let $E(\widetilde{Y})^+ \equiv E(\widetilde{Y}(\mu_L)(Y))^+$, then when the term $E(X-r)^+$ in Eq. (2) is changed to be $E(\widetilde{Y})^+$, we obtain the following theorem.

Theorem 1. *In Eq.* (2), *when the crisp random variable X with the probability distribution* $N(\mu_L,\sigma_L)$ *is changed to be the fuzzy random variable* $\widetilde{X}$ (*as expressed in Eq.* (8)), *we obtain the total expected annual cost in the fuzzy sense*

$$EAC^*(Q,L;\Delta_1,\Delta_2) = \frac{D}{Q}\{A + [\pi + \pi_0(1-\beta)]E(\widetilde{Y})^+ + U(L)\} + h\left[\frac{Q}{2} + k\sigma\sqrt{L} + (1-\beta)E(\widetilde{Y})^+\right], \tag{14}$$

for $Q>0$, $L\in[L_i, L_{i-1}]$, $i=1,2,\ldots,n$.

As mentioned earlier, due to various uncertainties, the annual average demand may have a little fluctuation, especially in a perfect competitive market. Therefore, it is difficult for the decision-maker to assess the annual average demand by a crisp value D, but easier to determine it by an interval $[D-\Delta_3, D+\Delta_4]$. Similar to the previous approach, corresponding to the interval $[D-\Delta_3, D+\Delta_4]$, we can set the following triangular fuzzy number

$$\widetilde{D} = (D-\Delta_3, D, D+\Delta_4), \tag{15}$$

where Δ_3 and Δ_4 are determined by the decision-maker and should satisfy the conditions: $0<\Delta_3<D$ and $0<\Delta_4$.

Again, by the centroid method, we get

$$C(\widetilde{D}) = D + \tfrac{1}{3}(\Delta_4 - \Delta_3) > 0, \tag{16}$$

which is the estimate of total demand in the fuzzy sense.

Theorem 2. *Fuzzifying the annual average demand D in Eq.* (14) *to be the triangular fuzzy number* $\widetilde{D}$ *as showed in Eq.* (15), *then we obtain*:

(i) *the fuzzy total cost as*

$$F_{(Q,L)}(\widetilde{D}) = \frac{\widetilde{D}}{Q}\{A + [\pi + \pi_0(1-\beta)]E(\widetilde{Y})^+ + U(L)\} + h\left[\frac{Q}{2} + k\sigma\sqrt{L} + (1-\beta)E(\widetilde{Y})^+\right], \tag{17}$$

(ii) *the estimate of total expected annual cost in the fuzzy sense as*

$$K(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4) \equiv C(F_{(Q,L)}(\widetilde{D})) = EAC^*(Q,L;\Delta_1,\Delta_2) + \frac{(\Delta_4-\Delta_3)}{3Q}\{A+[\pi+\pi_0(1-\beta)]E(\widetilde{Y})^+ + U(L)\}, \tag{18}$$

for $Q>0$, $L\in[L_i, L_{i-1}]$, $i=1,2,\ldots,n$.

Proof

(i) For each $Q>0$, $L\in[L_i, L_{i-1}]$, $i=1,2,\ldots,n$, from Eq. (14), we set $F_{(Q,L)}(D)\equiv EAC^*(Q,L;\Delta_1,\Delta_2)$ and fuzzify D to be the fuzzy number $\widetilde{D}$ as in Eq. (15), then the result showed in Eq. (17) is obtained.

(ii) Since $Q>0$ and $A+[\pi+\pi_0(1-\beta)]E(\widetilde{Y})^+ + U(L)>0$, hence we can get the following triangular number:

$$F_{(Q,L)}(\widetilde{D}) = (F_1,F_2,F_3), \tag{19}$$

where

$$F_1 = \frac{D-\Delta_3}{Q}\{A+[\pi+\pi_0(1-\beta)]E(\widetilde{Y})^+ + U(L)\} + h\left[\frac{Q}{2}+k\sigma\sqrt{L}+(1-\beta)E(\widetilde{Y})^+\right]$$
$$= EAC^*(Q,L;\Delta_1,\Delta_2) - \frac{\Delta_3}{Q}\{A+[\pi+\pi_0(1-\beta)]E(\widetilde{Y})^+ + U(L)\},$$

$$F_2 = EAC^*(Q,L;\Delta_1,\Delta_2),$$

$$F_3 = \frac{D+\Delta_4}{Q}\{A+[\pi+\pi_0(1-\beta)]E(\widetilde{Y})^+ + U(L)\} + h\left[\frac{Q}{2}+k\sigma\sqrt{L}+(1-\beta)E(\widetilde{Y})^+\right]$$
$$= EAC^*(Q,L;\Delta_1,\Delta_2) + \frac{\Delta_4}{Q}\{A+[\pi+\pi_0(1-\beta)]E(\widetilde{Y})^+ + U(L)\}.$$

Utilizing the centroid method to defizzify $F_{(Q,L)}(\widetilde{D})$ leads to

$$C(F_{(Q,L)}(\widetilde{D})) = \tfrac{1}{3}(F_1+F_2+F_3). \tag{20}$$

Substituting the above F_1, F_2 and F_3 into Eq. (20), we obtain the result, which is denoted by $K(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4)$, as showed in Eq. (18).

4. The optimal solution

This section provides the solution procedure for the problem of determining the optimal order quantity and the optimal lead time such that the total expected annual cost in the fuzzy sense has a minimum value, while the decision-maker takes Δ_1, Δ_2, Δ_3, Δ_4 satisfying the conditions: $0<\Delta_1<\mu_{L_n}$ $(=\mu L_n)$, $k\sigma_{L_0}(=k\sigma\sqrt{L_0})<\Delta_2$, $0<\Delta_3<D$ and $0<\Delta_4$.

Let $S=\{L|L\in[L_i, L_{i-1}], i=1,2,\ldots,n\}$. Also, from Eq. (1), let

$$U_i(L) \equiv U(L) = c_i(L_{i-1}-L) + \sum_{j=1}^{i-1} c_j(b_j-a_j), \quad i=1,2,\ldots,n \quad \text{and} \quad \sum_{j=1}^{0} c_j(b_j-a_j) = 0.$$

Further, from Eqs. (14) and (18), let

$$\begin{aligned} G_i(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4) &= \frac{D}{Q}\{A+[\pi+\pi_0(1-\beta)]E(\widetilde{Y})^+ + U_i(L)\} + h\left[\frac{Q}{2}+k\sigma\sqrt{L}+(1-\beta)E(\widetilde{Y})^+\right] \\ &\quad + \frac{(\Delta_4-\Delta_3)}{3Q}\{A+[\pi+\pi_0(1-\beta)]E(\widetilde{Y})^+ + U_i(L)\}, \end{aligned} \tag{21}$$

for $L \in [L_i, L_{i-1}]$, $i=1,2,\ldots,n$, and $Q>0$.

Then, the mathematical expression of our problem is to find

$$\begin{aligned} MK &\equiv \min_{Q2;0, L\in S} K(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4) = \min_{L\in S}\min_{Q2;0} K(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4) \\ &= \min_{1\leqslant i\leqslant n}\ \min_{L\in[L_i,L_{i-1}]}\ \min_{Q2;0} G_i(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4). \end{aligned} \tag{22}$$

Now, we first find $\min_{Q>0} G_i(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4)$. For fixed $i \in \{1,2,\ldots,n\}$, and $L \in [L_i, L_{i-1}]$, we take the first and second partial derivatives of Eq. (21) with respect to Q, and obtain

$$\begin{aligned} \frac{\partial}{\partial Q} G_i(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4) = &-\frac{D}{Q^2}\{A+[\pi+\pi_0(1-\beta)]E(\widetilde{Y})^+ + U_i(L)\} + \frac{h}{2} \\ &- \frac{(\Delta_4-\Delta_3)}{3Q^2}\{A+[\pi+\pi_0(1-\beta)]E(\widetilde{Y})^+ + U_i(L)\}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial Q^2} G_i(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4) = &\frac{2D}{Q^3}\{A+[\pi+\pi_0(1-\beta)]E(\widetilde{Y})^+ + U_i(L)\} \\ &+ \frac{2(\Delta_4-\Delta_3)}{3Q^3}\{A+[\pi+\pi_0(1-\beta)]E(\widetilde{Y})^+ + U_i(L)\}, \end{aligned}$$

respectively.

Because $\partial^2 G_i(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4)/\partial Q^2 > 0$, thus for fixed $i \in \{1,2,\ldots,n\}$, and $L \in [L_i, L_{i-1}]$, the minimum value of $G_i(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4)$ will occur at the point Q that satisfies $\partial G_i(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4)/\partial Q = 0$. Solving this equation for Q (denote the value by $Q_i^{(0)}(L)$), we obtain

$$Q_i^{(0)}(L) = \sqrt{\frac{2(3D+\Delta_4-\Delta_3)}{3h}\{A+[\pi+\pi_0(1-\beta)]E(\widetilde{Y})^+ + U_i(L)\}}. \tag{23}$$

And the minimum value of $G_i(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4)$ is $G_i(Q_i^{(0)}(L),L;\Delta_1,\Delta_2,\Delta_3,\Delta_4)$, i.e., $G_i(Q_i^{(0)}(L),L;\Delta_1,\Delta_2,\Delta_3,\Delta_4) = \min_{Q2;0} G_i(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4)$.

Therefore, from Eq. (22), the problem reduces to find

$$MK = \min_{1\leqslant i\leqslant n}\ \min_{L\in[L_i,L_{i-1}]} G_i\left(Q_i^{(0)}(L),L;\Delta_1,\Delta_2,\Delta_3,\Delta_4\right). \tag{24}$$

Next, for fixed $i \in \{1,2,\ldots,n\}$, by the numerical analysis method, we can find $L_i^0 \in [L_i, L_{i-1}]$ such that

$$G_i\left(Q_i^{(0)}(L_i^{(0)}),L_i^{(0)};\Delta_1,\Delta_2,\Delta_3,\Delta_4\right) = \min_{L\in[L_i,L_{i-1}]} G_i\left(Q_i^{(0)}(L),L;\Delta_1,\Delta_2,\Delta_3,\Delta_4\right). \tag{25}$$

Furthermore, for each $i=1,2,\ldots,n$, we evaluate the value of $G_i(Q_i^{(0)}(L_i^{(0)}),L_i^{(0)};\Delta_1,\Delta_2,\Delta_3,\Delta_4)$ and find $\min_{1\leqslant i\leqslant n}G_i(Q_i^{(0)}(L_i^{(0)}),L_i^{(0)};\Delta_1,\Delta_2,\Delta_3,\Delta_4)$. If $G_m(Q_m^{(0)}(L_m^{(0)}),L_m^{(0)};\Delta_1,\Delta_2,\Delta_3,\Delta_4)=\min_{1\leqslant i\leqslant n}G_i(Q_i^{(0)}(L_i^{(0)}),L_i^{(0)};\Delta_1,\Delta_2,\Delta_3,\Delta_4)$, then we have

$$MK=G_m\big(Q_m^{(0)}(L_m^{(0)}),L_m^{(0)};\Delta_1,\Delta_2,\Delta_3,\Delta_4\big). \tag{26}$$

Thus, the optimal lead time is $L^*=L_m^{(0)}$, and the optimal order quantity is $Q^*=Q_m^{(0)}(L_m^{(0)})$.

4.1. Numerical example

To illustrate the results of proposed model and compare them with those obtained from the crisp model, let us consider an inventory system with the data used in [10]: $D=600$ units/year, $A=\$200$ per order, $h=\$20$ per unit per year, $\pi=\$50$ per unit short, $\pi_0=\$150$, $\sigma=7$ units/week, $q=0.2$ (hence, $k=0.8416$), and the lead time has three components with data shown in Table 1.

Using the data given in Table 1, we have the following length of lead time with some components crashed to their minimum duration: $L_0=56$ days, $L_1=56-14=42$ days, $L_2=42-14=28$ days, $L_3=28-7=21$ days. Hence, $L_3=\min L_i=21$ days (=3 weeks), $L_0=\max\ L_i=56$ days (=8 weeks), $0<\Delta_1<\mu L_3=34.62$, $\Delta_2>k\sigma\sqrt{L_0}=16.66$, $0<\Delta_3<D=600$, $0<\Delta_4$. Also, the lead time crashing costs are as follows.

For fixed $i\in\{1,2,3\}$ and $L\in[L_i,\ L_{i-1}]$, from Eq. (23), we get the order quantity

$$Q_i^{(0)}(L)=\sqrt{\frac{1800+\Delta_4-\Delta_3}{30}\{200+[50+150(1-\beta)]E(\widetilde{Y})^++U_i(L)\}},$$

and the corresponding total cost (from Eq. (21))

$$\begin{aligned}&G_i(Q_i^{(0)}(L),L;\Delta_1,\Delta_2,\Delta_3,\Delta_4)\\&\quad=\frac{600}{Q_i^{(0)}(L)}\{200+[50+150(1-\beta)]E(\widetilde{Y})^++U_i(L)\}+20\left[\frac{1}{2}Q_i^{(0)}(L)+5.8912\sqrt{L}+(1-\beta)E(\widetilde{Y})^+\right]\\&\qquad+\frac{(\Delta_4-\Delta_3)}{3Q_i^{(0)}(L)}\{200+[50+150(1-\beta)]E(\widetilde{Y})^++U_i(L)\},\qquad L\in[L_i,L_{i-1}],\quad i=1,2,3,\end{aligned}$$

where

$$E(\widetilde{Y})^+=\Phi\left(\frac{\Delta_2}{7\sqrt{L}}\right)-\Phi(0.8416)+\frac{7\sqrt{L}}{\Delta_2}\left[\phi\left(\frac{\Delta_2}{7\sqrt{L}}\right)-\phi(0.8416)\right].$$

Note that β, Δ_1, Δ_2, Δ_3 and Δ_4 are given parameters. When L is specified, we can use the formulas listed in Table 2 to calculate $U_i(L)$, and by checking the standard normal distribution table or using the software

Table 1
Lead time data

Lead time component i	Normal duration b_i (days)	Minimum duration a_i (days)	Unit crashing cost c_i (\$/day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

Table 2
Lead time crashing cost

i	$U_i(L)$	
1	$0.4(56-L)=22.4-0.4L$,	for $42 \leqslant L \leqslant 56$
2	$1.2(42-L)+0.4\times 14=56-1.2L$,	for $28 \leqslant L \leqslant 42$
3	$5(28-1)+0.4\times 14+1.2\times 14=162.4-5L$,	for $21 \leqslant L \leqslant 28$

such as Microsoft Excel to find the values of $\Phi\left(\frac{\Delta_2}{7\sqrt{L}}\right)$, $\Phi(0.8416)$, $\phi\left(\frac{\Delta_2}{7\sqrt{L}}\right)$ and $\phi(0.8416)$, then calculate $E(\widetilde{Y})^+$. Once $U_i(L)$ and $E(\widetilde{Y})^+$ are obtained, the values of $Q_i^{(0)}(L)$ and $G_i(Q_i^{(0)}(L), L; \Delta_1, \Delta_2, \Delta_3, \Delta_4)$, can be found easily.

For example, consider a case where $\beta=0$, $\Delta_1=10$, $\Delta_2=20$, $\Delta_3=25$, $\Delta_4=50$. For $i=1$, $L\in[L_1, L_0]=[42, 56]$, using $U_1(L)=22.4-0.4L$ and above procedure, we obtain the results listed in Table 3.

From Table 3, we find that the minimum value of $G_1(Q_1^{(0)}(L), L; \Delta_1, \Delta_2, \Delta_3, \Delta_4)$, for $L\in[42, 56]$, is \$2538.03, which occurred at $L=42$ days and $Q_1^{(0)}(L) = 112.46$ units. From Eq. (25), this solution is denoted by $L_1^{(0)} = 42$, $Q_1^{(0)}(L_1^{(0)}) = 112.46$ and $G_1(Q_1^{(0)}(L_1^{(0)}), L_1^{(0)}; \Delta_1, \Delta_2, \Delta_3, \Delta_4) = 2538.03$.

For $i=2$, $L\in[L_2, L_1]=[28, 42]$, $U_2(L)=56-1.2L$; and $i=3$, $L\in[L_3, L_2]=[21, 28]$, $U_3(L)=162.4-5L$, we use the same procedure and obtain the following results: $L_2^{(0)} = 42$, $Q_2^{(0)}(L_2^{(0)}) = 112.46$, $G_2(Q_2^{(0)}(L_2^{(0)}), L_2^{(0)}; \Delta_1, \Delta_2, \Delta_3, \Delta_4) = 2538.03$, and $L_3^{(0)} = 28$, $Q_3^{(0)}(L_3^{(0)}) = 117.78$, $G_3(Q_3^{(0)}(L_3^{(0)}), L_3^{(0)}; \Delta_1, \Delta_2, \Delta_3, \Delta_4) = 2591.76$.

By comparing the values of $G_i(Q_i^{(0)}(L_i^{(0)}), L_i^{(0)}; \Delta_1, \Delta_2, \Delta_3, \Delta_4)$, for $i=1,2,3$, we get the minimum value $MK = G_1(Q_1^{(0)}(L_1^{(0)}), L_1^{(0)}; \Delta_1, \Delta_2, \Delta_3, \Delta_4) = G_2(Q_2^{(0)}(L_2^{(0)}), L_2^{(0)}; \Delta_1, \Delta_2, \Delta_3, \Delta_4) = 2538.03$. Thus, the optimal lead time is $L^* = L_1^{(0)} = L_2^{(0)} = 42$ days (=6 weeks), and the optimal order quantity is $Q^* = Q_1^{(0)}(L_1^{(0)}) = Q_2^{(0)}(L_2^{(0)}) = 113$ units (truncated).

Now, let us consider the cases for $\beta=0$, 0.5, 0.8, 1 with various sets of $(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$. By the procedure outlined above, we obtain the computed results as showed in Table 4.

Moreover, in order to compare the results with those obtained from crisp model (crisp random lead-time demand and crisp annual demand), we first list the optimal solution of crisp model in Table 5.

Table 3
The results of solution procedure for $L\in[L_1,L_0]=[42,56]$

L (days)	$U_1(L)$	$E(\widetilde{Y})^+$	$Q_1^{(0)}(L)$	$G_1(Q_1^{(0)}(L), L; \Delta_1, \Delta_2, \Delta_3, \Delta_4)$
56	0.0	0.00375	110.51	2543.51
55	0.4	0.00412	110.64	2543.14
54	0.8	0.00451	110.77	2542.76
53	1.2	0.00493	110.90	2542.38
52	1.6	0.00537	111.04	2541.99
51	2.0	0.00584	111.17	2541.60
50	2.4	0.00634	111.31	2541.21
49	2.8	0.00686	111.45	2540.81
48	3.2	0.00742	111.59	2540.42
47	3.6	0.00801	111.73	2540.02
46	4.0	0.00863	111.87	2539.62
45	4.4	0.00929	112.01	2539.22
44	4.8	0.00998	112.16	2538.82
43	5.2	0.01072	112.31	2538.42
42	5.6	0.01149	112.46	2538.03

Table 4
The optimal solutions of proposed fuzzy model (L^* in weeks)

Given parameters				$\beta=0$			$\beta=0.5$		
Δ_1	Δ_2	Δ_3	Δ_4	L^*	Q^*	MK	L^*	Q^*	MK
10	20	25	50	6	113	2538.03	6	112	2533.25
10	20	30	100	6	114	2565.59	6	114	2560.75
10	20	35	150	6	115	2592.82	6	115	2587.92
10	25	25	50	6	113	2558.44	6	113	2546.00
10	25	30	100	8	113	2585.92	6	114	2573.66
10	25	35	150	8	114	2612.85	6	116	2600.99
10	30	25	50	8	112	2577.09	6	113	2559.06
10	30	30	100	8	114	2604.57	6	115	2586.88
10	30	35	150	8	115	2631.73	6	116	2614.36
10	20	50	25	6	111	2507.00	6	111	2502.29
10	20	100	30	6	109	2478.70	6	109	2474.05
10	20	150	35	6	108	2450.04	6	108	2445.44
10	25	50	25	6	112	2527.13	6	111	2514.87
10	25	100	30	6	110	2498.58	6	110	2486.47
10	25	150	35	6	109	2469.66	6	108	2457.70
10	30	50	25	8	111	2546.15	6	112	2527.75
10	30	100	30	8	109	2517.92	6	111	2499.19
10	30	150	35	8	108	2489.33	6	109	2470.26
				$\beta=0.8$			$\beta=1$		
10	20	25	50	6	112	2530.37	6	112	2528.46
10	20	30	100	6	113	2557.84	6	113	2555.90
10	20	35	150	6	115	2584.98	6	115	2583.02
10	25	25	50	6	112	2538.51	6	112	2533.50
10	25	30	100	6	114	2566.08	6	114	2561.01
10	25	35	150	6	115	2593.32	6	115	2588.19
10	30	25	50	6	113	2546.86	6	113	2538.69
10	30	30	100	6	114	2574.53	6	114	2566.26
10	30	35	150	6	116	2601.86	6	115	2593.50
10	20	50	25	6	111	2499.45	6	110	2497.56
10	20	100	30	6	109	2471.25	6	109	2469.38
10	20	150	35	6	108	2442.67	6	108	2440.83
10	25	50	25	6	111	2507.48	6	111	2502.54
10	25	100	30	6	110	2479.17	6	109	2474.29
10	25	150	35	6	108	2450.49	6	108	2445.68
10	30	50	25	6	111	2515.71	6	111	2507.65
10	30	100	30	6	110	2487.30	6	110	2479.34
10	30	150	35	6	108	2458.52	6	108	2450.66

Table 5
The optimal solutions of crisp model (from [10])

β	L_s	Q_s	$EAC(Q_s,L_s)$
0.0	3	177	3780.00
0.5	4	158	3408.93
0.8	4	144	3123.70
1.0	4	134	2917.82

Then, the relative error of lead time, order quantity, and minimum total expected annual cost in the fuzzy sense can be measured by: $\mathrm{Rel}\,L=[(L^*-L_s)/L_s]\times 100\%$, $\mathrm{Rel}\,Q=[(Q^*-Q_s)/Q_s]\times 100\%$, and $\mathrm{Rel}\,TC=[(MK-EAC)/EAC]\times 100\%$, respectively. Using the values in Tables 4 and 5, and these formulas, we obtain the results as summarized in Table 6.

In this example, for each case of β with various $(\Delta_1,\Delta_2,\Delta_3,\Delta_4)$, it can be observed that the solutions L^*,Q^* and MK of proposed fuzzy model are different from L_s,Q_s and EAC of crisp model ($L^*>L_s$, $Q^*<Q_s$,$MK<EAC$; and $\mathrm{Rel}\,L>0$, $\mathrm{Rel}\,Q<0$, $\mathrm{Rel}\,TC<0$). We note that the numerical results depend on the given values of problem parameters, which therefore, for other cases, it may get different results, In the following section, we will show that the solutions of crisp and fuzzy models are approximately equal at some certain cases.

Table 6
The relative error (%) of lead time, order quantity, and minimum total expected annual cost in the fuzzy sense

Given parameters				$\beta=0$			$\beta=0.5$		
Δ_1	Δ_2	Δ_3	Δ_4	Rel L	Rel Q	Rel TC	Rel L	Rel Q	Rel TC
10	20	25	50	100.00	−36.16	−32.86	50.00	−29.11	−25.69
10	20	30	100	100.00	−35.59	−32.13	50.00	−27.85	−24.88
10	20	35	150	100.00	−35.03	−31.41	50.00	−27.22	−24.08
10	25	25	50	100.00	−36.16	−32.32	50.00	−28.48	−25.31
10	25	30	100	166.67	−36.16	−31.59	50.00	−27.85	−24.50
10	25	35	150	166.67	−35.59	−30.88	50.00	−26.58	−23.70
10	30	25	50	166.67	−36.72	−31.82	50.00	−28.48	−24.93
10	30	30	100	166.67	−35.59	−31.10	50.00	−27.22	−24.11
10	30	35	150	166.67	−35.03	−30.38	50.00	−26.58	−23.31
10	20	50	25	100.00	−37.29	−33.68	50.00	−29.75	−26.60
10	20	100	30	100.00	−38.42	−34.43	50.00	−31.01	−27.42
10	20	150	35	100.00	−38.98	−35.18	50.00	−31.65	−28.26
10	25	50	25	100.00	−36.72	−33.14	50.00	−29.75	−26.23
10	25	100	30	100.00	−37.85	−33.90	50.00	−30.38	−27.06
10	25	150	35	100.00	−38.42	−34.67	50.00	−31.65	−27.90
10	30	50	25	166.67	−37.29	−32.64	50.00	−29.11	−25.85
10	30	100	30	166.67	−38.42	−33.39	50.00	−29.75	−26.69
10	30	150	35	166.67	−38.98	−34.14	50.00	−31.01	−27.54
				$\beta=0.8$			$\beta=1$		
10	20	25	50	50.00	−22.22	−18.99	50.00	−16.42	−13.34
10	20	30	100	50.00	−21.53	−18.12	50.00	−15.67	−12.40
10	20	35	150	50.00	−20.14	−17.25	50.00	−14.18	−11.47
10	25	25	50	50.00	−22.22	−18.73	50.00	−16.42	−13.17
10	25	30	100	50.00	−20.83	−17.85	50.00	−14.93	−12.23
10	25	35	150	50.00	−20.14	−16.98	50.00	−14.18	−11.30
10	30	25	50	50.00	−21.53	−18.47	50.00	−15.67	−12.99
10	30	30	100	50.00	−20.83	−17.58	50.00	−14.93	−12.05
10	30	35	150	50.00	−19.44	−16.71	50.00	−14.18	−11.12
10	20	50	25	50.00	−22.92	−19.98	50.00	−17.91	−14.40
10	20	100	30	50.00	−24.31	−20.89	50.00	−18.66	−15.37
10	20	150	35	50.00	−25.00	−21.80	50.00	−19.40	−16.35
10	25	50	25	50.00	−22.92	−19.73	50.00	−17.16	−14.23
10	25	100	30	50.00	−23.61	−20.63	50.00	−18.66	−15.20
10	25	150	35	50.00	−25.00	−21.55	50.00	−19.40	−16.18
10	30	50	25	50.00	−22.92	−19.46	50.00	−17.16	−14.06
10	30	100	30	50.00	−23.61	−20.37	50.00	−17.91	−15.03
10	30	150	35	50.00	−25.00	−21.29	50.00	−19.40	−16.01

5. Discussions

[A] The relationship between Theorems 1 and 2.

We evaluate the difference of the total expected annual cost in the fuzzy sense obtained in Theorems 1 and 2 (ii) as follows.

$$|K(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4) - EAC^*(Q,L;\Delta_1,\Delta_2)| = \frac{1}{3Q}\{A+[\pi+\pi_0(1-\beta)]E(\widetilde{Y})^+ + U(L)\}|\Delta_4-\Delta_3|.$$

It is clear when $|\Delta_4-\Delta_3|$ is getting smaller, $K(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4)$ and $EAC^*(Q,L;\Delta_1,\Delta_2)$ are getting closer, and if $\Delta_3=\Delta_4$, then $K(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4)=EAC^*(Q,L;\Delta_1,\Delta_2)$. Thus, it can be seen that Theorem 1 is a special case of Theorem 2.

[B] The fuzzy probability of fuzzy set $\widetilde{Y}^*(\mu_L)$.

From the definition of fuzzy probability and Eq. (11), we obtain

$$\widetilde{P}(\widetilde{Y}^*(\mu_L)) = \frac{1}{\sqrt{2\pi}\sigma_L}\int_{\mu_L-r-\Delta_1}^{\mu_L-r}\left(\frac{y-(\mu_L-r-\Delta_1)}{\Delta_1}\mathrm{e}^{-\frac{(y+r-\mu_L)^2}{2\sigma_L^2}}\right)\mathrm{d}y + \frac{1}{\sqrt{2\pi}\sigma_L} \times \int_{\mu_L-r}^{\mu_L-r+\Delta_2}\left(\frac{(\mu_L-r+\Delta_2)-y}{\Delta_2}\mathrm{e}^{-\frac{(y+r-\mu_L)^2}{2\sigma_L^2}}\right)\mathrm{d}y. \tag{27}$$

Let $w=(y+r-\mu_L)/\sigma_L$ and $W=(Y+r-\mu_L)/\sigma_L$, we obtain

$$\begin{aligned}\widetilde{P}(\widetilde{Y}^*(\mu_L)) &= \frac{1}{\sqrt{2\pi}\Delta_1}\int_{-\frac{\Delta_1}{\sigma_L}}^{0}(\Delta_1+\sigma_L w)\mathrm{e}^{-\frac{w^2}{2}}\mathrm{d}w + \frac{1}{\sqrt{2\pi}\Delta_2}\int_0^{\frac{\Delta_2}{\sigma_L}}(\Delta_2-\sigma_L w)\mathrm{e}^{-\frac{w^2}{2}}\mathrm{d}w \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\Delta_1/\sigma_L}^{\Delta_2/\sigma_L}\mathrm{e}^{-\frac{w^2}{2}}\mathrm{d}w + \frac{\sigma_L}{\sqrt{2\pi}\Delta_1}\left(\mathrm{e}^{-\frac{\Delta_1^2}{2\sigma_L^2}}-1\right) + \frac{\sigma_L}{\sqrt{2\pi}\Delta_2}\left(\mathrm{e}^{-\frac{\Delta_2^2}{2\sigma_L^2}}-1\right) \\ &= P\left(\frac{-\Delta_1}{\sigma_L}\leqslant W\leqslant\frac{\Delta_2}{\sigma_L}\right) - \frac{\sigma_L}{\sqrt{2\pi}}\left[\frac{1}{\Delta_1}\left(1-\mathrm{e}^{-\frac{\Delta_1^2}{2\sigma_L^2}}\right)+\frac{1}{\Delta_2}\left(1-\mathrm{e}^{-\frac{\Delta_2^2}{2\sigma_L^2}}\right)\right] \\ &= P(\mu_L-r-\Delta_1\leqslant Y\leqslant\mu_L-r+\Delta_2) - \frac{\sigma_L}{\sqrt{2\pi}}\left[\frac{1}{\Delta_1}\left(1-\mathrm{e}^{-\frac{\Delta_1^2}{2\sigma_L^2}}\right)+\frac{1}{\Delta_2}\left(1-\mathrm{e}^{-\frac{\Delta_2^2}{2\sigma_L^2}}\right)\right],\end{aligned} \tag{28}$$

where $Y\sim N(\mu_L-r,\sigma_L)$ and $P(\mu_L-r-\Delta_1\leqslant Y\leqslant\mu_L-r+\Delta_2)$ is the crisp probability of the random variable Y that belongs to the confidence interval $[\mu_L-r-\Delta_1,\ \mu_L-r+\Delta_2]$.

Remark. Because $\widetilde{Y}^*(\mu_L)=(\mu_L-r-\Delta_1,\mu_L-r,\mu_L-r+\Delta_2)$, therefore, the Eq. (28) can be described as the probability of the confidence interval $[\mu_L-r-\Delta_1,\mu_L-r+\Delta_2]$ in the fuzzy sense. Furthermore, from Eq. (28), we have $\widetilde{P}(\widetilde{Y}^*(\mu_L)) < P(\mu_L-r-\Delta_1\leqslant Y\leqslant\mu_L-r+\Delta_2)$.

[C] Compare the results between $E(\widetilde{Y})^+$ and $E(Y)^+(\equiv E(X-r)^+$ in Eq. (2)).

For the crisp random variable $Y=X-r$ with pdf showed in Eq. (9), by $r=\mu_L+k\sigma_L$ and $w=(y+r-\mu_L)/\sigma_L$,we get

$$E(Y)^+ = \int_0^\infty y\frac{1}{\sqrt{2\pi}\sigma_L}\mathrm{e}^{-\frac{(y+r-\mu_L)^2}{2\sigma_L^2}}\mathrm{d}y = \frac{\sigma_L}{\sqrt{2\pi}}\int_k^\infty(w-k)\mathrm{e}^{-\frac{w^2}{2}}\mathrm{d}w = \sigma_L\{\phi(k)-k[1-\Phi(k)]\}.$$

Besides, in Eq. (13), we have derived $E(\widetilde{Y})^+ = \Phi\left(\frac{\Delta_2}{\sigma_L}\right)-\Phi(k)+\frac{\sigma_L}{\Delta_2}\left[\phi\left(\frac{\Delta_2}{\sigma_L}\right)-\phi(k)\right]$.

Recall that $\Phi(\cdot)$ and $\phi(\cdot)$ stands for cdf and pdf of standard normal distribution, respectively. Therefore, we can measure the difference between $E(\widetilde{Y})^+$ and $E(Y)^+$ as

$$E(\widetilde{Y})^+ - E(\widetilde{Y})^+ = \Phi\left(\frac{\Delta_2}{\sigma_L}\right) - \Phi(k) + \frac{\sigma_L}{\Delta_2}\left[\phi\left(\frac{\Delta_2}{\sigma_L}\right) - \phi(k)\right] - \sigma_L\{\phi(k) - k[1-\Phi(k)]\}$$
$$= \Phi\left(\frac{\Delta_2}{\sigma_L}\right) + k\sigma_L - (1+k\sigma_L)\Phi(k) + \frac{\sigma_L}{\Delta_2}\phi\left(\frac{\Delta_2}{\sigma_L}\right) - \left(1+\frac{1}{\Delta_2}\right)\sigma_L\phi(k). \tag{29}$$

Since $0<r-\mu_L=k\sigma_L<k\sigma_{L_0}<\Delta_2$ and $k=(r-\mu_L)/\sigma_L=(r-\mu_L)/(\sigma\sqrt{L})$, hence, when $\sigma\to 0^+$, $k\to\infty$, and then $\Delta_2\to\infty$. In this case, from Eq. (29), we have

$$\lim_{\Delta_2\to\infty}[E(\widetilde{Y})^+ - E(Y)^+] = (1+k\sigma_L)[1-\Phi(k)]\sigma_L\phi(k). \tag{30}$$

Furthermore, from Eq. (30), we obtain

$$\lim_{k\to\infty}\lim_{\Delta_2\to\infty}[E(\widetilde{Y})^+ - E(Y)^+] = \lim_{k\to\infty}\{(1+k\sigma_L)[1-\Phi(k)] - \sigma_L\phi(k)\} = \lim_{k\to\infty}\frac{1-\Phi(k)}{1/(1+k\sigma_L)} - \lim_{k\to\infty}\sigma_L\phi(k)$$
$$= \lim_{k\to\infty}\frac{\phi(k)}{\sigma_L/(1+k\sigma_L)^2}\text{(by L'Hospital rule)} = \lim_{k\to\infty}\frac{(1+k\sigma_L)^2}{\sqrt{2\pi}\sigma_L e^{k^2/2}}$$
$$= \lim_{k\to\infty}\frac{2(1+k\sigma_L)}{\sqrt{2\pi}k e^{k^2/2}} = 0.$$

Similarly, we can show $\lim_{\Delta_2\to\infty}\lim_{k\to\infty}[E(\widetilde{Y})^+ - E(Y)^+] = 0$.

Besides, instead of taking $\Delta_2\to\infty$ and $k\to\infty$ in above mathematical analysis, we show numerically that $E(\widetilde{Y})^+$ and $E(Y)^+$ is quite close when some finite values of Δ_2 and/or k are chosen. For example, consider $k=3.5, 4, 4.5, 4.9$. From the table of standard normal distribution, we find $\Phi(k)$ and $\phi(k)$, and obtain the result of Eq. (30), i.e., $(1+k\sigma_L)[1-\Phi(k)]-\sigma_L\phi(k)$, as follows.

k	$\Phi(k)$	$\phi(k)$	$(1+k\sigma_L)[1-\Phi(k)]-\sigma_L\phi(k)$
3.5	0.99976733	0.00087268	$0.00023267-0.00005833\sigma_L$
4.0	0.99996831	0.00013383	$0.00003169-0.00000709\sigma_L$
4.5	0.99999660	0.00001598	$0.00000340-0.00000068\sigma_L$
4.9	0.99999952	0.00000244	$0.00000048-0.00000009\sigma_L$

On the other hand, we take $\Delta_2=4.9\sigma_L$ and $k=4.5$, and calculate the result of Eq. (29). We obtain

$$E(\widetilde{Y})^+ - E(Y)^+ = \Phi(4.9) + 4.5\sigma_L - (1+4.5\sigma_L)\Phi(4.5) + \frac{1}{4.9}\phi(4.9) - \left(1+\frac{1}{4.9\sigma_L}\right)\sigma_L\phi(4.5)$$
$$= 0.99999952 + 4.5\sigma_L - (1+4.5\sigma_L)0.9999966 + \frac{0.00000244}{4.9}$$
$$-\left(\sigma_L + \frac{1}{4.9}\right)0.00001598$$
$$= 0.00000016 - 0.00000068\sigma_L.$$

This result shows that when $\Delta_2=4.9\sigma_L$ and $k=4.5$ (which satisfies $k\sigma_L<\Delta_2$) are taken, it will imply a very small value of $E(\widetilde{Y})^+ - E(Y)^+$, i.e., $E(\widetilde{Y})^+$ and $E(Y)^+$ is approximate equal, thus, the total expected annual cost in the fuzzy sense $EAC^*(Q,L;\Delta_1,\Delta_2)$ obtained in Eq. (14) will approximate equal to that of crisp case $EAC(Q,L)$ showed in Eq. (2), so do their solutions. Moreover, by further taking $\Delta_3=\Delta_4$, then $K(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4)$ will close to $EAC(Q,L)$ also, because $K(Q,L;\Delta_1,\Delta_2,\Delta_3,\Delta_4)=EAC^*(Q,L;\Delta_1,\Delta_2)$ when $\Delta_3=\Delta_4$ (as discussed in [A]). In this case, the relative error of lead time, order quantity, and minimum total expected annual cost in the fuzzy sense will approach zero, i.e., $\mathrm{Rel}\,L\to 0$, $\mathrm{Rel}\,Q\to 0$, $\mathrm{Rel}\,TC\to 0$.

6. Conclusions

For a mixture inventory model with variable lead time, Ouyang and Yao [11] has applied the fuzzy sets theory to deal with the uncertain annual average demand, while the lead-time demand is treated as an ordinary (crisp) random variable with unknown form of probability distribution. In this paper, we also consider a mixture inventory model and address the issue of lead-time reduction in the fuzzy environments. Building upon Ouyang et al.'s [10] model in which the annual average demand D is a crisp value and the random lead-time demand X is normally distributed, we first fuzzify X to be a fuzzy random variable $\widetilde{X}$ and derive the total expected annual cost in the fuzzy sense. Then, we further fuzzify D to be the triangular fuzzy number $\widetilde{D}$ and obtain the fuzzy total cost. After defuzzification, we derive the estimate of total expected annual cost in the fuzzy sense and obtain the corresponding optimal order quantity and lead time.

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