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Empirical Bayes estimation of the guarantee lifetime in a two-parameter exponential distribution

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Abstract

We study empirical Bayes estimation of the guarantee lifetime θ in a two-parameter exponential distribution having a probability density $p(x|\theta,\beta) = (1/\beta)\exp(-(x-\theta)/\beta)I(x-\theta)$ with unknown scale parameter β. An empirical Bayes estimator φ_n^* is proposed and its associated asymptotic optimality is studied. It is shown that φ_n^* is asymptotically optimal in the sense that its regret converges to zero at a rate $n^{-2r/(2r+1)}$, where n is the number of past observations available and r is a positive integer related to the prior distribution G.

MSC: Primary 62C12; secondary 62F10

Keywords: Asymptotic optimality; Rate of convergence; Regret

1. Introduction

Recently, there is a growing interest in empirical Bayes theory for some family of distributions. For instance, Nogami (1988) and Huang and Liang (1997a,b) study empirical Bayes procedures for uniform distributions. Singh and Prasad (1989) and Prasad and Singh (1990) investigate empirical Bayes procedures for estimating the guarantee lifetime in a two-parameter exponential distribution. Tiwari and Zalkikar (1990) and Liang (1993) consider empirical Bayes estimation problems for Pareto distributions. Datta (1991, 1994) and Li and Gupta (2001, 2003) study empirical Bayes procedures for truncation-parameter distributions. Huang (1995) and Huang and Liang (1997a,b) study empirical Bayes procedures for truncation-parameter distributions using linex error loss. Balakrishnan and Ma (2002) and Liang (2003) study empirical Bayes procedures for a location parameter in a shifted gamma distribution.

In this paper, we consider a two-parameter exponential distribution having a probability density function

$$p(x|\theta,\beta) = \frac{1}{\beta}\exp\left(\frac{-(x-\theta)}{\beta}\right)I(x-\theta), \tag{1.1}$$

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where $\theta > 0$ is the guarantee lifetime parameter and $I(x) = 1$ if $x > 0$, and 0 otherwise. Singh and Prasad (1989) and Prasad and Singh (1990) consider the problem of estimating the guarantee lifetime parameter θ. They propose some empirical Bayes estimators for θ under the situation that the scale parameter β is known. However, it is noted that in many practical applications, the value of the parameter β may not be known. Therefore, it is useful and important to consider the problem of estimation for the guarantee time parameter θ when β is unknown.

In the empirical Bayes setup, the parameter θ is a realization of a positive random variable Θ having an unknown prior distribution G. Throughout the paper, we assume the following :

Assumption A. The prior distribution G has a density g satisfying

(A1) g is decreasing in $\theta > 0$ and $g(\theta) = 0$ for $\theta > b$ for some known value b, $0 < b < \infty$.
(A2) g is $(r-1)$ times differentiable and $g^{(r-1)}(\theta)$ is continuous on $[0, b]$.

The paper is organized as follows. The estimation problem is formulated in Section 2 and a Bayes estimator is derived. Then the empirical Bayes framework of this estimation problem is introduced in Section 3. By mimicking the form of the Bayes estimator, an empirical Bayes estimator φ_n^* is constructed. The asymptotic optimality of φ_n^* is investigated in Section 4. Under Assumption A, φ_n^* is shown to be asymptotically optimal that its corresponding regret converges to zero at a rate $n^{-2r/(2r+1)}$, where n is the number of past observations.

2. Bayes estimation

Let $X_1, \ldots, X_m$ be a sample of size m from a two-parameter exponential distribution having a probability density $p(x|\theta, \beta)$ given by (1.1). Let $Y = \min(X_1, \ldots, X_m)$ and $W = \sum_{i=1}^m X_i - mY$. For a given (θ, β), Y follows a two-parameter exponential distribution with probability density

$$f(y|\theta, \beta) = \frac{m}{\beta} \exp\left(\frac{-m(y-\theta)}{\beta}\right) I(y-\theta), \tag{2.1}$$

and distribution of $2W/\beta$ follows $\chi^2(2(m-1))$, the chi-square with df $2(m-1)$. We denote the probability density of W by $g(w|\beta)$. Y and W are then independent, and (Y, W) is sufficient for the parameters (θ, β). It is assumed that the value of the scale parameter β is fixed but unknown, and the parameter θ is a realization of a positive random variable Θ, which follows an unknown prior distribution G fulfilling Assumption A. Thus, $f_G(y|\beta) = \int f(y|\theta, \beta)\,\mathrm{d}G(\theta)$ is the marginal probability density of Y. Let $F_G(y|\beta)$ denote the accumulative distribution associated with $f_G(y|\beta)$. Following Prasad and Singh (1990), the prior distribution G can be expressed as

$$G(\theta) = F_G(\theta|\beta) + \frac{\beta}{m} f_G(\theta|\beta). \tag{2.2}$$

Since we are interested in Bayes and empirical Bayes estimators of the parameter θ, we may consider estimators based on (Y, W). Using the square error loss, given $(Y, W) = (y, w)$, the Bayes estimator $\varphi_G(y, w)$ is the posterior mean of Θ, i.e.

$$\begin{aligned} \varphi_G(y, w) = E[\Theta|(Y, W) = (y, w)] &= \frac{\int_0^y \theta f(y|\theta, \beta) g(w|\beta)\,\mathrm{d}G(\theta)}{\int_0^y f(y|\theta, \beta) g(w|\beta)\,\mathrm{d}G(\theta)} \\ &= \frac{\int_0^y \theta f(y|\theta, \beta)\mathrm{d}G(\theta)}{\int_0^y f(y|\theta, \beta)\mathrm{d}G(\theta)} = \frac{\int_0^y \theta \mathrm{e}^{m\theta/\beta}\,\mathrm{d}G(\theta)}{\int_0^y \mathrm{e}^{m\theta/\beta}\mathrm{d}G(\theta)} \leqslant y. \end{aligned} \tag{2.3}$$

Note that $\varphi_G(y, w)$ is independent of w and hence is denoted by $\varphi_G(y)$.
Using the identity relationship (2.2), we can obtain

$$\int_0^y \theta f(y|\theta, \beta)\,\mathrm{d}G(\theta) = y f_G(y|\beta) - \int_0^y \exp\left(\frac{-m(y-\theta)}{\beta}\right) \mathrm{d}F_G(\theta|\beta).$$

Therefore,

$$\varphi_G(y) = \frac{\int_0^y \theta f(y|\theta,\beta)\,\mathrm{d}G(\theta)}{f_G(y|\beta)} = y - \frac{\alpha_G(y|\beta)}{f_G(y|\beta)} \leqslant y, \tag{2.4}$$

where

$$\alpha_G(y|\beta) = \int_0^y \exp\left(\frac{-m(y-\theta)}{\beta}\right) \mathrm{d}F_G(\theta|\beta). \tag{2.5}$$

It should be noted that under Assumption [A1], for $y > b$,

$$\varphi_G(y) = \frac{\int_0^y \theta \mathrm{e}^{m\theta/\beta}\mathrm{d}G(\theta)}{\int_0^y \mathrm{e}^{m\theta/\beta}\mathrm{d}G(\theta)} = \frac{\int_0^b \theta \mathrm{e}^{m\theta/\beta}\mathrm{d}G(\theta)}{\int_0^b e^{m\theta/\beta}\,\mathrm{d}G(\theta)} = \varphi_G(b). \tag{2.6}$$

The minimum Bayes risk of this estimation problem is then given by

$$R(G,\varphi_G) = E_{(Y,\Theta)}[\varphi_G(Y) - \Theta]^2. \tag{2.7}$$

Note that the Bayes estimator $\varphi_G(y)$ is a functional of the prior distribution G which is unknown. Therefore, it is impossible to implement φ_G for practical application. However, when a sequence of past data is available, we can estimate some related quantities by part of past data. In the following, we study the estimation problem by the empirical Bayes approach.

3. Empirical Bayes estimator

For the empirical Bayes framework, at stage i, we let $X_{i1}, \ldots, X_{im}$ be a sample of size m arising from a two-parameter exponential distribution with p.d.f $p(x|\theta_i,\beta)$, where θ_i is a realization of a positive random variable Θ_i. Here, it is assumed that Θ_i are iid and follow the unknown prior distribution G fulfilling Assumption A. Let $Y_i = \min(X_{i1},\ldots,X_{im})$ and $W_i = \sum_{j=1}^m X_{ij} - mY_i$. Thus, (Y_i, W_i, Θ_i), $i = 1,2,\ldots$ are iid copies of (Y, W, Θ), where (Y_i, W_i), $i = 1,2,\ldots,$ are observable, but Θ_i are not observable. Let $\underset{\sim}{Y}(n) = (Y_1,\ldots,Y_n)$, $\underset{\sim}{W}(n) = (W_1,\ldots,W_n)$. At the present stage, say stage $n+1$, $(\underset{\sim}{Y}(n), \underset{\sim}{W}(n))$ stands for the n past data and (Y_{n+1}, W_{n+1}) denotes the present random observation. Let θ_{n+1} be a realized value of the current random guarantee lifetime parameter Θ_{n+1}. We attempt to estimate θ_{n+1} by using the current observed value $Y_{n+1} = y$ and the past data $(\underset{\sim}{Y}(n), \underset{\sim}{W}(n))$. Thus, an empirical Bayes estimator $\varphi_n(y) \equiv \varphi_n(y, \underset{\sim}{Y}(n), \underset{\sim}{W}(n))$ is an estimator of θ_{n+1} based on the current observation $Y_{n+1} = y$ and the past data $(\underset{\sim}{Y}(n), \underset{\sim}{W}(n))$. The Bayes risk of φ_n is

$$R(G,\varphi_n) = E_n E_{(Y_{n+1},\Theta_{n+1})}[\varphi_n(Y_{n+1}) - \Theta_{n+1}]^2, \tag{3.1}$$

where the expectation E_n is taken with respect to $(\underset{\sim}{Y}(n), \underset{\sim}{W}(n))$. Since $R(G,\varphi_G)$ is the minimum Bayes risk, $R(G,\varphi_n) - R(G,\varphi_G) \geqslant 0$ for all n. Naturally, this nonnegative regret is thus often used as a measure of performance of the empirical Bayes estimator φ_n. An empirical Bayes estimator φ_n is said to be asymptotically optimal, relative to the prior distribution G, at a rate ε_n if $R(G,\varphi_n) - R(G,\varphi_G) = \mathrm{O}(\varepsilon_n)$, where $\{\varepsilon_n\}$ is a sequence of positive, decreasing numbers such that it converges to zero.

In the following, we seek a way to construct empirical Bayes estimators possessing the desired asymptotic optimality. The proposed empirical Bayes estimator will mimic the form of the Bayes estimator φ_G given in (2.4)–(2.5). Therefore, we need to estimate the parameter β and the two functions $\alpha_G(y|\beta)$ and $f_G(y|\beta)$.

Estimation of β: Note that $W_1,\ldots,W_n$ are mutually independent and $2W_i/\beta$ follows $\chi^2(2(m-1))$. So, $2(W_1 + \cdots + W_n)/\beta$ follows $\chi^2(2n(m-1))$. Define $\beta_n = (W_1 + \cdots + W_n)/n(m-1)$. Hence, $2n(m-1)\beta_n/\beta$ follows $\chi^2(2n(m-1))$. Therefore, $E_n[\beta_n] = \beta$, $Var(\beta_n) = \beta^2/n(m-1)$.

$$E_n\left[\frac{1}{\beta_n} - \frac{1}{\beta}\right]^2 = \frac{n(m-1)+2}{\beta^2[n(m-1)-1][n(m-1)-2]} \leqslant \frac{2}{\beta^2 n}$$

and

$$E_n\left[\left|\frac{1}{\beta_n}-\frac{1}{\beta}\right|\right] \leqslant \left[E_n\left[\frac{1}{\beta_n}-\frac{1}{\beta}\right]^2\right]^{1/2} \leqslant \frac{\sqrt{2}}{\beta\sqrt{n}}.$$

Also, note that β_n and $\underset{\sim}{Y}(n)$ are independent.

Let K be a kernel function satisfying the following conditions:

$(K1)$ Support of $K = [0, 1]$,

$(K2)$ $$\int_0^1 x^{\ell} K(x)\,dx = \begin{cases} 1 & \text{if } \ell = 0, \\ 0 & \text{if } \ell = 1, \dots, r-1, \end{cases}$$

$(K3)$ $|K(x)| \leqslant k_0$ for all x.

Estimation of $\alpha_G(y|\beta)$ *and* $f_G(y|\beta)$:

For its simplicity, taking $h_n = h$, define

$$\alpha_n(y) = \frac{1}{n}\sum_{j=1}^{n} \exp\left(\frac{-m(y-Y_j)}{\beta_n}\right) I(y - Y_j),$$

$$f_n(y) = \frac{1}{nh}\sum_{j=1}^{n} K\left(\frac{Y_j - y}{h}\right), \tag{3.2}$$

where $\{h \equiv h_n\}$ is a positive sequence which decreases to zero. The exact value of h will be given later. From Lemma A.1, we have

(a) $|E_n f_n(y) - f_G(y|\beta)| \leqslant c_1 h^r$, where $c_1 = k_0/r! \sup\{|f_G^{(r)}(y|\beta)|; y > 0\} < \infty$ under Assumption [A2].

(b) $Var(f_n(y)) \leqslant c_2/nh$, where $c_2 = mk_0^2/\beta$ (k_0 is given by [K3]).

(c) $|E_n\alpha_n(y) - \alpha_G(y|\beta)| \leqslant yc_3/\sqrt{n}$ where $c_3 = m\sqrt{2}/\beta$.

(d) $$Var(\alpha_n(y)) \leqslant \frac{\exp(-nc_4)}{n} + \frac{\alpha_G(y|\beta)}{n}, \tag{3.3}$$

where $c_4 = (m-1)(1 - \ln 2)$. Then, $\alpha_n(y)$ and $f_n(y)$ are consistent estimators of $\alpha_G(y|\beta)$ and $f_G(y|\beta)$, respectively. By mimicking the form of the Bayes estimator $\varphi_G(y)$ of (2.4), we propose an empirical Bayes estimator $\varphi_n^*(y)$ as follows:

For $Y_{n+1} = y$

$$\varphi_n^*(y) = \begin{cases} y - \left[\left(\dfrac{\alpha_n(y)}{f_n(y)} \vee 0\right) \wedge y\right] & \text{for } 0 < y \leqslant b, \\ \varphi_n^*(b) & \text{for } y > b. \end{cases} \tag{3.4}$$

The Bayes risk of φ_n^* is

$$R(G, \varphi_n^*) = E_n E_{(Y_{n+1},\Theta_{n+1})}[\varphi_n^*(Y_{n+1}) - \Theta_{n+1}]^2. \tag{3.5}$$

4. Asymptotic optimality

Without loss of generality, we may assume that $b \equiv \sup\{\theta > 0 | g(\theta) > 0\} > 1$. The proposed empirical Bayes estimator φ_n^* then possesses the following asymptotic optimality.

Theorem 4.1. *Assume the prior distribution G fulfilling Assumptions* [A1]–[A2]. *Then, the empirical Bayes estimator* φ_n^* *is asymptotically optimal in the sense that* $R(G, \varphi_n^*) - R(G, \varphi_G) = O(n^{-2r/(2r+1)})$.

Proof. From (2.7),(3.5) and by noting that for $y \geqslant b$, $\varphi_G(y) = \varphi_G(b)$ and $\varphi_n^*(y) = \varphi_n^*(b)$, the regret of the empirical Bayes estimator φ_n^* can be expressed as follows:

$$\begin{aligned} R(G,\varphi_n^*) - R(G,\varphi_G) &= E_n E_{Y_{n+1}}[\varphi_n^*(Y_{n+1}) - \varphi_G(Y_{n+1})]^2 \\ &= \int_0^1 E_n[\varphi_n^*(y) - \varphi_G(y)]^2 f_G(y|\beta)\,\mathrm{d}y \\ &\quad + \int_1^b E_n[\varphi_n^*(y) - \varphi_G(y)]^2 f_G(y|\beta)\,\mathrm{d}y \\ &\quad + E_n[\varphi_n^*(b) - \varphi_G(b)]^2[1 - F_G(b|\beta)] \\ &= \mathbf{I}(n) + \mathbf{II}(n) + \mathbf{III}(n). \end{aligned} \tag{4.1}$$

From (2.4), $0 \leqslant \alpha_G(y|\beta)/f_G(y|\beta) \leqslant y$. For each $0<y\leqslant b$ and for $1<\lambda\leqslant 2$, it follows from Lemma A.2 (see Appendix) that

$$\begin{aligned} E_n[\varphi_n^*(y) - \varphi_G(y)]^2 &= E_n\left[\left(y - \left(\frac{\alpha_n(y)}{f_n(y)} \vee 0\right) \wedge y\right) - \left(y - \frac{\alpha_G(y|\beta)}{f_G(y|\beta)}\right)\right]^2 \\ &= E_n\left[\left(\frac{\alpha_n(y)}{f_n(y)} \vee 0\right) \wedge y - \frac{\alpha_G(y|\beta)}{f_G(y|\beta)}\right]^2 \\ &\leqslant y^{2-\lambda} E_n\left[\left|\left(\frac{\alpha_n(y)}{f_n(y)} \vee 0\right) \wedge y - \frac{\alpha_G(y|\beta)}{f_G(y|\beta)}\right|^\lambda\right] \\ &\leqslant \frac{2y^{2-\lambda}}{f_G^\lambda(y|\beta)}\{E_n[|\alpha_n(y) - \alpha_G(y|\beta)|^\lambda] \\ &\quad + (2y)^\lambda E_n[|f_n(y) - f_G(y|\beta)|^\lambda]\}. \end{aligned} \tag{4.2}$$

Substituting (4.2) into $\mathbf{I}(n)$ of (4.1), and by Lemma A.1, we obtain

$$\begin{aligned} \mathbf{I}(n) &= \int_0^1 E_n[\varphi_n^*(y) - \varphi_G(y)]^2 f_G(y|\beta)\,\mathrm{d}y \\ &\leqslant \int_0^1 \frac{2y^{2-\lambda}}{f_G^{\lambda-1}(y|\beta)} \times \frac{\exp(-\lambda n c_4/2)}{n^{\lambda/2}}\,\mathrm{d}y + \int_0^1 \frac{2y^{2-\lambda}}{f_G^{\lambda-1}(y|\beta)} \times \frac{\alpha_G^{\lambda/2}(y|\beta)}{n^{\lambda/2}}\,\mathrm{d}y \\ &\quad + \int_0^1 \frac{2y^{2-\lambda}}{f_G^{\lambda-1}(y|\beta)} \times \frac{c_3^\lambda y^\lambda}{n^{\lambda/2}}\,\mathrm{d}y + \int_0^1 \frac{8y^2}{f_G^{\lambda-1}(y|\beta)} \times \frac{c_2^{\lambda/2}}{(nh)^{\lambda/2}}\,\mathrm{d}y \\ &\quad + \int_0^1 \frac{8y^2 h^{\lambda r} c_1^\lambda}{f_G^{\lambda-1}(y|\beta)}\,\mathrm{d}y \\ &= \sum_{i=1}^5 \mathbf{I}_i(n). \end{aligned} \tag{4.3}$$

Note that for each $0<y<1$, under Assumption [A1]

$$\begin{aligned} f_G(y|\beta) &= \int_0^y \frac{m}{\beta} \mathrm{e}^{-m(y-\theta)/\beta} g(\theta)\,\mathrm{d}\theta \geqslant g(1) \int_0^y \frac{m}{\beta} \mathrm{e}^{-m(y-\theta)/\beta}\,\mathrm{d}\theta \\ &= g(1)[1 - \mathrm{e}^{-my/\beta}] \geqslant \frac{g(1)my}{3\beta} = c_5 y, \end{aligned} \tag{4.4}$$

where $c_5 = mg(1)/3\beta$.

Substituting (4.4) into $\mathbf{I}_1(n)$, we have

$$\mathbf{I}_1(n) \leqslant \frac{\exp(-\lambda n c_4/2)}{n^{\lambda/2}} \int_0^1 \frac{2y^{2-\lambda}}{y^{\lambda-1} c_5^{\lambda-1}}\,\mathrm{d}y = \frac{\exp(-\lambda n c_4/2)}{n^{\lambda/2} c_5^{\lambda-1}(2-\lambda)}. \tag{4.5}$$

Also, since $0 \leqslant \alpha_G(y|\beta)/f_G(y|\beta) \leqslant y$, we have,

$$\mathbf{I}_2(n) = \frac{2}{n^{\lambda/2}} \int_0^1 \frac{y^{2-\lambda}}{f_G^{\lambda/2-1}(y|\beta)} \left(\frac{\alpha_G(y|\beta)}{f_G(y|\beta)}\right)^{\lambda/2} \mathrm{d}y \leqslant \frac{2}{n^{\lambda/2}} \int_0^1 y^{2-\lambda} y^{\lambda/2} f_G^{1-\lambda/2}(y|\beta)\,\mathrm{d}y$$
$$\leqslant \frac{2}{n^{\lambda/2}} \int_0^1 f_G^{1-\lambda/2}(y|\beta)\,\mathrm{d}y \leqslant \frac{2}{n^{\lambda/2}}, \tag{4.6}$$

$$\mathbf{I}_3(n) \leqslant \frac{2c_3^{\lambda}}{n^{\lambda/2}} \int_0^1 \frac{y^2}{(c_5 y)^{\lambda-1}}\,\mathrm{d}y \leqslant \frac{2c_3^{\lambda}}{n^{\lambda/2} c_5^{\lambda-1}}, \tag{4.7}$$

$$\mathbf{I}_4(n) \leqslant \frac{8c_2^{\lambda/2}}{(nh)^{\lambda/2}} \int_0^1 \frac{y^2}{(c_5 y)^{\lambda-1}}\,\mathrm{d}y \leqslant \frac{8c_2^{\lambda/2}}{(nh)^{\lambda/2} c_5^{\lambda-1}}, \tag{4.8}$$

$$\mathbf{I}_5(n) \leqslant 8h^{\lambda r} c_1^{\lambda} \int_0^1 \frac{y^2}{(c_5 y)^{\lambda-1}}\,\mathrm{d}y \leqslant \frac{8h^{\lambda r} c_1^{\lambda}}{c_5^{\lambda-1}}. \tag{4.9}$$

Combining (4.4)–(4.9), it follows that

$$\mathbf{I}(n) = \mathrm{O}\left(\frac{\exp(-\lambda n c_4/2)}{n^{\lambda/2}(2-\lambda)}\right) + \mathrm{O}((nh)^{-\lambda/2}) + \mathrm{O}(h^{\lambda r}). \tag{4.10}$$

Under Assumption [A.1], for each $1 \leqslant y \leqslant b$,

$$f_G(y|\beta) = \int_0^y \frac{m}{\beta} \mathrm{e}^{-m(y-\theta)/\beta} g(\theta)\,\mathrm{d}\theta = \frac{m}{\beta} \mathrm{e}^{-my/\beta} \int_0^y \mathrm{e}^{m\theta/\beta}\,\mathrm{d}G(\theta)$$
$$\geqslant \frac{m}{\beta} \mathrm{e}^{-mb/\beta} \int_0^1 \mathrm{e}^{m\theta/\beta}\,\mathrm{d}G(\theta) \equiv c_6 > 0. \tag{4.11}$$

Taking $\lambda = 2$, and substituting (4.2) into $\mathbf{II}(n)$, again it follows from Lemma A.1 that

$$\mathbf{II}(n) \leqslant \int_1^b \frac{2}{f_G(y|\beta)} \times \frac{\exp(-nc_4)}{n}\,\mathrm{d}y + \int_1^b \frac{2}{f_G(y|\beta)} \times \frac{\alpha_G(y|\beta)}{n}\,\mathrm{d}y + \int_1^b \frac{2}{f_G(y|\beta)} \times \frac{c_3^2 y^2}{n}\,\mathrm{d}y$$
$$+ \int_1^b \frac{8y^2}{f_G(y|\beta)} \times \frac{c_2}{nh}\,\mathrm{d}y + \int_1^b \frac{8y^2 h^{2r} c_1^2}{f_G(y|\beta)}\,\mathrm{d}y$$
$$= \sum_{i=1}^{5} \mathbf{II}_i(n). \tag{4.12}$$

By (4.11),

$$\mathbf{II}_1(n) \leqslant \frac{2b\exp(-nc_4)}{nc_6}, \tag{4.13}$$

$$\mathbf{II}_2(n) = \frac{1}{n} \int_1^b \frac{2\alpha_G(y|\beta)}{f_G(y|\beta)}\,\mathrm{d}y = \frac{1}{n} \int_1^b 2y\,\mathrm{d}y \leqslant \frac{b^2}{n}, \tag{4.14}$$

$$\mathbf{II}_3(n) \leqslant \frac{2c_3^2}{nc_6} \int_1^b y^2\,\mathrm{d}y \leqslant \frac{c_3^2 b^3}{nc_6}, \tag{4.15}$$

$$\mathbf{II}_4(n) \leqslant \frac{8c_2}{nhc_6} \int_1^b y^2\,\mathrm{d}y \leqslant \frac{3c_2 b^3}{nhc_6}, \tag{4.16}$$

$$\mathbf{II}_5(n) \leqslant \frac{c_1^2 h^{2r}}{c_6} \int_1^b 8y^2\,\mathrm{d}y \leqslant \frac{3h^{2r} c_1^2 b^3}{c_6}. \tag{4.17}$$

Combining (4.12)–(4.17) it yields that

$$\mathbf{II}(n) = \mathrm{O}((nh)^{-1}) + O(h^{2r}). \tag{4.18}$$

Let $c_7 = 1 - F_G(b|\beta)$. Thus,

$$\begin{aligned}\mathbf{III}(n) &= E_n[\varphi_n^*(b) - \varphi_G(b)]^2 c_7 = E_n\left[\left(\frac{\alpha_n(b)}{f_n(b)} \vee 0\right) \wedge b - \frac{\alpha_G(b|\beta)}{f_G(b|\beta)}\right]^2 c_7 \\ &\leqslant \frac{2}{f_G^2(b|\beta)}\{E_n[\alpha_n(b) - \alpha_G(b|\beta)]^2 + 4b^2 E_n[f_n(b) - f_G(b|\beta)]^2\} \\ &\leqslant \frac{2}{c_6^2}\left[\frac{\exp(-nc_4)}{n} + \frac{\alpha_G(b|\beta)}{n} + \frac{c_3^2 b^2}{n} + \frac{b^2 c_2}{nh} + b^2 c_1^2 h^{2r}\right] \\ &= \mathrm{O}((nh)^{-1}) + \mathrm{O}(h^{2r}).\end{aligned} \tag{4.19}$$

Taking (4.1), (4.10), (4.18) and (4.19) together it leads to $R(G, \varphi_n^*) - R(G, \varphi_G) = \mathrm{O}(\exp(-\lambda n c_4/2)/n^{\lambda/2}(2-\lambda)) + \mathrm{O}((nh)^{-\lambda/2}) + \mathrm{O}(h^{\lambda r})$. If we choose $h = n^{-1/(2r+1)}$ and $\lambda = \lambda_n = 2(1 - 1/\ln n)$, then, $R(G, \varphi_n^*) - R(G, \varphi_G) = \mathrm{O}(n^{-2r/(2r+1)})$. $\square$

5. Concluding remark

Both Singh and Prasad (1989) and Prasad and Singh (1990) have investigated empirical Bayes estimators for the guarantee lifetime parameter θ in the two-parameter exponential distributions when β is known. Prasad and Singh (1990) have proved that their proposed empirical Bayes estimator φ_n^{PS} possesses the asymptotic optimality; however, the corresponding rate of convergence has not been studied. Singh and Prasad (1989) have studied an empirical Bayes estimator φ_n^{SP}. Under the assumption that $f_G(y|\beta)$ is r-times differentiable and $|\theta| \leqslant a$ for some known finite a (see Singh and Prasad, 1989; Prasad and Singh, 1990), they have shown that φ_n^{SP} is asymptotically optimal with a rate

$$R(G, \varphi_n^{\mathrm{SP}}) - R(G, \varphi_G) = \mathrm{O}((n^{-r/(r+1)} \ln \ln n)^{1/2-\varepsilon}) \tag{5.1}$$

for $\varepsilon > 0$. From (5.1), it can be seen that the best possible rate for φ_n^{SP} is $\mathrm{O}((n^{-1} \ln \ln n)^{1/2})$ when r is sufficiently large. In this paper, we extend this empirical Bayes estimation problem to the case that β is unknown. The achievable rate of convergence of our proposed empirical Bayes estimator φ_n^* is $\mathrm{O}(n^{-2r/(2r+1)})$, which is faster than that of φ_n^{SP}.

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Appendix A. Appendix A

Lemma A.1. *Suppose Assumptions* [A1]–[A2] *hold. Then,*

(a) $|E_n f_n(y) - f_G(y|\beta)| \leqslant c_1 h^r$, *where* $c_1 = (k_0/r!) \sup\{|f_G^{(r)}(y|\beta)|; y > 0\} < \infty$.
(b) $Var(f_n(y)) \leqslant c_2/nh$, *where* $c_2 = mk_0^2/\beta$.
(c) $|E_n \alpha_n(y) - \alpha_G(y|\beta)| \leqslant y c_3/\sqrt{n}$ *where* $c_3 = m\sqrt{2}/\beta$.
(d) $Var(\alpha_n(y)) \leqslant \exp(-nc_4)/n + \alpha_G(y|\beta)/n$, *where* $c_4 = (m-1)(1 - \ln 2)$.
For $0 < \lambda \leqslant 2$.
(e) $E_n[|f_n(y) - f_G(y|\beta)|^\lambda] \leqslant (c_2/nh)^{\lambda/2} + (c_1 h^r)^\lambda$.
(f) $E_n[|\alpha_n(y) - \alpha_G(y|\beta)|^\lambda] \leqslant (\exp(-nc_4)/n)^{\lambda/2} + (\alpha_G(y|\beta)/n)^{\lambda/2} + (yc_3/\sqrt{n})^\lambda$.

Proof. (e) and (f) can be obtained from (a), (b), (c) and (d) and an application of C_r-inequality. (a) and (b) can be obtained through a straightforward computation. It remains to show only parts (c) and (d).

(c) Note that β_n and $\underset{\sim}{Y}(n)$ are independent. Also, $|e^{-x} - e^{-y}| \leqslant |x - y|$ for $x>0$, $y>0$. Thus,

$$\begin{aligned}
&|E_n\alpha_n(y) - \alpha_G(y|\beta)| \\
&= \left|\int_0^y E_n e^{-m(y-t)/\beta_n}\, dF_G(t|\beta) - \int_0^y e^{-m(y-t)/\beta}\, dF_G(t|\beta)\right| \\
&\leqslant \int_0^y m(y-t) E_n\left[\left|\frac{1}{\beta_n} - \frac{1}{\beta}\right|\right] dF_G(t|\beta) \\
&\leqslant \int_0^y m(y-t)\frac{\sqrt{2}}{\beta\sqrt{n}}\, dF_G(t|\beta) \\
&\leqslant \frac{\sqrt{2}my}{\beta\sqrt{n}} = \frac{yc_3}{\sqrt{n}}.
\end{aligned}$$

(d) Note that $\alpha_n(y) = (1/n)\sum_{j=1}^n \exp(-m(y - Y_j)/\beta_n) I(y - Y_j)$. Recall that $Y_1, \ldots, Y_n$ and β_n are mutually independent. Thus, given β_n, $\exp(-m(y - Y_j)/\beta_n) I(y - Y_j)$, $j = 1, \ldots, n$, are iid. So, the conditional variance of $\alpha_n(y)$ given β_n is

$$\begin{aligned}
Var(\alpha_n(y)|\beta_n) &= \frac{1}{n} Var\left(\exp\left(\frac{-m(y - Y_j)}{\beta_n}\right) I(y - Y_j)|\beta_n\right) \\
&\leqslant \frac{1}{n} E_n\left[\exp\left(\frac{-2m(y - Y_j)}{\beta_n}\right) I(y - Y_j)|\beta_n\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
Var(\alpha_n(y)) &= E_n[Var(\alpha_n(y)|\beta_n)] \\
&\leqslant \frac{1}{n} E_n E_n\left[\exp\left(\frac{-2m(y - Y_j)}{\beta_n}\right) I(y - Y_j)|\beta_n\right] \\
&= \frac{1}{n} E_n\left[\exp\left(\frac{-2m(y - Y_j)}{\beta_n}\right) I(y - Y_j)\right] \\
&= \frac{1}{n}\int_0^y E_n[e^{-2m(y-t)/\beta_n}]\, dF_G(t|\beta),
\end{aligned}$$

where

$$\begin{aligned}
&E_n\left[\exp\left(\frac{-2m(y-t)}{\beta_n}\right)\right] \\
&= E_n\left[\exp\left(\frac{-2m(y-t)}{\beta_n}\right) I\left(\frac{1}{\beta} - \frac{2}{\beta_n}\right)\right] + E_n\left[\exp\left(\frac{-2m(y-t)}{\beta_n}\right) I\left(\frac{2}{\beta_n} - \frac{1}{\beta}\right)\right] \\
&\equiv A(y,n) + B(y,n).
\end{aligned}$$

For $0<t<y$,

$$\begin{aligned}
A(y,n) &\leqslant P\left\{\frac{1}{\beta} - \frac{2}{\beta_n} > 0\right\} = P\left\{\frac{\beta_n}{\beta} - 1 > 1\right\} \\
&\leqslant \exp(-n(m-1)[1 - \ln 2]) = \exp(-nc_4),
\end{aligned}$$

where the inequality is obtained by an application of Lemma 4.1 of Liang (1997) and the fact that $2n(m-1)\beta_n/\beta$ follows $\chi^2(2n(m-1))$;

$$B(y,n) = E_n\left[\exp\left(\frac{-2m(y-t)}{\beta_n}\right) I\left(\frac{2}{\beta_n} - \frac{1}{\beta}\right)\right] \leqslant \exp\left(\frac{-m(y-t)}{\beta}\right).$$

Therefore,

$$Var(\alpha_n(y)) \leqslant \frac{1}{n}\int_0^y \left[\exp(-nc_4) + \exp\left(\frac{-m(y-t)}{\beta}\right)\right] \mathrm{d}F_G(t|\beta)$$
$$\leqslant \frac{\exp(-nc_4)}{n} + \frac{\alpha_G(y|\beta)}{n}. \qquad \square$$

The following Lemma is from Singh (1977).

Lemma A.2. *For a pair of random variables* (Y, Z) *and real values* y, $z \neq 0$, $0 < c < \infty$ *and* $0 < \lambda \leqslant 2$,

$$E\left[\left|\frac{Y}{Z} - \frac{y}{z}\right| \wedge c\right]^{\lambda} \leqslant \frac{2}{|z|^{\lambda}}\left\{E\left[\left|Y - y\right|^{\lambda}\right] + \left(\left|\frac{y}{z}\right| + c\right)^{\lambda} E\left[|Z - z|^{\lambda}\right]\right\}.$$

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