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Fuzzy Flexibility and Product Variety in Lot-Sizing

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In terms of flexibility and product variety in lot-sizing systems of crisp cases, the average demand of per unit of time (m_j), the relative duration of setup (q_j), and the unit cost of production (c_j) are considered. Instead of using the usual method that the m_j, q_j, and c_j in the total cost function are respectively fuzzified by the triangular fuzzy numbers to derive fuzzy total cost, in this paper, we construct three different intervals to include m_j, q_j, and c_j, respectively, and then consider the fuzzification of the system from these three different intervals directly. And finally the fuzzy total cost is obtained. By applying respectively the signed distance and centroid method for defuzzification, two different total cost functions are obtained, and thus the respective optimal solutions are computed.

Keywords: fuzzy sets, signed distance, fuzzy total cost, lot-sizing, flexibility, centroid, product variety

1. INTRODUCTION

In early literature addressing flexibility and product variety in lot-sizing problems, Hadley and Whitin [2] proposed a useful multi-product capacitated EOQ model and provided the solution by using a Lagrangian algorithm. Parsons [10] first solved the problem in a closed-form in 1966. Recently, several studies have discussed flexibility and product variety in lot-sizing problem. Spence and Porteus [12] formulated a model of increased effective capacity resulting from reduced setup times, and they also considered overtime and lot-sizing, which is normally considered only in aggregate planning models. Xavier de Groote [1] performed a sensitivity analysis of the multi-product capacitated lot-sizing problems formulated by Hadley and Whitin [2]. The sensitivity is analyzed by many aggregate parameters that can be interpreted as a measure of the variety of the product line and the flexibility of the production process in which the definition of flexibility is also considered. Wang and Fang [13] proposed a novel fuzzy linear programming method for solving the aggregate production planning problem where the market demands and unit cost to subcontract are fuzzy in nature. Later, Wang and Fang [14] applied the same method to solve the aggregate production planning with multiple objectives, where fac-

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tors such as the product price and unit cost to subcontract are fuzzy in nature. Hsieh [3] developed two fuzzy production inventory models and applied an extension of the Lagrangean algorithm to solve the inequality constraint problems to find optimal solutions. Hsieh and Chiang [4] established a manufacturing-to-sale planning model by using possibility linear programming techniques to deal with uncertain manufacturing factors. However, while some discussion of the above models is constrained to the crisp cases, the others focus on fuzzy linear programming methods. Recently, the defuzzification problems of production inventory have been considered [8, 9]. In the total cost function of crisp production inventory [8], total demand and production quantity per day are fuzzified into triangular fuzzy sets to generate the fuzzy total cost function. After defuzzification using centroid, the estimator of total cost functions can be found, and thus is an optimal solution. In the total cost function of crisp production inventory [9], production quantity per cycle is fuzzified into trapezoidal fuzzy sets to compute the fuzzy total cost. The estimator of total cost function is also discerned employing centroid to defuzzify the fuzzy total cost; furthermore, an optimal solution is found as well. Both the works previously mentioned used fuzzification of total cost function of crisp case.

The average demand of per unit time (m_j), relative duration of setup (q_j), and unit cost of production (c_j) are fixed in flexibility and product variety in lot-sizing model [1, 12]. However, in reality, the average demand of per unit time, relative duration of setup, and unit cost of production may have some minor instability due to the uncertain nature of future production processes and fluctuations in demand. Therefore, this study applied fuzzy set theory as first proposed by Zadeh [16]; if the fuzzy average demand of per unit time, fuzzy relative duration of setup and fuzzy unit cost of production are expressed respectively as the neighborhood of the fixed average demand of per unit time, fixed relative duration of setup, and fixed unit cost of production, then it could better correlate with the real situation. This study presents fuzzy approaches to modify the model [1, 12]. Instead of usual point fuzzification method, we propose an interval fuzzification approach for this problem. So far, we have not found any related studies using this proposed method in this area.

In section 2, we quote some definitions and propositions which are used in section 3. In section 3, as mentioned in abstract; instead of the usual method, a new creative method is proposed for considering the fuzzification problem of flexibility and product variety in lot-sizing system of crisp case. Numerical examples are provided in section 4. In section 5, the comparison of using sign distance for defuzzification and that of centroid is discussed. In addition, the situation that the optimal solution of crisp case is the special condition of optimal solution of fuzzy case is discussed too. In section 6, the usual fuzzification method and the new creative method of this paper are compared. Also, the advantages of this new proposed method are addressed.

2. PRELIMINARIES

The essential definitions of fuzzy set below are used in section 3.

Definition 1 (Pu and Liu [11]) Let $\tilde{0}$ denote a fuzzy set on $R = (-\infty, \infty)$, then, $\tilde{0}$ is called a fuzzy point, if its membership function is defined in the following:

$$\mu_{\tilde{0}}(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

Definition 2 For $p < q$, $0 \le \lambda \le 1$, the fuzzy set $[p, q; \lambda]$ on R is labeled a level λ fuzzy interval if its membership function is defined in the following:

$$\mu_{[p,q;\lambda]}(x) = \begin{cases} \lambda, & p \le x \le q, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3 Let $\tilde{A}$ be a fuzzy set on R, then, $\tilde{A} = (a, b, c)$, $a < b < c$, denotes a triangular fuzzy set if its membership function is defined in the following:

$$\mu_{\tilde{A}}(x) = \begin{cases} (x-a)/(b-a), & a \le x \le b, \\ (c-x)/(c-b), & b \le x \le c, \\ 0, & \text{otherwise.} \end{cases}$$

By definition in Klir and Yuan [6], Centroid of $\tilde{A}$ is written in the following:

$$C(\tilde{A}) = \left[\int_{-\infty}^{\infty} x\mu_{\tilde{A}}(x)dx\right] \Big/ \left[\int_{-\infty}^{\infty} \mu_{\tilde{A}}(x)dx\right] = \frac{1}{3}(a+b+c). \tag{1}$$

Let F_T denote a family of all triangular fuzzy sets on R. Let $\tilde{A} = (a, b, c) \in F_T$, from decomposition theory, $\tilde{A}$ can be written as $\tilde{A} = \bigcup_{0 \le \lambda \le 1} \lambda I_{A(\lambda)}$, where $A(\lambda) = \{x \mid \mu_{\tilde{A}}(x) \ge \lambda\}$ is the λ-level set of $\tilde{A}$ and $I_{A(\lambda)}$ is the characteristic function of $A(\lambda)$, where $A(\lambda)$ can also be expressed as $A(\lambda) = [\tilde{A}_L(\lambda), \tilde{A}_U(\lambda)]$, $0 \le \lambda \le 1$, $\tilde{A}_L(\lambda) = a + (b-a)\lambda$, $\tilde{A}_U(\lambda) = c - (c - b)\lambda$.

From Definition 2, $\mu_{\lambda I_A(\lambda)}(x) = \mu_{[\tilde{A}_L(\lambda), \tilde{A}_U(\lambda); \lambda]}(x)$ $\forall x \in R$. Hence,

$$\tilde{A} = \bigcup_{0 \le \lambda \le 1} [\tilde{A}_L(\lambda), \tilde{A}_U(\lambda); \lambda]. \tag{2}$$

From Yao and Wu [15], the signed distance of fuzzy set $\tilde{A} = (a, b, c)$ on F_T is expressed in the following.

Definition 4 Let $a, 0 \in R$, then $d_0(a, 0) = a$ is defined as the signed distance of a measured from the origin 0.

If $a > 0$, the distance from a to 0 is $d_0(a, 0) = a$. Similarly, if $a < 0$, the distance from a to 0 is $-d_0(a, 0) = -a$. Hence, $d_0(a, 0) = a$ is named the signed distance of a from 0.

Let $\tilde{A} = (a, b, c) \in F_T$, for each $\lambda \in [0, 1]$, $A(\lambda) = [\tilde{A}_L(\lambda), \tilde{A}_U(\lambda)]$ is the λ-level set of $\tilde{A}$. From Definition 4, signed distance of interval $[\tilde{A}_L(\lambda), \tilde{A}_U(\lambda)]$ to 0 is defined by $d_0([\tilde{A}_L(\lambda), \tilde{A}_U(\lambda)], 0) = \frac{1}{2}(\tilde{A}_L(\lambda) + \tilde{A}_U(\lambda)) = \frac{1}{2}[a + c + (2b - a - c)\lambda]$. Since for all λ, $[\tilde{A}_L(\lambda), \tilde{A}_U(\lambda)] \leftrightarrow [\tilde{A}_L(\lambda), \tilde{A}_U(\lambda); \lambda]$ is a one-to-one mapping relationship, and $0 \leftrightarrow \tilde{0}$. Thus, signed distance of $[\tilde{A}_L(\lambda), \tilde{A}_U(\lambda); \lambda]$ measured from $\tilde{0}$ can be defined as

$$d([\tilde{A}_L(\lambda), \tilde{A}_U(\lambda); \lambda], \tilde{0}) = d_0([\tilde{A}_L(\lambda), \tilde{A}_U(\lambda)], 0) = \tfrac{1}{2}(a + c + (2b - a - c)\lambda), 0 \le \lambda \le 1. \tag{3}$$

The mean of Eq. (3) is calculated by applying the integration and from Eq. (2), we have the following definition.

Definition 5 Let $\tilde{A} = (a, b, c) \in F_T$, the signed distance of $\tilde{A}$ measured from $\tilde{0}$ is defined as

$$d(\tilde{A}, \tilde{0}) = \frac{1}{4}(2b + a + c) = \left(\frac{1}{2}\int_0^1 [\tilde{A}_L(\lambda) + \tilde{A}_U(\lambda)]d\lambda\right). \tag{4}$$

From Klir and Bo Yuan [6, 7], four basic arithmetic operations on fuzzy numbers are used throughout the paper, *i.e.*, +, –, ·, / are denoted as addition, subtraction, multiplication, and division, respectively. Kaufmann and Gupta [5] propose the following.

Proposition 1 Let $\tilde{A} = (a, b, c)$, $\tilde{B} = (p, q, r) \in F_T$, $k \in R$, then we have

(1) $\tilde{A} + \tilde{B} = (a + p, b + q, c + r) \in F_T$,

(2) $k\tilde{A} = \begin{cases} (ka, kb, kc) & \text{if } k > 0, \\ (kc, kb, ka) & \text{if } k < 0, \ k\tilde{A} \in F_T. \end{cases}$

From Proposition 1 and Definition 5, the following proposition can be obtained.

Proposition 2 Let $\tilde{A}$, $\tilde{B} \in F_T$, $k \in R$, then,

(1) $d(\tilde{A} + \tilde{B}, \tilde{0}) = d(\tilde{A}, \tilde{0}) + d(\tilde{B}, \tilde{0})$,
(2) $d(k\tilde{A}, \tilde{0}) = kd(\tilde{A}, \tilde{0})$.

The following ranking of fuzzy numbers on F_T is defined in [15] in the following.

Definition 6 Let $\tilde{A} = (a, b, c)$, $\tilde{B} = (p, q, r)$ both belong to F_T, the following ordering is defined.

$\tilde{A} \prec \tilde{B}$ if and only if $d(\tilde{A}, \tilde{0}) < d(\tilde{B}, \tilde{0})$,
$\tilde{A} \approx \tilde{B}$ if and only if $d(\tilde{A}, \tilde{0}) = d(\tilde{B}, \tilde{0})$.

Proposition 3 For arbitrary $\tilde{A}, \tilde{B}, \tilde{C} \in F_T$, the following properties follow:

(1) Law of trichotomy: Exactly one and only one of the relations $\tilde{A} \prec \tilde{B}$, $\tilde{B} \prec \tilde{A}$ or $\tilde{A} \approx \tilde{B}$ holds.
(2) Law of reflexivity: $\tilde{A} \prec\approx \tilde{A}$ holds.
(3) Law of antisymmetry: $\tilde{A} \prec\approx \tilde{B}$ and $\tilde{B} \prec\approx \tilde{A}$ imply $\tilde{A} \approx \tilde{B}$.
(4) Law of transitivity: $\tilde{A} \prec\approx \tilde{B}, \tilde{B} \prec\approx \tilde{C}$ imply $\tilde{A} \prec\approx \tilde{C}$.

From Proposition 3, "$\prec, \approx$" is the linear order of F_T.

Definition 7 For $\tilde{A}_k = 1, 2, \ldots, n, \in F_T$, define $\underset{1\le k\le n}{\text{Min}}\ \tilde{A}_k = \tilde{A}_j$, $j \in \{1, 2, \ldots, n\}$, if and only if $\tilde{A}_j \prec\approx \tilde{A}_k\ \forall k \in \{1, 2, \ldots, n\}$, or equivalently $d(\tilde{A}_j, \tilde{0}) \le d(\tilde{A}_k, \tilde{0})\ \forall k \in \{1, 2, \ldots, n\}$.

From Kaufmann and Gupta [5], the following interval operations exist.

$$a < b,\ c < d,$$
$$[a, b] + [c, d] = [a + c, b + d],$$
$$k[a, b] = \begin{cases} [ka, kb] & \text{if } k > 0, \\ [kb, ka] & \text{if } k < 0. \end{cases}$$

If $0 \le a < b$ and $0 \le c < d$, then

$$[a, b] \times [c, d] = [ac, bd]. \tag{5}$$

For $\tilde{A} = (a, b, c) \in F_T$, let the midpoint of interval $[a, c]$ be represented by $M = \frac{1}{2}(a + c)$, then from Eq. (1), Centroid of $\tilde{A}$ is $C(\tilde{A}) = \frac{1}{3}(a + b + c)$; from Eq. (4), the signed distance of $\tilde{A}$ is $d(\tilde{A}, \tilde{0}) = \frac{1}{4}(2b + a + c)$. This results in the following.

$$M - C(\tilde{A}) = \frac{1}{3}(M - b),\ C(\tilde{A}) - d(\tilde{A}, \tilde{0}) = \frac{1}{6}(M - b),\ d(\tilde{A}, \tilde{0}) - b = \frac{1}{2}(M - b).$$

Consider the following cases:

Case 1: If $b < M$, then $b < d(\tilde{A}, \tilde{0}) < C(\tilde{A}) < M$ (in Fig. 1).
Case 2: If $M < b$, then $M < C(\tilde{A}) < d(\tilde{A}, \tilde{0}) < b$ (in Fig. 2).
Case 3: If $M = b$, then $M = C(\tilde{A}) = d(\tilde{A}, \tilde{0}) = b$.

Figs. 1 and 2 show that in cases 1 and 2, $\mu_{\tilde{A}}(C(\tilde{A})) < \mu_{\tilde{A}}(d(\tilde{A}, \tilde{0})) < \mu_{\tilde{A}}(b) = 1$. In case 3, $\mu_{\tilde{A}}(C(\tilde{A})) = \mu_{\tilde{A}}(d(\tilde{A}, \tilde{0})) = \mu_{\tilde{A}}(b) = 1$. Hence, the following proposition is found.

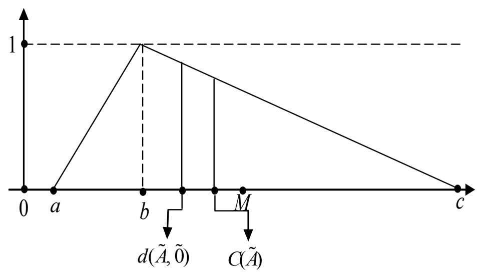

Fig. 1. Case 1.

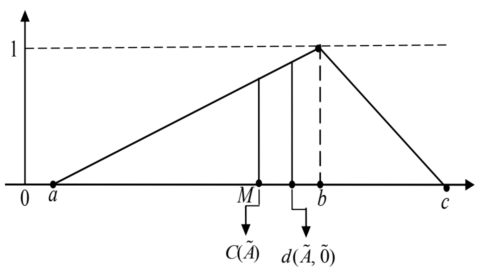

Fig. 2. Case 2.

Proposition 4 For triangular fuzzy sets $\tilde{A} = (a, b, c) \in F_T$, according to the principle of maximum membership grade, the method based on signed distance to defuzzy $\tilde{A}$ triangular fuzzy number is better than that of the centroid method.

Let p, q, $r \in R$ and $p < q < r$, q is an any fixed point in $[p, r]$, corresponding to the interval $[p, r]$, q can consider for fuzzification as the triangular fuzzy number $\tilde{q}$ in the following. Decision maker takes a point from the interval $[p, r]$, if the point is q, the error between the point and fixed point q is zero. Based on the confidence level concept; if the error is zero, then the confidence level is the maximum value and set to 1. If the point is taken from the interval $[p, q)$, when the point moves away from q, then the error between the point and q becomes larger, *i.e.*, the confidence level becomes smaller. Additionally, if the point is equal to p, the confidence level attains the minimum value and thus set to 0. Similarly, If the point is taken from the interval $(q, r]$, when the point moves away from q, the confidence level becomes smaller. Additionally, if the point is equal to r, the confidence level attains to 0. Hence, corresponding to the interval $[p, r]$, the following triangular fuzzy number $\tilde{q}$ is set.

In Fig. 3, it shows that the membership grade is 1 when triangular fuzzy number $\tilde{q}$ locates at q. However, in the interval of $[p, q)$ or $(q, r]$, the membership grade decreases when $\tilde{q}$ moves away from q. The membership grade is 0 when $\tilde{q}$ takes exactly either one points of p or r. Therefore, the membership grade shares the same properties of confidence level. If we view confidence level as the membership grade, corresponding to the interval $[p, r]$, it is reasonable to set triangular fuzzy number $\tilde{q}$. Hence, we have the following proposition

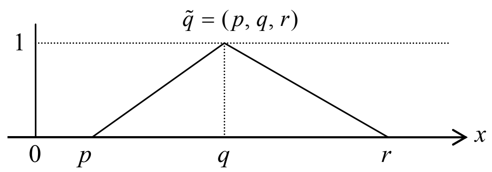

Fig. 3. Triangular fuzzy number $\tilde{q}$.

Proposition 5 Let p, q, $r \in R$ and $p < q < r$, q is an any fixed point in $[p, r]$, then corresponding to the interval $[p, r]$, q can be fuzzified into the triangular fuzzy number $\tilde{q} = (p, q, r)$.

3. FUZZY FLEXIBILITY AND PRODUCT VARIETY IN LOT-SIZING

3.1 Crisp Case

To formulate the flexibility and product variety in lot-sizing from Groote [1], the following notations are applied for each product j, $j \in \{1, 2, \ldots, n\}$.

n: number of product types in manufacturing,
Q_j: product lot-size ($Q_j > 0$),
m_j: average demand (per unit of time),
r_j: finite production rate,
p: fraction of time in which the facility is available for processing ($0 \le p \le 1$),
S: nominal setup time,

c_S: direct setup cost (per unit of setup time),
f: fixed cost (per unit of time),
q_j: relative duration of setup,
c_j: unit cost of production (labor and material),
i: opportunity cost of capital per unit of time,

$$\alpha = \sum_{j=1}^{n} (m_m / r_j).$$

The cost of the j^{th} product is written in the following:

$$F(Q_j; m_j, q_j, c_j) = \frac{1}{Q_j} c_S S m_j q_j + \frac{1}{2} i c_j Q_j + c_j m_j, \; j = 1, 2, \ldots, n. \tag{6}$$

The total cost of flexibility and product variety in lot-sizing model can be expressed as:

$$\sum_{j=1}^{n} F(Q_j; m_j, q_j, c_j) + f. \tag{7}$$

Note 1. The detail of the fixed cost f is considered in section 5.3.

Groote [1] formulated the problem in the following.

$$\operatorname*{Min}_{\bar{Q}} \sum_{j=1}^{n} F(Q_j; m_j, q_j, c_j), \tag{8}$$

subject to $\sum_{j=1}^{n} \frac{m_j q_j}{Q_j} \le \frac{p-\alpha}{S}$, where $\bar{Q} = \{Q_j > 0, j = 1, 2, \ldots, n\}$. (9)

The solution to problems (in Eqs. (8) and (9)) has been derived by Parsons [10]. It simply applies the Kuhn-Tucker theorem. The optimal cost (per unit time) is given by Spence and Porteus [12] in the following.

$$\begin{cases} \sum_{j=1}^{n} \sqrt{2 c_S S i c_j m_j q_j} + \sum_{j=1}^{n} c_j m_j + f, \text{ when } \sum_{j=1}^{n} \sqrt{\frac{i c_j m_j q_j}{2 c_S S}} \le \frac{p-\alpha}{S}; \\ \frac{\left(\sum_{j=1}^{n} \sqrt{2 S i c_j m_j q_j} \right)^2}{4(p-\alpha)} + c_S (p-\alpha) + \sum_{j=1}^{n} c_j m_j + f, \text{ otherwise.} \end{cases} \tag{10}$$

3.2 Fuzzy Problem and Optimal Solution without Fuzzification of Crisp Total Cost Function $F(Q_j; m_j, q_j, c_j)$

This section derives the fuzzy problem without fuzzification of parameters in crisp total cost function $F(Q_j; m_j, q_j, c_j)$, which differs from the fuzzification of $F(Q_j; m_j, q_j, c_j)$

by $F(Q_j; \tilde{m}_j, \tilde{q}_j, \tilde{c}_j)$, where $\tilde{m}_j, \tilde{q}_j, \tilde{c}_j$ are triangular fuzzy numbers. The following problems are considered for each product j, $j \in \{1, 2, \ldots, n\}$. In perfectly competitive markets, the estimate of average demand m_j per unit time that is achieved from the past data may fluctuate most of time when present or future data are applied. Hence, the estimate can be written as "average demand per unit of time in the neighborhood of m_j" (fuzzy language). Therefore, the average demand per unit of time which located in the interval $[m_j - \Delta_{j11}, m_j + \Delta_{j12}]$ should be considered. Similarly, deciding on a value q_j relative duration of setup is usually more difficult than that considering the relative duration of setup locating in the interval $[q_j - \Delta_{j21}, q_j + \Delta_{j22}]$. Therefore, this study considers the relative duration of setup locating in the interval $[q_j - \Delta_{j21}, q_j + \Delta_{j22}]$ and unit cost of production locating in the interval of $[c_j - \Delta_{j31}, c_j + \Delta_{j32}]$. The decision maker takes reasonable Δ_{jtk}, $t = 1, 2, 3$, $k = 1, 2$ which satisfy the following conditions.

$$0 < \Delta_{j11} < m_j,\ 0 < \Delta_{j21} < q_j,\ 0 < \Delta_{j31} < c_j,\ 0 < \Delta_{jt2}\ ,\ t = 1, 2, 3. \tag{11}$$

Henceforth, we always take $j \in \{1, 2, \ldots, n\}$, $t = 1, 2, 3$, $k = 1, 2$. For each product j, Q_j represent unknown decision variables, m_j, q_j, c_j, r_j and c_S, S, i, p, f are known parameters. For any average demand m_j^* in the interval of $[m_j - \Delta_{j11}, m_j + \Delta_{j12}]$, any relative duration of setup q_j^* in $[q_j - \Delta_{j21}, q_j + \Delta_{j22}]$ and any unit cost of production of setup c_j^* in $[c_j - \Delta_{j31}, c_j + \Delta_{j32}]$, Eq. (6) implies that for each product j and Q_j, the following equation of cost for product j is given by

$$F(Q_j; m_j^*, q_j^*, c_j^*) = \frac{1}{Q_j} c_S S m_j^* q_j^* + \frac{1}{2} i c_j^* Q_j + c_j^* m_j^*. \tag{12}$$

For product j, because

$$m_j - \Delta_{j11} \le m_j^* \le m_j + \Delta_{j12},\ q_j - \Delta_{j21} \le q_j^* \le q_j + \Delta_{j22},\ c_j - \Delta_{j31} \le c_j^* \le c_j + \Delta_{j32},$$
$$\text{and } c_S > 0,\ S > 0,\ Q_j > 0,\ i > 0. \tag{13}$$

So, for each Q_j, $F(Q_j; m_j^*, q_j^*, c_j^*)$ is thus in the following interval.

$$[F(Q_j; m_j - \Delta_{j11}, q_j - \Delta_{j21}, c_j - \Delta_{j31}), F(Q_j; m_j + \Delta_{j12}, q_j + \Delta_{j22}, c_j + \Delta_{j32})]. \tag{14}$$

Since $m_j \in [m_j - \Delta_{j11}, m_j + \Delta_{j12}]$, $q_j \in [q_j - \Delta_{j21}, q_j + \Delta_{j22}]$, $c_j \in [c_j - \Delta_{j31}, c_j + \Delta_{j32}]$, thus the crisp case's Eq. (6) $F(Q_j; m_j, q_j, c_j) \in$ interval Eq. (14). If we put $q = F(Q_j; m_j, q_j, c_j)$ in Proposition 5, then from Proposition 5, corresponding to the interval Eq. (14), the triangular fuzzy number $\tilde{F}(Q_j)$ is given by

$$\tilde{F}(Q_j) = (F(Q_j; m_j - \Delta_{j11}, q_j - \Delta_{j21}, c_j - \Delta_{j31}), F(Q_j; m_j, q_j, c_j), F(Q_j; m_j + \Delta_{j12}, q_j + \Delta_{j22}, c_j + \Delta_{j32})) \in F_T, \tag{15}$$

where $\tilde{F}(Q_j)$ is called fuzzy total cost.

3.2.1 Defuzzification of fuzzy total cost $\tilde{F}(Q_j)$ based on signed distance

Using the principle of maximum membership grade and by Proposition 4, signed distance is clearly better than centroid for defuzzfication the triangular fuzzy set Eq. (15). Therefore, from Definition 5, if we let $a = F(Q_j; m_j - \Delta_{j11}, q_j - \Delta_{j21}, c_j - \Delta_{j31})$, $b = F(Q_j; m_j, q_j, c_j)$, and $c = F(Q_j; m_j + \Delta_{j12}, q_j + \Delta_{j22}, c_j + \Delta_{j32})$ (in Eq. (4)), then for each product j, we have

$$F^*(Q_j; \Delta_{jtk}) \equiv d(\tilde{F}(Q_j), \tilde{0}) = F(Q_j; m_j, q_j, c_j) + \frac{1}{4}[R_2(Q_j, \Delta_{jt2}) - R_1(Q_j; \Delta_{jt1})], \tag{16}$$

where

$$R_1(Q_j; \Delta_{jt1}) = \frac{1}{Q_j} c_s S(m_j\Delta_{j21} + q_j\Delta_{j11} - \Delta_{j11}\Delta_{j21}) + \frac{1}{2} i\Delta_{j31}Q_j + (c_j\Delta_{j11} + m_j\Delta_{j31} - \Delta_{j11}\Delta_{j31}), \tag{17}$$

$$R_2(Q_j; \Delta_{jt2}) = \frac{1}{Q_j} c_s S(m_j\Delta_{j22} + q_j\Delta_{j12} + \Delta_{j12}\Delta_{j22}) + \frac{1}{2} i\Delta_{j32}Q_j + (c_j\Delta_{j12} + m_j\Delta_{j32} + \Delta_{j12}\Delta_{j32}). \tag{18}$$

Since $c_S > 0$, $S > 0$, $s > 0$, and $i > 0$, it follows from Eqs. (11) and (16) that $F^*(Q_j; \Delta_{jtk})$ is positive for all j, t, k and therefore it is an estimate of cost for product j in the fuzzy sense derived by signed distance.

From Proposition 2 and Eq. (16), we see that $d\left(\sum_{j=1}^{n} \tilde{F}(Q_j), \tilde{0}\right) = \sum_{j=1}^{n} F^*(Q_j; \Delta_{jtk})$ $= \sum_{j=1}^{n} F(Q_j; m_j, q_j, c_j) + \frac{1}{4}\sum_{j=1}^{n}[R_2(Q_j; \Delta_{jt2}) - R_1(Q_j; \Delta_{jt1})]$. Hence, the total cost of flexibility and product variety in lot-sizing of fuzzy case are given in the following (similar as Eq. (7)).

$$\sum_{j=1}^{n} F^*(Q_j; \Delta_{jtk}) + f. \tag{19}$$

The value $m_j q_j$ (in Eq. (9)) is replaced by $[m_j - \Delta_{j11}, m_j + \Delta_{j12}] \times [q_j - \Delta_{j21}, q_j + \Delta_{j22}] = [(m_j - \Delta_{j11})(q_j - \Delta_{j21}), (m_j + \Delta_{j12})(q_j + \Delta_{j22})]$ (in Eqs. (5) and (11)), for each j, $0 < \frac{1}{Q_j}(m_j - \Delta_{j11})(q_j - \Delta_{j21}) < \frac{1}{Q_j}(m_j + \Delta_{j12})(q_j + \Delta_{j22})$. Hence, Eq. (9) can be rewritten as $\sum_{j=1}^{n}\frac{1}{Q_j}(m_j + \Delta_{j12})(q_j + \Delta_{j22}) \le \frac{p-\alpha}{S}$. Finally, the following theorem can be concluded.

Theorem 1 The flexibility and product variety in lot-sizing of fuzzy case based on signed distance can be written in the following.

$$\underset{Q}{\text{Min}} \sum_{j=1}^{n} F^*(Q_j; \Delta_{jtk}), \tag{20}$$

subject to $\sum_{j=1}^{n} \frac{1}{Q_j}(m_j + \Delta_{j12})(q_j + \Delta_{j22}) \le \frac{p-\alpha}{S}$. (21)

Remark 1: By Definition 6, Definition 7, Proposition 3 and Eq. (15), Eq. (16), Eq. (20), thereby we obtain $\underset{\bar{Q}}{\text{Min}} \sum_{j=1}^{n} F^*(Q_j; \Delta_{jtk}) = \underset{\bar{Q}}{\text{Min}} \sum_{j=1}^{n} d(\tilde{F}(Q_j), \tilde{0})$ which is equivalent to $\underset{\bar{Q}}{\text{Min}} \sum_{j=1}^{n} \tilde{F}(Q_j)$, where $\sum_{j=1}^{n} \tilde{F}(Q_j)$ represents $\tilde{F}(Q_1) + \tilde{F}(Q_2) + \ldots + \tilde{F}(Q_n)$.

The optimal solution of Theorem 1 is considered below:

Applying Eq. (6), Eqs. (17) and (18), for each product j, $F^*(Q_j; \Delta_{jtk})$ (in Eq. (16)) can be recast in the following

$$F^*(Q_j; \Delta_{jtk}) = A_j \frac{1}{Q_j} + B_j Q_j + C_j, \tag{22}$$

where

$$\begin{aligned}
A_j &\equiv c_S S[m_j q_j + \frac{1}{4} m_j(\Delta_{j22} - \Delta_{j21}) + \frac{1}{4} q_j(\Delta_{j12} - \Delta_{j11}) + \frac{1}{4}(\Delta_{j12}\Delta_{j22} + \Delta_{j11}\Delta_{j21})] \\
&= c_S S a_j, \\
B_j &\equiv \frac{1}{2} i[c_j + \frac{1}{4}(\Delta_{j32} - \Delta_{j31})] = \frac{1}{2} i b_j, \\
C_j &\equiv c_j m_j + \frac{1}{4} c_j(\Delta_{j12} - \Delta_{j11}) + \frac{1}{4} m_j(\Delta_{j32} - \Delta_{j31}) + \frac{1}{4}(\Delta_{j12}\Delta_{j32} + \Delta_{j11}\Delta_{j31})], \\
a_j &\equiv m_j q_j + \frac{1}{4} m_j(\Delta_{j22} - \Delta_{j21}) + \frac{1}{4} q_j(\Delta_{j12} - \Delta_{j11}) + \frac{1}{4}(\Delta_{j12}\Delta_{j22} + \Delta_{j11}\Delta_{j21}), \\
b_j &= c_j + \frac{1}{4}(\Delta_{j32} - \Delta_{j31}).
\end{aligned} \tag{23}$$

From Eq. (11), for each product j,

$$A_j > 0, B_j > 0, C_j > 0, a_j > 0, b_j > 0. \tag{24}$$

In Theorem 1, Eqs. (20) and (21) can be written in the following via Eqs. (22) and (23).

$$\begin{aligned}
\underset{\bar{Q}}{\text{Min}} \sum_{j=1}^{n} F^*(Q_j; \Delta_{jtk}) &= \underset{\bar{Q}}{\text{Min}} \sum_{j=1}^{n} (A_j \frac{1}{Q_j} + B_j Q_j + C_j) \\
&= \underset{\bar{Q}}{\text{Min}} \sum_{j=1}^{n} [c_S S a_j \frac{1}{Q_j} + \frac{1}{2} i b_j Q_j + C_j]
\end{aligned} \tag{25}$$

subject to $$\sum_{j=1}^{n} \frac{1}{Q_j}(m_j + \Delta_{j12})(q_j + \Delta_{j22}) \leq \frac{p - \alpha}{S}. \tag{26}$$

Theorem 2 The optimal solution of flexibility and product variety in lot-sizing of fuzzy case of Theorem 1 (Eqs. (20) and (21)) or (Eqs. (25) and (26)) is given in the following respective conditions.

(1) $\frac{iSD^2}{(P-\alpha)^2} < c_S$, where $D = \sum_{j=1}^{n} \left(\frac{b_j}{2a_j} \right)^{1/2} (m_j + \Delta_{j12})(q_j + \Delta_{j22})$.

For each product j, the optimal lot-size is given by $Q_j^{(0)} = \left[\frac{2c_S Sa_j}{ib_j} \right]^{1/2}$, and its minimum total cost is $F^{(0)} = \sum_{j=1}^{n} [A_j \frac{1}{Q_j^{(0)}} + B_j Q_j^{(0)} + C_j] + f$.

(2) $\frac{iSD^2}{(P-\alpha)^2} \geq c_S$

For each product j, the optimal lot-size is given by $Q_j^{(1)} = \frac{SD}{p-\alpha} \left[\frac{2a_j}{b_j} \right]^{1/2}$, its minimum total cost is given by $F^{(1)} = \sum_{j=1}^{n} [A_j \frac{1}{Q_j^{(1)}} + B_j Q_j^{(1)} + C_j] + f$, where A_j, B_j, C_j, a_j, b_j are defined in Eq. (23).

Proof:

(1) For each product j, without condition (26), the optimal solution of Eq. (25) is the unique optimal solution which satisfying $\frac{\partial}{\partial Q_j} \sum_{j=1}^{n} [c_S Sa_j \frac{1}{Q_j} + \frac{1}{2} ib_j Q_j + C_j] = - c_S Sa_j \frac{1}{Q_j^2} + \frac{1}{2} ib_j = 0$. Therefore, the optimal production lot-size for product j is $Q_j^{(0)} = \left[\frac{2c_S Sa_j}{ib_j} \right]^{1/2}$, and if $\frac{iSD^2}{(P-\alpha)^2} < c_S$, then condition (26) is satisfied by $Q_j^{(0)}$. In addition, from Eq. (19), the minimum total cost is $\sum_{j=1}^{n} [A_j \frac{1}{Q_j^{(0)}} + B_j Q_j^{(0)} + C_j] + f$. This completes the proof.

(2) Two situations are considered.

(i) Consider $\frac{iSD^2}{(P-\alpha)^2} = c_S$, Eq. (25) is recast as $\underset{Q}{\text{Min}} \sum_{j=1}^{n} \left[\frac{iS^2 D^2 a_j}{(p-\alpha)^2} \frac{1}{Q_j} + \frac{1}{2} ib_j Q_j + C_j \right]$.

Without condition (26), the optimal production lot-size of product j is $Q_j^{(1)} = \frac{SD}{p-\alpha} \left(\frac{2a_j}{b_j} \right)^{1/2}$. Taking $Q_j = Q_j^{(1)}$, the left-hand side of condition (26) is given by $\sum_{j=1}^{n} \frac{1}{Q_j^{(1)}} (m_j + \Delta_{j12})(q_j + \Delta_{j22}) = \frac{p-\alpha}{S}$ (*). Hence condition (26) is satisfied and the equality holds. Therefore, if $\frac{iSD^2}{(P-\alpha)^2} = c_S$, then the optimal production lot-size of product j is $Q_j^{(1)} = \frac{SD}{p-\alpha} \left(\frac{2a_j}{b_j} \right)^{1/2}$.

(ii) Consider $\frac{iSD^2}{(P-\alpha)^2} > c_S$

From Eq. (25), for each j, let $L(Q_j) \equiv c_S Sa_j \frac{1}{Q_j} + \frac{1}{2} ib_j Q_j + C_j$ and we have $\frac{\partial}{\partial Q_j} \sum_{j=1}^{n} [c_S Sa_j \frac{1}{Q_j} + \frac{1}{2} ib_j Q_j + C_j] = \frac{d}{dQ_j} L(Q_j) = \frac{1}{Q_j^2} \left[\frac{1}{2} ib_j Q_j^2 - c_S Sa_j \right]$. Hence, if $Q_j < \left[\frac{2c_S Sa_j}{ib_j} \right]^{1/2}$ $(= Q_j^{(0)})$, then $L(Q_j)$ monotonically decreases with respect to Q_j; and if $Q_j > \left[\frac{2c_S Sa_j}{ib_j} \right]^{1/2}$, then $L(Q_j)$ monotonically increases with respect to Q_j. Since ${Q_j^{(1)}}^2 - {Q_j^{(0)}}^2$

$$= \frac{2Sa_j}{ib_j}\left(\frac{iSD^2}{(p-\alpha)^2} - c_S\right) > 0, \text{ thus, } Q_j^{(0)} < Q_j^{(1)} \text{ (see Fig. 4).}$$

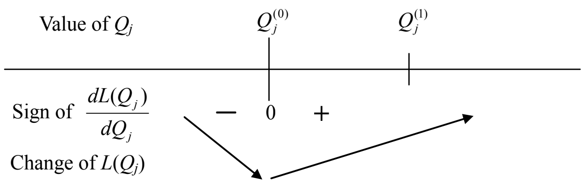

Fig. 4. Change of $L(Q_j)$.

For each product j, take any Q_j^* such that $0 < Q_j^* < Q_j^{(1)}$ in Fig. 4. Substituting Q_j^* into the left-hand side of Eq. (26) with $m_j + \Delta_{j12} > 0$, $q_j + \Delta_{j22} > 0$ reveals that $\sum_{j=1}^{n} \frac{1}{Q_j^*}(m_j + \Delta_{j12})(q_j + \Delta_{j22}) > \sum_{j=1}^{n} \frac{1}{Q_j^{(1)}}(m_j + \Delta_{j12})(q_j + \Delta_{j22}) = \frac{p-\alpha}{S}$ (by (*) in (i) of (2)). Hence, condition (26) is again violated and Q_j^* is not a solution. So, the optimal solution is in interval $Q_j \geq Q_j^{(1)}$. Therefore, from Fig. 4, if $\frac{iSD^2}{(p-\alpha)^2} > c_S$, the optimal production lot-size for product j can be shown to be $Q_j^{(1)}$ and the minimum total cost can be obtained by Eq. (19). This completes the proof. ❑

3.2.2 Defuzzification of fuzzy total cost $\tilde{F}(Q_j)$ by the centroid

From Eq. (1), we have

$$C(\tilde{F}(Q_j); \Delta_{jtk}) = F(Q_j; m_j, q_j, c_j) + \frac{1}{3}[R_2(Q_j, \Delta_{jt2}) - R_1(Q_j; \Delta_{jt1})], \tag{27}$$

where $C(\tilde{F}(Q_j); \Delta_{jtk})$ is an estimate of cost for product j in the fuzzy sense by the centroid method. From Eqs. (16), (27), (17) and (18), we have

$$\begin{aligned} &C(\tilde{F}(Q_j); \Delta_{jtk}) - F^*(Q_j; \Delta_{jtk}) \\ &= \frac{1}{12}\left\{\frac{1}{Q_j} c_s S[m_j(\Delta_{j22} - \Delta_{j21}) + q_j(\Delta_{j12} - \Delta_{j11}) + (\Delta_{j11}\Delta_{j21} + \Delta_{j12}\Delta_{j22})] \right. \\ &\left. + \frac{1}{2} iQ_j(\Delta_{j32} - \Delta_{j31}) + c_j(\Delta_{j12} - \Delta_{j11}) + m_j(\Delta_{j32} - \Delta_{j31}) + (\Delta_{j11}\Delta_{j31} + \Delta_{j12}\Delta_{j32})\right\}. \end{aligned} \tag{28}$$

Similarly, the analogous methods are applied to Eqs. (22)-(24) of section 3.2.1. For convenience, the following notations are defined.

$$a_j^{(0)} \equiv m_j q_j + \frac{1}{3} m_j(\Delta_{j22} - \Delta_{j21}) + \frac{1}{3} q_j(\Delta_{j12} - \Delta_{j11}) + \frac{1}{3}(\Delta_{j12}\Delta_{j22} + \Delta_{j11}\Delta_{j21}),$$

$$A_j^{(0)} \equiv c_S S a_j^{(0)},\ b_j^{(0)} = c_j + \frac{1}{3}(\Delta_{j32} - \Delta_{j31}),\ B_j^{(0)} \equiv \frac{1}{2} ib_j^{(0)},$$

$$C_j^{(0)} \equiv c_j m_j + \frac{1}{3} c_j (\Delta_{j12} - \Delta_{j11}) + \frac{1}{3} m_j (\Delta_{j32} - \Delta_{j31}) + \frac{1}{3} (\Delta_{j12}\Delta_{j32} + \Delta_{j11}\Delta_{j31})], \quad (29)$$

$$D^{(0)} = \sum_{j=1}^{n} \left(\frac{b_j^{(0)}}{2a_j^{(0)}} \right)^{1/2} (m_j + \Delta_{j12})(q_j + \Delta_{j22}). \quad (30)$$

Similarly to Eq. (22), Eq. (27) can be rewritten in the following.

$$C(\tilde{F}(Q_j);\Delta_{jtk}) = A_j^{(0)} \frac{1}{Q_j} + B_j^{(0)} Q_j + C_j^{(0)}. \quad (31)$$

Applying the same method, Theorems 1 and 2 lead to the following conclusions.

Theorem 3 The flexibility and product variety in lot-sizing of fuzzy case based on centroid is expressed in the following.

$$\operatorname*{Min}_{Q} \sum_{j=1}^{n} C(\tilde{F}(Q_j);\Delta_{jtk}), \quad (32)$$

subject to $\displaystyle\sum_{j=1}^{n} \frac{1}{Q_j} (m_j + \Delta_{j12})(q_j + \Delta_{j22}) \le \frac{p-\alpha}{S}$. (33)

Theorem 4 The optimal solution of flexibility and product variety in lot-sizing of fuzzy case of Theorem 3 (Eqs. (32) and (33))) is given in the following respective conditions.

(1) $\dfrac{iSD^{(0)^2}}{(p-\alpha)^2} < c_S$

For each product j, optimal lot-size is given by $Q_j^{(2)} = \left[\dfrac{2c_S S a_j^{(0)}}{i b_j^{(0)}} \right]^{1/2}$, and its minimum total cost is $F^{(2)} = \displaystyle\sum_{j=1}^{n} [A_j^{(0)} \frac{1}{Q_j^{(2)}} + B_j^{(0)} Q_j^{(2)} + C_j^{(0)}] + f$.

(2) $\dfrac{iSD^{(0)^2}}{(p-\alpha)^2} \ge c_S$

For each product j, optimal lot-size is given by $Q_j^{(3)} = \dfrac{SD^{(0)}}{p-\alpha} \left[\dfrac{2a_j^{(0)}}{b_j^{(0)}} \right]^{1/2}$, and the minimum total cost is $F^{(3)} = \displaystyle\sum_{j=1}^{n} [A_j^{(0)} \frac{1}{Q_j^{(3)}} + B_j^{(0)} Q_j^{(3)} + C_j^{(0)}] + f$.

4. NUMERICAL EXAMPLES

Some optimal solutions of fuzzy cases were computed by respectively applying Theorems 2 and 4, The results are tabulated in Tables 1 to 3. Respective notations are depicted in each case.

(a) Optimal solutions in Theorems 2 and 4

In Theorem 2, $F^{(0)}$ is represented by F_2 with $k = 0$; $F^{(1)}$ is represented by F_2 with $k = 1$. In Theorem 4, $F^{(2)}$ is represented by F_4 with $h = 2$; $F^{(3)}$ is denoted by F_4 with $h = 3$.

Table 1. (a) Optimal solutions of theorems 2, 4 for example 1.

case	j	Δ_{j11}	Δ_{j21}	Δ_{j31}	Fuzzy case (Theorem 2)			Fuzzy case (Theorem 4)		
		Δ_{j12}	Δ_{j22}	Δ_{j32}	$Q_j^{(k)}$	$k = 0, 1$	F_2	$Q_j^{(h)}$	$h = 2, 3$	F_4
0	1	0.0000	0.0000	0.0000						
		0.0000	0.0000	0.0000	3.286	0		3.286	2	
	2	0.0000	0.0000	0.0000						
		0.0000	0.0000	0.0000	3.633			3.633		
	3	0.0000	0.0000	0.0000						
		0.0000	0.0000	0.0000	6.928		2126.0435	6.928		2126.0435
1	1	0.0003	0.0004	0.0003						
		0.0004	0.0005	0.0006	3.286	0		3.286	2	
	2	0.0007	0.0005	0.0005						
		0.0004	0.0009	0.0008	3.633			3.633		
	3	0.0006	0.0007	0.0005						
		0.0005	0.0012	0.0007	6.929		2126.0448	6.929		2126.0452
2	1	0.10	0.30	1.20						
		0.20	0.60	0.80	3.433	0		3.481	2	
	2	0.15	0.90	0.30						
		0.23	0.80	0.50	3.598			3.586		
	3	0.40	0.30	0.60						
		0.50	0.10	0.90	6.774		2129.6291	6.722		2130.8241
3	1	0.03	0.05	0.11						
		0.01	0.03	0.08	3.277	0		3.274	2	
	2	0.12	0.10	0.09						
		0.07	0.09	0.04	3.629			3.627		
	3	0.06	0.02	0.05						
		0.06	0.02	0.05	6.928		2125.0354	6.928		2124.6993
4	1	1.40	0.88	2.60						
		1.35	0.60	2.50	3.220	1		3.354	3	
	2	1.50	1.09	3.00						
		1.30	1.00	2.90	3.653			3.839		
	3	2.16	1.15	1.70						
		2.15	0.80	1.60	6.801		2127.7199	7.090		2128.2799
5	1	2	0.7	3						
		1.5	0.6	1.7	3.315	0		3.325	2	
	2	2.6	0.8	2						
		1.9	0.3	1	3.472			3.417		
	3	1.9	0.6	1.2						
		1.3	1.8	2.3	7.746		2115.2942	7.994		2111.7100
6	1	0.9	0.85	0.4						
		0.8	0.7	1.4	3.359	1		3.353	3	
	2	0.8	1	1.7						
		0.7	0.9	1.5	3.753			3.758		
	3	1.6	1.18	2.1						
		1.1	1	2	7.122		2126.9001	7.121		2127.1855
7	1	7.5	0.8	15						
		1	0.3	2	3.513	1		3.424	3	
	2	10	1.099	20						
		6	1.08	4	4.435			4.519		
	3	12.5	1.1	7.5						
		9	0.7	7	7.713		2016.0865	7.625		1979.4302

Table 1. (b) The relative percentages for table 1 (a).

case	$R_{F_{24}}$(%)	R_{F_2}(%)	R_{F_4}(%)
0	0.0000	0.0000	0.0000
1	0.0000	0.0001	0.0001
2	− 0.0561	0.1687	0.2249
3	0.0158	− 0.0474	− 0.0632
4	− 0.0263	0.0789	0.1052
5	0.1697	− 0.5056	− 0.6742
6	− 0.0134	0.0403	0.0537
7	1.8519	− 5.1719	− 6.8961

Table 2. (a) Optimal solutions of theorems 2, 4 for example 2.

case	j	Δ_{j11}	Δ_{j21}	Δ_{j31}	Fuzzy case (Theorem 2)			Fuzzy case (Theorem 4)		
		Δ_{j12}	Δ_{j22}	Δ_{j32}	$Q_j^{(k)}$	$k = 0, 1$	F_2	$Q_j^{(h)}$	$h = 2, 3$	F_4
0	1	0.0000	0.0000	0.0000						
		0.0000	0.0000	0.0000	60.000	1		60.000	3	
	2	0.0000	0.0000	0.0000						
		0.0000	0.0000	0.0000	66.332			66.332		
	3	0.0000	0.0000	0.0000						
		0.0000	0.0000	0.0000	126.491		2144.0520	126.491		2144.0520
1	1	0.0003	0.0004	0.0003						
		0.0004	0.0005	0.0006	60.001	1		60.001	3	
	2	0.0007	0.0005	0.0005						
		0.0004	0.0009	0.0008	66.335			66.336		
	3	0.0006	0.0007	0.0005						
		0.0005	0.0012	0.0007	126.497		2144.0540	126.500		2144.0547
2	1	0.10	0.30	1.20						
		0.20	0.60	0.80	62.683	1		63.555	3	
	2	0.15	0.90	0.30						
		0.23	0.80	0.50	65.688			65.473		
	3	0.40	0.30	0.60						
		0.50	0.10	0.90	123.675		2147.6908	122.728		2148.9006
3	1	0.03	0.05	0.11						
		0.01	0.03	0.08	59.832	1		59.775	3	
	2	0.12	0.10	0.09						
		0.07	0.09	0.04	66.254			66.227		
	3	0.06	0.02	0.05						
		0.06	0.02	0.05	126.492		2143.0170	126.493		2142.6720
4	1	1.40	0.88	2.60						
		1.35	0.60	2.50	561.343	1		559.071	3	
	2	1.50	1.09	3.00						
		1.30	1.00	2.90	636.867			639.906		
	3	2.16	1.15	1.70						
		2.15	0.80	1.60	1185.643		2217.7345	1181.617		2218.2505
5	1	2	0.7	3						
		1.5	0.6	1.7	549.612	1		547.177	3	
	2	2.6	0.8	2						
		1.9	0.3	1	575.708			562.321		
	3	1.9	0.6	1.2						
		1.3	1.8	2.3	1284.299		2203.5174	1315.745		2199.7769
6	1	0.9	0.85	0.4						
		0.8	0.7	1.4	559.882	1		558.759	3	
	2	0.8	1	1.7						
		0.7	0.9	1.5	625.434			626.360		
	3	1.6	1.18	2.1						
		1.1	1	2	1186.956		2216.4090	1186.782		2216.7318
7	1	7.5	0.8	15						
		1	0.3	2	585.535	1		570.735	3	
	2	10	1.099	20						
		6	1.08	4	739.161			753.226		
	3	12.5	1.1	7.5						
		9	0.7	7	1285.568		2108.0989	1270.861		2068.7884

Table 2. (b) The relative percentages for table 2 (a).

case	$R_{F_{24}}(\%)$	$R_{F_2}(\%)$	$R_{F_4}(\%)$
0	0.0000	0.0000	0.0000
1	0.0000	0.0001	0.0001
2	− 0.0563	0.1697	0.2261
3	0.0161	− 0.0483	− 0.0644
4	− 0.0233	3.4366	3.4607
5	0.1700	2.7735	2.5990
6	− 0.0146	3.3748	3.3898
7	1.9002	− 1.6769	− 3.5103

Table 3. (a) Optimal solutions of theorems 2, 4 for example 3.

case	j	Δ_{j11} / Δ_{j12}	Δ_{j21} / Δ_{j22}	Δ_{j31} / Δ_{j32}	Fuzzy case (Theorem 2) $Q_j^{(k)}$	$k = 0, 1$	F_2	Fuzzy case (Theorem 4) $Q_j^{(h)}$	$h = 2, 3$	F_4
0	1	0.0000	0.0000	0.0000						
		0.0000	0.0000	0.0000	3270.434	1		3270.434	3	
	2	0.0000	0.0000	0.0000						
		0.0000	0.0000	0.0000	4820.801			4820.801		
	3	0.0000	0.0000	0.0000						
		0.0000	0.0000	0.0000	4136.808		43165.1814	4136.808		43165.1814
1	1	0.0003	0.0004	0.0003						
		0.0004	0.0005	0.0006	3273.052	1		3273.023	3	
	2	0.0007	0.0005	0.0005						
		0.0004	0.0009	0.0008	4824.811			4824.819		
	3	0.0006	0.0007	0.0005						
		0.0005	0.0012	0.0007	4140.278		43166.5802	4140.294		43166.5883
2	1	0.10	0.30	1.20						
		0.20	0.60	0.80	5030.842	1		5099.441	3	
	2	0.15	0.90	0.30						
		0.23	0.80	0.50	7036.526			7013.214		
	3	0.40	0.30	0.60						
		0.50	0.10	0.90	5978.623		43969.6497	5938.306		43974.7561
3	1	0.03	0.05	0.11						
		0.01	0.03	0.08	3416.469	1		3414.551	3	
	2	0.12	0.10	0.09						
		0.07	0.09	0.04	5044.493			5044.487		
	3	0.06	0.02	0.05						
		0.06	0.02	0.05	4333.486		43237.4384	4335.060		43235.8087
4	1	1.40	0.88	2.60						
		1.35	0.60	2.50	5748.035	1		5721.765	3	
	2	1.50	1.09	3.00						
		1.30	1.00	2.90	8719.686			8768.702		
	3	2.16	1.15	1.70						
		2.15	0.80	1.60	7290.623		44449.0469	7264.592		44444.5220
5	1	2	0.7	3						
		1.5	0.6	1.7	5754.816	1		5690.885	3	
	2	2.6	0.8	2						
		1.9	0.3	1	8082.887			7852.815		
	3	1.9	0.6	1.2						
		1.3	1.8	2.3	8216.815		44435.8759	8418.242		44401.2307
6	1	0.9	0.85	0.4						
		0.8	0.7	1.4	5935.933	1		5924.600	3	
	2	0.8	1	1.7						
		0.7	0.9	1.5	8843.164			8858.593		
	3	1.6	1.18	2.1						
		1.1	1	2	7536.920		44535.5635	7532.294		44534.9589
7	1	7.5	0.8	15						
		1	0.3	2	5448.354	1		5358.944	3	
	2	10	1.099	20						
		6	1.08	4	8711.166			8827.158		
	3	12.5	1.1	7.5						
		9	0.7	7	7047.019		42980.0411	6997.031		42507.9291

Table 3. (b) The relative percentages for table 3 (a).

case	$R_{F_{24}}$(%)	R_{F_2}(%)	R_{F_4}(%)
0	0.0000	0.0000	0.0000
1	0.0000	0.0032	0.0033
2	− 0.0116	1.8637	1.8755
3	0.0038	0.1674	0.1636
4	0.0102	2.9743	2.9638
5	0.0780	2.9438	2.8635
6	0.0014	3.1747	3.1733
7	1.1106	− 0.4289	− 1.5226

(b) Three relative percentages are assumed in the following

$$R_{F_{24}} = \frac{F_2 - F_4}{F_4} \times 100(\%),\ R_{F_2} = \frac{F_2 - F}{F} \times 100(\%),\ R_{F_4} = \frac{F_4 - F}{F} \times 100(\%),$$

where F is minimum total cost of optimal solution of crisp case.

Example 1: To illustrate the optimal solution procedure, we consider the flexibility and product variety in lot-sizing problem with the following data: $n = 3$, $S = 0.036$, $c_s = 1$, $f = 500$, $(m_j, q_j, c_j) = (15, 0.9, 30), (20, 1.1, 40), (25, 1.2, 15)$, $j = 1, 2, 3$, respectively. By using Theorem 2 and 4, the results are shown in Tables 1 (a) and (b).

Example 2: Let $n = 3$, $S = 6$, $c_s = 2$, $f = 500$, with m_j, q_j and c_j ($j = 1, 2, 3$) having the same values as in Example 1. By using Theorems 2 and 4, the results are shown in Tables 2 (a) and (b).

Example 3: Let $n = 3$, $S = 6$, $c_s = 2$, $f = 500$, $(m_j, q_j, c_j) = (100, 0.9, 80), (200, 1.1, 90), (150, 1.2, 100)$, $j = 1, 2, 3$, respectively, then by using Theorems 2 and 4, the results are as shown in Tables 3 (a) and (b).

From Tables 1 to 3, we find the following results.

(a) In case 0 and 1 of Tables 1 to 3, Δ_{jtk} are very small values, it shows that the optimal products $Q_j^{(k)}, Q_j^{(h)}, Q_j$ of product j are very close and the minimum total costs F_2, F_4, F are very close too (see section 5.2).
(b) In Tables 1 to 3, for the same cases have the same Δ_{jtk} so, if we change S, c_s, m_j, q_j, c_j ($j = 1, 2, 3$) in Examples 1 to 3, then the quantities of $Q_j^{(k)}, F_2, Q_j^{(h)}, F_4$ also change.

5. DISCUSSION

5.1 The Comparisons between Optimal Solutions by Signed Distance Defuzzification with that of Defuzzification by Centroid

(a) According to the principle of maximum membership grade in Proposition 4, Theorem 2 is better than Theorem 4.
(b) According to the approach which considers smaller values of minimum total cost to be better.

Following Eqs. (16) and (28), we have

$$\begin{aligned}
&\sum_{j=1}^{n} [C(\tilde{F}(Q_j); \Delta_{jtk}) - F^*(Q_j; \Delta_{jtk})] \\
&= \frac{1}{12} \sum_{j=1}^{n} \left\{ \frac{1}{Q_j} c_s S[\, m_j(\Delta_{j22} - \Delta_{j21}) + q_j(\Delta_{j12} - \Delta_{j11}) + (\Delta_{j11}\Delta_{j21} + \Delta_{j12}\Delta_{j22})] \right. \\
&\left. + \frac{1}{2} iQ_j(\Delta_{j32} - \Delta_{j31}) + c_j(\Delta_{j12} - \Delta_{j11}) + m_j(\Delta_{j32} - \Delta_{j31}) + (\Delta_{j11}\Delta_{j31} + \Delta_{j12}\Delta_{j32}) \right\}.
\end{aligned} \tag{34}$$

Both in Theorems 2 and 4, the constraints are $\sum_{j=1}^{n} \frac{1}{Q_j}(m_j + \Delta_{j12})(q_j + \Delta_{j22}) \leq \frac{p-\alpha}{S}$.

For each j, Q_j, m_j, q_j, c_j are revealed to be positive values and $c_S > 0$, $S > 0$, hence, when $\Delta_{jt1} < \Delta_{jt2}$, for all j, t, the following inequality is derived by Eq. (34) $\sum_{j=1}^{n} F^*(Q_j; \Delta_{jtk}) < \sum_{j=1}^{n} C(\tilde{F}(Q_j); \Delta_{jtk})$. It shows that the Theorem 2 is better since it considers that the smaller values of minimum total cost are better. For the other conditions of Δ_{jtk}, the minimum total costs of optimal solution must be computed by Theorems 2 and 4 separately. Then by taking the point of view of smaller value, if the minimum total cost of optimal solution in Theorem 2 is less than that of Theorem 4, Theorem 2 is considered to be better than Theorem 4. Under the opposite conditions, Theorem 4 is better.

5.2 The Problem (in Eqs. (8) and (9)) of Crisp Case is Respective a Special Condition of Theorems 1, 2 and Theorems 3, 4

In the crisp case, for each product j, if $\Delta_{jtk} = 0$, then from section 6.2, we see that the problem (in Eqs. (8), (9)) of crisp case is shown to be the special condition of problem (in Eqs. (20), (21)) of Theorem 1. In Theorem 2, for each j, let $\Delta_{jtk} = 0$, then (1) of Theorem 2 shows that $\frac{iSD^2}{(P-\alpha)^2} < c_S$ becomes to $\sum_{j=1}^{n} \sqrt{\frac{ic_j m_j q_j}{2c_S S}} \leq \frac{p-\alpha}{S}$ and $Q_j^{(0)} = \left[\frac{2c_S S m_j q_j}{ic_j}\right]^{1/2}$. The minimum total cost is $\sum_{j=1}^{n}[A_j \frac{1}{Q_j^{(0)}} + B_j Q_j^{(0)} + C_j] + f = \sum_{j=1}^{n} \sqrt{2c_S Sic_j m_j q_j} + \sum_{j=1}^{n} c_j m_j + f$, revealing that the result is the first formula of Eq. (10). Using Eq. (2) of Theorem 2, the $\frac{iSD^2}{(P-\alpha)^2} \geq c_S$ can be replaced by $\sum_{j=1}^{n} \sqrt{\frac{ic_j m_j q_j}{2c_S S}} \geq \frac{p-\alpha}{S}$ and $Q_j^{(1)} = \frac{S}{p-\alpha}\left(\frac{2m_j q_j}{c_j}\right)^{1/2} \sum_{j=1}^{n}\left(\frac{c_j m_j q_j}{2}\right)^{1/2}$. Therefore, the minimum total cost is constructed as $\sum_{j=1}^{n}[A_j \frac{1}{Q_j^{(1)}} + B_j Q_j^{(1)} + C_j] + f = \frac{\left(\sum_{j=1}^{n}\sqrt{2Sic_j m_j q_j}\right)^2}{4(p-\alpha)} + c_S(p-\alpha) + \sum_{j=1}^{n} c_j m_j + f$, which indicates that the result is the second formula of Eq. (10). Hence, the optimal solution of problem (in Eqs. (8) and (9)) of the crisp case is the special condition of the optimal solution of problem (in Eqs. (25) and (26)) of fuzzy case in Theorem 2.

5.3 Fuzzification of Fixed Cost f

Because the fixed cost f is an estimate in Eq. (7), it may change slightly after the finishing production process. Therefore, similar to section 3.2, the following consideration is made. Assume that the fixed cost is located in the interval $[f - \Delta_1, f + \Delta_2]$, and decision maker takes reasonable values of Δ_1 and Δ_2 which fulfill $0 < \Delta_1 < f$ and $0 < \Delta_2$. By proposition 5, corresponding to the interval $[f - \Delta_1, f + \Delta_2]$, the triangular fuzzy number $\tilde{f} = (f - \Delta_1, f, f + \Delta_2)$ is set. Hence, by taking signed distance and the centroid method respectively for defuzzification of $\tilde{f}$, $f^* = d(\tilde{f}, \tilde{0}) = f + \frac{1}{4}(\Delta_2 - \Delta_1)$, and $f^{**} = C(\tilde{f}) = f + \frac{1}{3}(\Delta_2 - \Delta_1)$ are obtained. Therefore, the fixed cost f of Theorems 2 and 4 can be recast in the following.

Theorems 1-4 reveals that the fixed cost f of optimal solution of minimum total cost has no relationship with Q_j. Hence, all the fixed cost f that are in minimum total cost $F^{(0)}$, $F^{(1)}$ of Theorem 2 can be rewritten as f^*. Same as in minimum total cost $F^{(2)}$, $F^{(3)}$ of Theorem 4, all the fixed cost f can be replaced by f^{**}.

6. CONCLUSION

In this section, we compare the usual method for fuzzification of the crisp total cost function $F(Q_j; m_j, q_j, c_j)$ (in Eq. (6)) of section 3.1 to that of the new creative method of this paper for the jth product in the following three steps. Also, the advantages of this new proposed method are addressed as follows.

6.1 The First Step of Fuzzification

For each product j, the average demand of per unit time (m_j), relative duration of setup (q_j), and unit cost of production (c_j) can not be fixed as a value during the planning period. Hence, the fuzzification problem is emerged. In the following, we consider the usual fuzzification method and the method of this paper respectively.

(A1) The usual method is that the m_j, q_j, c_j are fuzzified respectively as the triangular fuzzy numbers $\tilde{m}_j = (m_j - w_{j11}, m_j, m_j + w_{j12})$, $\tilde{q}_j = (q_j - w_{j21}, q_j, q_j + w_{j22})$ and $\tilde{c}_j = (c_j - w_{j31}, c_j, c_j + w_{j32})$.

(B1) For the new method, we consider the quantities of m_j, q_j and c_j are located respectively in the interval of $[m_j - \Delta_{j11}, m_j + \Delta_{j12}]$, $[q_j - \Delta_{j21}, q_j + \Delta_{j22}]$ and $[c_j - \Delta_{j31}, c_j + \Delta_{j32}]$.

Therefore, the advantage of this new method can be stated as follows:

(C1.1) In the usual fuzzification method of (A1), there is not objective method to decide the value of w_{jtk}, $t = 1, 2, 3$, $k = 1, 2$.

(C1.2) In the method of this paper of (B1), we estimate the value of Δ_{jtk}, $t = 1, 2, 3$, $k = 1, 2$ by the interval which is obtained by applying statistical method on the past data and thus it is supposed to be more practical and objective than that of the usual method.

6.2 The Second Step of Fuzzification

(A2) In usual method, for jth product, the m_j, q_j, c_j of $F(Q_j; m_j, q_j, c_j)$ are fuzzified by $\tilde{m}_j, \tilde{q}_j, \tilde{c}_j,$ and then we obtain the following fuzzy total cost.

$$F(Q_j; \tilde{m}_j, \tilde{q}_j, \tilde{c}_j) = \frac{1}{Q_j} c_s S \tilde{m}_j \tilde{q}_j + \frac{1}{2} i Q_j \tilde{c}_j + \tilde{c}_j \tilde{m}_j. \tag{35}$$

(B2) For the new method, as discussed in section 3.2, the fuzzy total cost (in Eq. (15)) can be obtained via the interval of section 6.1 (B1) and applying proposition 5.

So, the advantage of this method is stated as follows:

(C2.1) In the usual method of (A2), we must fuzzify the parameters m_j, q_j, c_j of crisp total cost function $F(Q_j; m_j, q_j, c_j)$ to derive the fuzzy total cost function (in Eq. (35)).

(C2.2) However, for the new method, we can obtain the fuzzy total cost function (in Eq. (15)) by method of (B2) without fuzzification for the parameters m_j, q_j, c_j of crisp total cost function $F(Q_j; m_j, q_j, c_j)$.

6.3 The Defuzzification in Third Step of Fuzzification

(A3) In usual method, we need to derive the membership function before the fuzzy total cost $F(Q_j; \tilde{m}_j, \tilde{q}_j, \tilde{c}_j)$ is defuzzified by centroid. Though $\tilde{m}_j, \tilde{q}_j, \tilde{c}_j$ are triangular fuzzy numbers, but, fuzzy sets $\tilde{m}_j\tilde{q}_j$ and $\tilde{c}_j\tilde{m}_j$ are not. Therefore, the extension principle is employed for deriving the membership function $\mu_{\tilde{m}_j\tilde{q}_j}(x) = \sup_{ts=x} \mu_{\tilde{m}_j}(t)$ $\mu_{\tilde{m}_j\tilde{q}_j}(x) = \sup_{ts=x} \mu_{\tilde{m}_j}(t) \wedge \mu_{\tilde{q}_j}(s)$ and $\mu_{\tilde{c}_j\tilde{m}_j}(u) = \sup_{ts=u} \mu_{\tilde{c}_j}(t) \wedge \mu_{\tilde{m}_j}(s)$. From Eq. (35) and applying extension principle, let $a_j = c_S S/Q_j$, $b_j = i/2$, we obtain

$$\mu_{F(Q_j;\tilde{m}_j,\tilde{q}_j,\tilde{c}_j)}(z) = \sup_{a_jx+b_jy+u=z} \mu_{\tilde{m}_j\tilde{q}_j}(x) \wedge \mu_{\tilde{c}_j}(y) \wedge \mu_{\tilde{c}_j\tilde{m}_j}(u). \quad (36)$$

From Eq. (36), the centroid of Eq. (35) is given as follows:

$$C(F(Q_j; \tilde{m}_j, \tilde{q}_j, \tilde{c}_j)) = \int_{-\infty}^{\infty} z\mu_{F(Q_j;\tilde{m}_j,\tilde{q}_j,\tilde{c}_j)}(z)dz \Big/ \int_{-\infty}^{\infty} \mu_{F(Q_j;\tilde{m}_j,\tilde{q}_j,\tilde{c}_j)}(z)dz. \quad (37)$$

(B3) For the new method, since Eq. (15) is triangular fuzzy number, we have $C(\tilde{F}(Q_j)$; $\Delta_{jtk})$ in Eq. (27) of section 3.2.2.

Accordingly, the advantage of the new method against that of the usual can be concluded as follows:

(C3.1) In the usual method of (A3), from Eq. (36), we know that it is very difficult to derive the membership function of fuzzy total cost (in Eq. (35)). Hence, it is hard to obtain the centroid (in Eq. (37)) of fuzzy total cost. Therefore, it is difficult to consider the optimal solution in the fuzzy sense.

(C3.2) However, for the method of this paper of (B3), we can derive the cerntroid (in Eq. (27)) easily. From section 3.2.2, the optimal is obtained in Theorem 4. In the section 3.2.1, the signed distance is applied for defuzzification, and then we attain the optimal solution in Theorem 2.

On the other hand, the exact optimal solutions can be computed via Theorems 2 and 4. Therefore, the simulation method or genetic algorithm method are not needed for approximate solutions.

REFERENCES

1. X. de Groote, "Flexibility and product variety in lot-sizing models," *European Journal of Operational Research*, Vol. 75, 1994, pp. 264-274.
2. G. Hadley and T. Whitin, *Analysis of Inventory System*, Prentice Hall, Englewood Cliffs, New Jersey, 1963.
3. C. H. Hsieh, "Optimization of fuzzy production inventory models," *Information Sciences*, Vol. 146, 2002, pp. 29-40.
4. S. Hsieh and C. C. Chiang, "Manufacturing-to-sale planning model for fuel oil production," *The International Journal of Advanced Manufacturing Technology*, Vol. 18, 2001, pp. 303-311.
5. A. Kaufmann and M. M. Gupta, *Introduction to Fuzzy Arithmetic Theory and Applications*, Van Nostrand Reinhold, New York, 1991.
6. G. J. Klir and Bo Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.
7. G. J. Klir, U. St. Clair, and B. Yuan, *Fuzzy Set Theory-Foundations and Applications*, Prentice Hall, Upper Saddle River, New Jersey, 1997.
8. H. M. Lee and J. S. Yao, "Economic production quantity for fuzzy demand and fuzzy production quantity," *European Journal of Operational Research*, Vol. 109, 1998, pp. 203-211.
9. D. C. Lin and J. S. Yao, "Fuzzy economic production for production inventory," *Fuzzy Sets and Systems*, Vol. 111, 2000, pp. 465-495.
10. J. A. Parsons, "Multiproduct lot size determination when certain restrictions are active," *Journal of Industrial Engineering*, Vol. 27, 1966, pp. 360-365.
11. P. M. Pu and Y. M. Liu, "Fuzzy topology 1, neighborhood structure of a fuzzy point and Moore-Smith convergence," *Journal of Mathematics Analysis and Applications*, Vol. 76, 1980, pp. 571-599.
12. A. M. Spence and E. L. Porteus, "Setup reduction and increased effective capacity," *Management Science*, Vol. 35, 1987, pp. 1291-1301.
13. R. C. Wang and H. H. Fang, "Aggregate production planning with multiple objectives in a fuzzy environment," *European Journal of Operational Research*, Vol. 133, 2001, pp. 521-536.
14. R. C. Wang and H. H. Fang, "Aggregate production planning with fuzzy variables," *International Journal of Industrial Engineering*, Vol. 8, 2001, pp. 37-44.
15. J. S. Yao and K. M. Wu, "Ranking fuzzy number based on decomposition principle and signed distance," *Fuzzy Sets and Systems*, Vol. 116, 2000, pp. 275-288.
16. L. A. Zadeh, "Fuzzy sets," *Information and Control*, Vol. 8, 1965, pp. 338-353.
17. H. J. Zimmermann, *Fuzzy Set Theory and Its Applications*, Kluwer Academic Publishers, Boston, 1991.

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