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A comparison between two pricing and lot-sizing models with partial backlogging and deteriorated items

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Abstract

Recently, Abad [2003. Optimal pricing and lot-sizing under conditions of perishability, finite production and partial backordering and lost sale, European Journal of Operational Research, 144, 677–685] studied the pricing and lot-sizing problem for a perishable good under finite production, exponential decay, partial backordering and lost sale. In this article, we extend his model by adding not only the backlogging cost but also the cost of lost goodwill. We then analytically compare the net profits per unit time between Abad's (2003) model and Goyal and Giri's [2003. The production-inventory problem of a product with time varying demand, production and deterioration rates. European Journal of Operational Research, 147, 549–557] model. In Abad's model, the cycle starts with an instant production to accumulate stocks, then stops production to use up stocks, and finally restarts production to meet the unsatisfied demands. By contrast, in Goyal and Giri's model, the cycle begins with a period of shortages, then starts production until accumulated inventory reaches certain level, and finally stops production and uses up inventory. Our theoretical results show that there is no dominant one between these two models. Furthermore, we provide certain conditions under which one model has more net profit per unit time than the other. Finally, we give several numerical examples to illustrate the results.
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Keywords: Inventory; Pricing; Partial backlogging; Deteriorating items

1. Introduction

Many researchers have studied inventory models for deteriorating items such as volatile liquids, blood banks, medicines, electronic components and fashion goods. Ghare and Schrader (1963) were the first proponents for developing a model for an exponentially decaying inventory. They categorized decaying inventory into three types: Direct spoilage, physical depletion and deterioration. Next, Misra (1975) developed an economic order quantity (i.e., EOQ) model with a Weibull deterioration rate for the perishable product but he did not consider backordering. Dave and Patel (1981) considered an EOQ model for deteriorating items

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with time-proportional demand when shortages were prohibited. Sachan (1984) then generalized the EOQ model to allow for shortages. Later, Hariga (1996) generalized the demand pattern to any log-concave function. Teng et al. (1999) and Yang et al. (2001) further generalized the demand function to include any non-negative, continuous function that fluctuates with time. Recently, Papachristos and Skouri (2003) extended Wee's (1999) deteriorating EOQ model with quantity discount, pricing and partial backordering to allow for the demand rate to be a convex decreasing function of the selling price.

Abad (1996) established the optimal pricing and lot-sizing EOQ policies under conditions of perishability and partial backordering. Then Abad (2000) extended the optimal pricing and lot-sizing EOQ model to an economic production quantity (i.e., EPQ) model. Balkhi and Benkherouf (1996) developed a general EPQ model for deteriorating items where demand and production rates are time varying, but the rate of deterioration is constant. Balkhi (2001) then further generalized the EPQ model to allow for time-varying deterioration rate. Concurrently, Yan and Cheng (1998) considered a perishable single-item EPQ model in which production rate, demand rate and deterioration rate are assumed to be functions of time, and shortages are partially backlogged. Other recent articles related to this research area were written by Abad (2001), Chang and Dye (1999), Papachristos and Skouri (2000), Skouri and Papachristos (2003), Teng et al. (2002), Yang and Wee (2003) and Wee and Law (1999). In addition, Raafat (1991), and Goyal and Giri (2001) wrote two excellent surveys on the recent trends in modeling of continuously deteriorating inventory.

Recently, Abad (2003) studied the pricing and lot-sizing problem for a perishable good under finite production, exponential decay and partial backordering and lost sale. He assumed that customers are impatient and the backlogging rate is a negative exponential function of the waiting time. In addition, he assumed that the customers are served on first come first served basis during the shortage period. Then he provided a solution procedure to obtain the optimal price and lot-size that maximizes the net profit per unit time. However, he did not include the shortage cost for backlogged items and the cost of lost goodwill due to lost sales into the objective. If the objective does not include these two costs, then it will alter the optimal solution and overestimate the net profit. To correct them, in this paper, we add both the shortage cost for backlogged items and the cost of lost goodwill due to lost sales into the objective suggested by Abad (2003).

In Abad (2003), the production-inventory model starts with an instant production to accumulate stocks, then stops production to use up stocks, and finally restarts production to meet the unsatisfied demands. In fact, Abad's production-inventory model is similar to that in Balkhi and Benkherouf (1996). Lately, Goyal and Giri (2003) investigated a similar production-inventory problem in which the demand, production and deterioration rates of a product were assumed to vary with time. However, pricing was not under consideration and the backlogging rate was assumed to be a constant fraction. They then proposed a new production-inventory model in which the cycle begins with a period of shortages, then starts production until accumulated inventory reaches a certain level, and finally stops production and uses up inventory. Finally, Goyal and Giri (2003) provided a numerical example to show that their model outperforms Balkhi and Benkherouf's model (1996) in terms of the least expensive total cost per unit time.

In this paper, we first extend Abad's (2003) pricing and lot-sizing model by adding not only the shortage cost for backlogged items but also the cost of lost goodwill due to lost sales into the objective. Next, we establish a new modeling approach as in Goyal and Giri (2003) to the same pricing and lot-sizing inventory problem. We then characterize the optimal solution to both distinct models, and prove that both two models provide the same profit if all parameters are constant. However, if any single parameter is varying with time, then the performances of these two models are varied. Furthermore, we obtain some theoretical results that show the conditions under which one model has more net profit per unit time than the other. Finally, we provide several numerical examples to illustrate the results, and conclusions are made.

2. Assumptions and notations

The following assumptions are similar to those in Abad's (2003) model.

(1) The planning horizon is infinite.
(2) The initial and final inventory levels are both zero.

(3) Shortages are allowed. However, the longer the waiting time, the smaller the backlogging rate. Hence, we assume that the fraction of shortages backordered $B(\tau)$ is a decreasing and differentiable function of τ, where τ is the waiting time up to the next replenishment.
(4) The demand rate is a decreasing function of the selling price and it is twice differentiable.
(5) The production rate, which is finite, is higher than the demand rate.
(6) A constant fraction of the on-hand inventory deteriorates per unit of time and there is no repair or replacement of the deteriorated inventory. Hence, there is no salvage value for the deteriorated items.
(7) The unit cost, the holding cost, the shortage cost for backlogged items, and the cost of lost goodwill due to lost sales are assumed to be functions of time.

In addition, the following notations are used throughout this paper:

$I_i(t)$ on-hand stock level (or number of backorders during shortage periods) at time t in Phase i, $i = 1, 2, 3$, and 4
R production rate for the item (units/unit time)
K setup cost per setup
$v(t)$ unit cost at time t
$h(t)$ unit holding cost per unit time at time t
p unit selling price within the replenishment cycle (a decision variable), we assume that $p > v(t)$
$c_1(t)$ unit shortage cost per unit time for backlogged items at time t
$c_2(t)$ unit cost of lost goodwill due to lost sales at time t
$D(p)$ demand rate per unit time, which is a decreasing function of p. We assume $R > D(p)$. We will use D and $D(p)$ interchangeably
σ decay coefficient, which is a constant (i.e., exponential decay)
β duration of positive inventory before the end of production
T duration of positive inventory cycle (a decision variable)
ψ duration of negative inventory before the start of production
λ duration of negative inventory cycle (a decision variable)

In this paper, we assume that the vendor wants to determine the price p, the duration of positive inventory cycle T, and the duration of negative inventory cycle λ in order to maximize its net profit per unit time. As a result, we have three decision variables for this pricing and lot-sizing inventory shortage problem.

3. Mathematical formulations and theoretical results

In this section, we first establish Abad's modeling approach for the problem, then set up Goyal and Giri's modeling approach to the problem next, and finally compare the net profits per unit time obtained from these two models.

3.1. Model 1: Abad's (2003) approach

In this subsection, the behavior of the inventory in a cycle is shown in Fig. 1, as well as in Abad (2003). Consequently, the inventory cycle is described by the following four phases:

Phase 1: During the time interval $[0,\beta]$, the system is subject to the effect of production, demand and deterioration. Therefore, the change of the inventory level at time t, $I_1(t)$, is governed by

$$\frac{dI_1(t)}{dt} + \sigma I_1(t) = R - D \quad \text{with the boundary condition} \quad I_1(0) = 0. \tag{1}$$

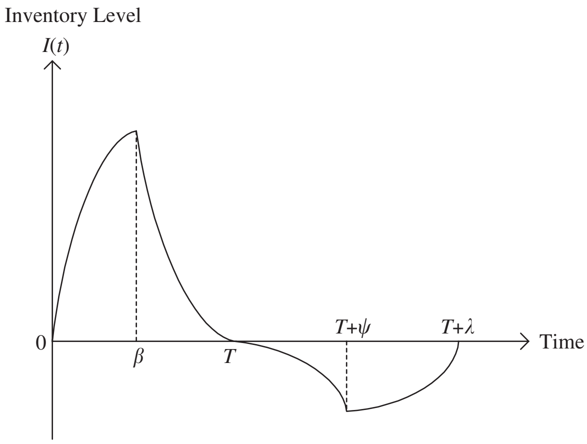

Fig. 1. A production-inventory cycle of Model 1.

Phase 2: In the time interval [β,T], the system is affected by the combined the demand and deterioration. Hence, the change of the inventory level at time t, $I_2(t)$, is governed by

$$\frac{dI_2(t)}{dt} + \sigma I_2(t) = -D \quad \text{with the boundary condition} \quad I_2(T) = 0. \tag{2}$$

The solution to differential equation (1) is

$$I_1(t) = \frac{(R-D)}{\sigma}\left(1 - e^{-\sigma t}\right), \quad t \in [0, \beta]. \tag{3}$$

Setting $t = \beta$ into Eq. (3), we obtain the maximum positive inventory in a cycle is

$$I_1(\beta) = \frac{(R-D)}{\sigma}\left(1 - e^{-\sigma\beta}\right). \tag{4}$$

Similarly, the solution to differential equation (2) is

$$I_2(t) = \frac{D}{\sigma}\left(e^{\sigma(T-t)} - 1\right), \quad t \in [\beta, T]. \tag{5}$$

Equating expressions (3) and (5) at $t = \beta$, we have

$$I_1(\beta) = I_2(\beta) = \frac{R-D}{\sigma}\left(1 - e^{-\sigma\beta}\right) = \frac{D}{\sigma}\left(e^{\sigma(T-\beta)} - 1\right). \tag{6}$$

Solving Eq. (6) for β, we have

$$\beta = \frac{1}{\sigma}\ln\left[\frac{R - D + De^{T\sigma}}{R}\right]. \tag{7}$$

Phase 3: For $t \in [T, T+\psi]$, similar to Abad's (2003) model, the backlogging rate $B(\tau)$ is a negative exponential function of the waiting time τ. Therefore, we have

$$B(\tau) = k_0 e^{-k_1\tau}, \quad k_0 \leqslant 1, \quad 0 \leqslant k_1. \tag{8}$$

Since customers are served on first come first served basis during shortage period, we know from Fig. 1 that the waiting time is given by $\tau = T+\psi-t-I_3(t)/R$, for $t \in [T, T+\psi]$. Therefore, the number of backorders at time $t, I_3(t)$, satisfies the following differential equation:

$$\frac{dI_3(t)}{dt} = -DB\left(T + \psi - t - I_3(t)/R\right) = -Dk_0 e^{-k_1\left(T+\psi-t-I_3(t)/R\right)}, \tag{9}$$

with the boundary condition $I_3(T)=0$. The solution to (9) is

$$I_3(t) = -\frac{R}{k_1}\left[\ln\left(\frac{Dk_0 e^{k_1(t-T-\psi)} + R - Dk_0 e^{-k_1\psi}}{R}\right)\right], \quad \text{for} \quad t \in [T, T+\psi]. \tag{10}$$

Setting $t = T+\psi$ into Eq. (10), we obtain the maximum number of backorders per cycle as follows:

$$-I_3(T+\psi) = \frac{R}{k_1}\left[\ln\left(\frac{Dk_0 + R - Dk_0 e^{-k_1\psi}}{R}\right)\right]. \tag{11}$$

Phase 4: For $t \in [T+\psi, T+\lambda]$, the waiting time is given by $\tau = -I_4(t)/R$. Therefore, the number of backorders at time t, $I_4(t)$, satisfies the following differential equation:

$$\frac{dI_4(t)}{dt} = R - DB\big(-I_4(t)/R\big) = R - Dk_0 e^{-k_1\left(-I_4(t)/R\right)}, \tag{12}$$

with the boundary condition $I_4(T+\lambda)=0$. The solution to (12) is

$$I_4(t) = -\frac{R}{k_1}\left[\ln\left(\frac{(R-Dk_0)e^{-k_1(t-T-\lambda)} + Dk_0}{R}\right)\right], \quad \text{for} \quad t \in [T+\psi, T+\lambda]. \tag{13}$$

Given the condition $I_3(T+\psi) = I_4(T+\psi)$, we get

$$\psi = \frac{1}{k_1}\left[\ln\left(\frac{Dk_0 + e^{k_1\lambda}(R-Dk_0)}{R}\right)\right]. \tag{14}$$

Applying Eq. (14) into Eq. (13), we can rewrite Eq. (13) as follows:

$$I_4(t) = -\frac{R}{k_1}\left[\ln\left(\frac{Re^{-k_1(t-T-\psi)} - Dk_0 e^{-k_1(t-T)} + Dk_0}{R}\right)\right]. \tag{15}$$

Next, the net profit per unit time consists of the following six elements:

(a) The revenue is given by

$$R_1 = pDT + pR(\lambda - \psi). \tag{16}$$

(b) The set up cost is given by

$$\mathrm{SC}_1 = K. \tag{17}$$

(c) The production cost is given by

$$\mathrm{PC}_1 = \int_0^{\beta} v(t)R\,dt + \int_{T+\psi}^{T+\lambda} v(t)R\,dt. \tag{18}$$

(d) The inventory holding cost is given by

$$\begin{aligned}\mathrm{HC}_1 &= \int_0^{\beta} h(t)I_1(t)\,dt + \int_{\beta}^{T} h(t)I_2(t)\,dt \\ &= \int_0^{\beta} h(t)\frac{(R-D)}{\sigma}\left(1-e^{-\sigma t}\right)dt + \int_{\beta}^{T} h(t)\frac{D}{\sigma}\left(e^{\sigma(T-t)}-1\right)dt.\end{aligned} \tag{19}$$

(e) The shortage cost for backlogged items is given by

$$\begin{aligned}\mathrm{BC}_1 &= \int_T^{T+\psi} c_1(t)[-I_3(t)]\,\mathrm{d}t + \int_{T+\psi}^{T+\lambda} c_1(t)[-I_4(t)]\,\mathrm{d}t \\ &= \frac{R}{k_1}\left\{\int_T^{T+\psi} c_1(t)\left[\ln\left(\frac{Dk_0\mathrm{e}^{k_1(tT-\psi)} + R - Dk_0\mathrm{e}^{-k_1\psi}}{R}\right)\right]\mathrm{d}t\right. \\ &\quad \left. + \int_{T+\psi}^{T+\lambda} c_1(t)\left[\ln\left(\frac{R\mathrm{e}^{-k_1(tT-\psi)} - Dk_0\mathrm{e}^{-k_1(t-T)} + Dk_0}{R}\right)\right]\mathrm{d}t\right\} \\ &= \frac{R}{k_1}\left\{\int_0^{\psi} c_1(t+T)\left[\ln\left(\frac{Dk_0\mathrm{e}^{k_1(t-\psi)} + R - Dk_0\mathrm{e}^{-k_1\psi}}{R}\right)\right]\mathrm{d}t\right. \\ &\quad \left. + \int_{\psi}^{\lambda} c_1(t+T)\left[\ln\left(\frac{R\mathrm{e}^{-k_1(t-\psi)} - Dk_0\mathrm{e}^{-k_1 t} + Dk_0}{R}\right)\right]\mathrm{d}t\right\}. \end{aligned} \tag{20}$$

(f) The cost of lost goodwill due to lost sales is given by

$$\begin{aligned}\mathrm{LC}_1 &= \int_T^{T+\psi} c_2(t)\big[1 - B\big(T+\psi-t-I_3(t)/R\big)\big]D\mathrm{d}t + \int_{T+\psi}^{T+\lambda} c_2(t)\big[1 - B\big(-I_4(t)/R\big)\big]D\,\mathrm{d}t \\ &= \int_0^{\lambda} c_2(t+T)\left(1 - \frac{Rk_0\mathrm{e}^{k_1(t-\psi)}}{R - Dk_0\mathrm{e}^{-k_1\psi} + Dk_0\mathrm{e}^{k_1(t-\psi)}}\right)D\,\mathrm{d}t. \end{aligned} \tag{21}$$

Given the above, the net profit during time-span $[0, T+\lambda]$ is

$$\begin{aligned} &F_1(p,T,\lambda) \\ &\quad = R_1 - \mathrm{SC}_1 - \mathrm{PC}_1 - \mathrm{HC}_1 - \mathrm{BC}_1 - \mathrm{LC}_1 \\ &\quad = [pDT + pR(\lambda-\psi)] - K - \left[\int_0^{\beta} v(t)R\,\mathrm{d}t + \int_{T+\psi}^{T+\lambda} v(t)R\,\mathrm{d}t\right] \\ &\qquad - \left[\int_0^{\beta} h(t)\frac{(R-D)}{\sigma}\left(1-\mathrm{e}^{-\sigma t}\right)\mathrm{d}t + \int_{\beta}^{T} h(t)\frac{D}{\sigma}\left(\mathrm{e}^{\sigma(T-t)}-1\right)\mathrm{d}t\right] \\ &\qquad - \frac{R}{k_1}\left\{\int_0^{\psi} c_1(t+T)\left[\ln\left(\frac{Dk_0\mathrm{e}^{k_1(t-\psi)} + R - Dk_0\mathrm{e}^{-k_1\psi}}{R}\right)\right]\mathrm{d}t\right. \\ &\qquad \left. + \int_{\psi}^{\lambda} c_1(t+T)\left[\ln\left(\frac{R\mathrm{e}^{-k_1(t-\psi)} - Dk_0\mathrm{e}^{-k_1 t} + Dk_0}{R}\right)\right]\mathrm{d}t\right\} \\ &\qquad - \int_0^{\lambda} c_2(t+T)\left(1 - \frac{Rk_0\mathrm{e}^{k_1(t-\psi)}}{R - Dk_0\mathrm{e}^{-k_1\psi} + Dk_0\mathrm{e}^{k_1(t-\psi)}}\right)D\,\mathrm{d}t, \end{aligned} \tag{22}$$

where $\beta \equiv \beta(p,T)$ is given by Eq. (7) and $\psi \equiv \psi(p,T)$ is given by Eq. (14). Hence, the net profit per unit time is

$$\Pi_1(p,T,\lambda) = \frac{F_1(p,T,\lambda)}{T+\lambda}, \tag{23}$$

where $F_1(p,\lambda,T)$ is given by Eq. (22). As a result, the problem faced by the vendor is

$$\text{(P1)}\ \max.\quad \Pi_1(p,T,\lambda) \tag{24a}$$

$$\begin{aligned} \text{s.t.}\quad & 0 < \psi < \lambda, \\ & 0 < \beta < T, \\ & v \leqslant p. \end{aligned} \tag{24b−d}$$

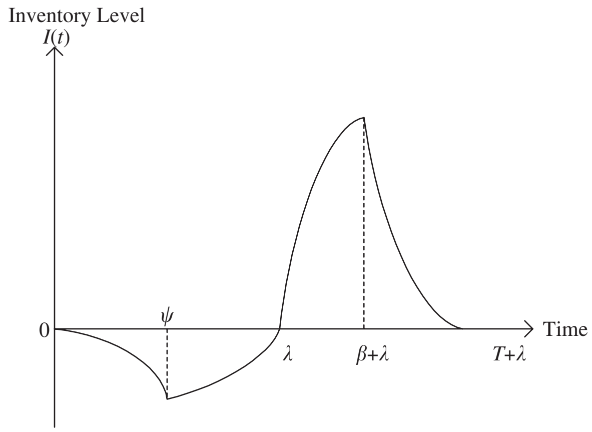

Fig. 2. A production-inventory cycle of Model 2.

3.2. Model 2: Goyal and Giri's (2003) approach

In this subsection, the behavior of the inventory in a cycle is depicted in Fig. 2, as well as in Goyal and Giri's (2003) model. Based on the assumptions in Section 2, and from Fig. 2, we know that the inventory is also described by the following four phases:

Phase 1: For $t\in[0,\psi]$,

$$\frac{dI_1(t)}{dt} = -DB\big(\psi - t - I_1(t)/R\big) = -Dk_0 e^{-k_1(\psi - t - I_1(t)/R)}, \tag{25}$$

with the boundary condition $I_1(0) = 0$.

Phase 2: For $t\in[\psi,\lambda]$,

$$\frac{dI_2(t)}{dt} = R - DB\big(-I_2(t)/R\big) = R - Dk_0 e^{-k_1(-I_2(t)/R)}, \tag{26}$$

with the boundary condition $I_2(\lambda) = 0$.

Phase 3: For $t\in[\lambda, \beta+\lambda]$,

$$\frac{dI_3(t)}{dt} + \sigma I_3(t) = R - D, \tag{27}$$

with the boundary condition $I_3(\lambda) = 0$.

Phase 4: For $t\in[\beta+\lambda, T+\lambda]$,

$$\frac{dI_4(t)}{dt} + \sigma I_4(t) = -D, \tag{28}$$

with the boundary condition $I_4(T+\lambda) = 0$.

The solutions of the above four ordinary differential equations are given as follows.

$$I_1(t) = -\frac{R}{k_1}\ln\left[\frac{Dk_0 e^{k_1(t-\psi)} + R - Dk_0 e^{-k_1\psi}}{R}\right], \quad t \in [0,\psi], \tag{29}$$

$$\begin{aligned} I_2(t) &= -\frac{R}{k_1}\ln\left[\frac{(R - Dk_0)e^{-k_1(t-\lambda)} + Dk_0}{R}\right] \\ &= -\frac{R}{k_1}\ln\left[\frac{Re^{-k_1(t-\psi)} - Dk_0 e^{-k_1 t} + Dk_0}{R}\right], \quad t \in [\psi,\lambda], \end{aligned} \tag{30}$$

$$I_3(t) = \frac{(R-D)}{\sigma}\left[1 - e^{-\sigma(t-\lambda)}\right], \quad t \in [\lambda, \beta+\lambda] \tag{31}$$

and

$$I_4(t) = \frac{D}{\sigma}\left[e^{\sigma(T+\lambda-t)} - 1\right], \quad t \in [\beta+\lambda, T+\lambda], \tag{32}$$

respectively. Solving the boundary conditions $I_1(\psi) = I_2(\psi)$ and $I_3(\beta+\lambda) = I_4(\beta+\lambda)$, we obtain the following equations which are the same as Eqs. (7) and (14), respectively:

$$\psi = \frac{1}{k_1}\left[\ln\left(\frac{Dk_0 + e^{k_1\lambda}(R - Dk_0)}{R}\right)\right] \quad \text{and} \quad \beta = \frac{1}{\sigma}\ln\left[\frac{R - D + De^{T\sigma}}{R}\right]. \tag{33}$$

Therefore, the net profit per unit time in Model 2 consists of the following elements:

(a) The revenue is given by

$$R_2 = pDT + pR(\lambda - \psi). \tag{34}$$

(b) The set up cost is given by

$$SC_2 = K. \tag{35}$$

(c) The production cost is given by

$$PC_2 = \int_{\psi}^{\lambda} v(t)R\,dt + \int_{\lambda}^{\beta+\lambda} v(t)R\,dt. \tag{36}$$

(d) The inventory holding cost is given by

$$\begin{aligned} HC_2 &= \int_{\lambda}^{\beta+\lambda} h(t)I_3(t)dt + \int_{\beta+\lambda}^{T+\lambda} h(t)I_4(t)\,dt \\ &= \int_{\lambda}^{\beta+\lambda} h(t)\frac{(R-D)}{\sigma}\left[1 - e^{-\sigma(t-\lambda)}\right] dt + \int_{\beta+\lambda}^{T+\lambda} h(t)\frac{D}{\sigma}\left[e^{\sigma(T+\lambda-t)} - 1\right] dt. \end{aligned} \tag{37}$$

(e) The shortage cost for backlogged items is given by

$$\begin{aligned} BC_2 &= \int_0^{\psi} c_1(t)[-I_1(t)]\,dt + \int_{\psi}^{\lambda} c_1(t)[-I_2(t)]\,dt \\ &= \frac{R}{k_1}\left\{\int_0^{\psi} c_1(t)\ln\left[\frac{Dk_0 e^{k_1(t-\psi)} + R - Dk_0 e^{-k_1\psi}}{R}\right] dt \right. \\ &\quad \left. + \int_{\psi}^{\lambda} c_1(t)\ln\left[\frac{Re^{-k_1(t-\psi)} - Dk_0 e^{-k_1 t} + Dk_0}{R}\right] dt\right\}. \end{aligned} \tag{38}$$

(f) The cost of lost goodwill due to lost sales is given by

$$\begin{aligned} LC_2 &= \int_0^{\psi} c_2(t)\left[1 - B\left(\psi - t - I_1(t)/R\right)\right]D\,dt + \int_{\psi}^{\lambda} c_2(t)\left[1 - B\left(-I_2(t)/R\right)\right]D\,dt \\ &= \int_0^{\lambda} c_2(t)\left(1 - \frac{Rk_0 e^{k_1(t-\psi)}}{R - Dk_0 e^{-k_1\psi} + Dk_0 e^{k_1(t-\psi)}}\right)D\,dt. \end{aligned} \tag{39}$$

Hence, the profit during time-span $[0,T+\lambda]$ is

$$\begin{aligned} F_2(p,T,\lambda) &= \mathrm{R}_2-\mathrm{SC}_2-\mathrm{PC}_2-\mathrm{HC}_2-\mathrm{BC}_2-\mathrm{LC}_2 \\ &= [pDT+pR(\lambda-\psi)]-K-\left[\int_\psi^\lambda v(t)R\,\mathrm{d}t+\int_\lambda^{\beta+\lambda} v(t)R\,\mathrm{d}t\right] \\ &\quad -\left\{\int_\lambda^{\beta+\lambda} h(t)\frac{(R-D)}{\sigma}\left[1-\mathrm{e}^{-\sigma(t-\lambda)}\right]\mathrm{d}t+\int_{\beta+\lambda}^{T+\lambda} h(t)\frac{D}{\sigma}\left[\mathrm{e}^{\sigma(T+\lambda-t)}-1\right]\mathrm{d}t\right\} \\ &\quad -\frac{R}{k_1}\left\{\int_0^\psi c_1(t)\ln\left[\frac{Dk_0\mathrm{e}^{k_1(t-\psi)}+R-Dk_0\mathrm{e}^{-k_1\psi}}{R}\right]\mathrm{d}t\right. \\ &\quad \left.+\int_\psi^\lambda c_1(t)\ln\left[\frac{Re^{-k_1(t-\psi)}-Dk_0\mathrm{e}^{-k_1t}+Dk_0}{R}\right]\mathrm{d}t\right\} \\ &\quad -\int_0^\lambda c_2(t)\left(1-\frac{Rk_0\mathrm{e}^{k_1(t-\psi)}}{R-Dk_0\mathrm{e}^{-k_1\psi}+Dk_0e^{k_1(t-\psi)}}\right)D\,\mathrm{d}t, \end{aligned} \tag{40}$$

where $\beta\equiv\beta(p,T)$ and $\psi\equiv\psi(p,T)$ are given by Eq. (33). Hence, the net profit per unit time is

$$\Pi_2(p,T,\lambda)=\frac{F_2(p,T,\lambda)}{T+\lambda}, \tag{41}$$

where $F_2(p,\lambda,T)$ is given by Eq. (40). Consequently, the problem faced by the vendor is

$$\text{(P2)}\quad \max \Pi_2(p,T,\lambda) \tag{42a}$$

$$\begin{aligned} \text{s.t.}\quad & 0<\psi<\lambda, \\ & 0<\beta<T, \\ & v\leqslant p. \end{aligned} \tag{42b−d}$$

3.3. A comparison between two models

Now, we compare the above two different models, and identify which model has more net profit per unit time than the other under what conditions. From the above analysis, we can obtain the following theorems.

Theorem 1. *Let (p_1,T_1,λ_1) and (p_2,T_2,λ_2) be the optimal solution for Models 1 and 2, respectively.*

(a) *If all parameters are constants (i.e., $v(t)=v$, $h(t)=h$, $c_1(t)=c_1$, and $c_2(t)=c_2$), then*

$$\text{Max. }\Pi_1(p,T,\lambda)=\Pi_1(p_1,T_1,\lambda_1)=\Pi_2(p_2,T_2,\lambda_2)=\text{Max.}\Pi_2(p,T,\lambda). \tag{43a}$$

(b) *If the holding cost $h(t)$ is non-decreasing with t, and the other parameters are constants (i.e., $v(t)=v$, $c_1(t)=c_1$, and $c_2(t)=c_2$), then*

$$\Pi_1(p_1,T_1,\lambda_1)\geqslant\Pi_2(p_2,T_2,\lambda_2). \tag{43b}$$

Otherwise, if the holding cost $h(t)$ is non-decreasing with t, then

$$\Pi_1(p_1,T_1,\lambda_1)\leqslant\Pi_2(p_2,T_2,\lambda_2).$$

(c) *If the shortage cost $c_1(t)$ is non-increasing with t, and the other parameters are constants (i.e., $v(t)=v$, $h(t)=h$, and $c_2(t)=c_2$,), then*

$$\Pi_1(p_1,T_1,\lambda_1)\geqslant\Pi_2(p_2,T_2,\lambda_2). \tag{43c}$$

Conversely, if the shortage cost $c_1(t)$ *is non-decreasing with* t, *then*

$$\Pi_1(p_1, T_1, \lambda_1) \leqslant \Pi_2(p_2, T_2, \lambda_2).$$

(d) *If the cost of lost goodwill* $c_2(t)$ *is non-increasing with* t, *and the other parameters are constants* (i.e.,$h(t) = h$, $v(t) = v$ *and* $c_1(t) = c_1$), *then*

$$\Pi_1(p_1, T_1, \lambda_1) \geqslant \Pi_2(p_2, T_2, \lambda_2). \tag{43d}$$

In contrast, if the cost of lost goodwill $c_2(t)$ *is non-decreasing with* t, *then*

$$\Pi_1(p_1, T_1, \lambda_1) \leqslant \Pi_2(p_2, T_2, \lambda_2).$$

Proof. See Appendix A. □

Theorem 2. *Let* (p_1,T_1,λ_1) *and* (p_2,T_2,λ_2) *be the optimal solution for Models* 1 *and* 2, *respectively. If the unit cost* $v(t)$ *is varying with time, and the other parameters are constants* (*i.e.,* $h(t) = h$, $c_1(t) = c_1$ *and* $c_2(t) = c_2$), *then we obtain that*

$$\text{if} \int_0^{\beta} [v(t) - v(t+\lambda)]\,\mathrm{d}t + \int_{\psi}^{\lambda} [v(t+T) - v(t)]\,\mathrm{d}t \leqslant 0,$$

then

$$\Pi_1(p_1, T_1, \lambda_1) \geqslant \Pi_2(p_2, T_2, \lambda_2), \text{ and vice versa.} \tag{44}$$

Proof. See Appendix B. □

In order to find the optimal values of p, λ and T, we have to solve the complex, nonlinear equations $\partial\prod_i(p,\lambda,T)/\partial p = 0$,$\partial\prod_i(p,\lambda,T)/\partial\lambda = 0$, and $\partial\prod_i(p,\lambda,T)/\partial T = 0$, and some additional complementary conditions, for $i = 1$ and 2. Although it is difficult to solve the problem analytically, the reader can follow the solution procedure proposed by Abad (2003) with proper software to solve the problem numerically.

4. Numerical examples

In this section, we use software MATHEMATICA version 4.1 to obtain the optimal solutions for both (P1) and (P2).

Example 1. To understand the effect of adding the shortage cost $c_1(t)$, and the cost of lost goodwill $c_2(t)$ to the net profit per unit time, we adopt the same example in Abad (2003). Therefore, we suppose $D(p) = 1\,600\,000p^{-3}$,$R = 1000$ units/week, $v(t) = \$10$/unit, $h(t) = \$1$/unit/week, $K = \$1000$/production run, $\sigma = 0.3$, $k_0 = 0.9$, and $k_1 = 0.6$. However, we add $c_1(t) = \$8$/unit, and $c_2(t) = \$5$/unit. We obtain the computational results as shown in Table 1. Comparing with the computational results in Abad (2003), we know that the optimal price would be lower while the net profit per unit would be higher if we do not include the shortage cost and the cost of lost goodwill into the model. Table 1 also verifies Part (a) of Theorem 1.

Table 1
The optimal values for Example 1

Model (i)	ψ	λ	β	$\beta+\lambda$	T	$T+\psi$	$T+\lambda$	p	Π_i
1	0.1650	0.2669	0.6602	0.9271	1.3329	1.4979	1.5998	15.3142	1039.02
2	0.1650	0.2669	0.6602	0.9271	1.3329	1.4979	1.5998	15.3142	1039.02

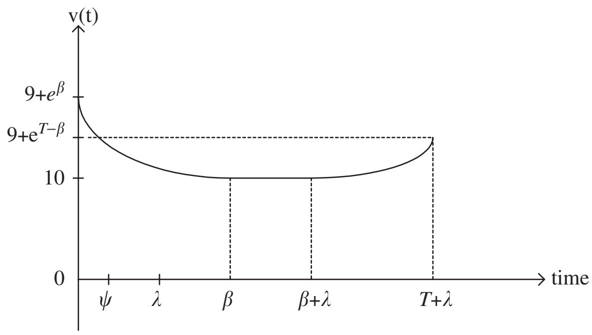

Fig. 3. The graph of the unit cost in Example 1.

Table 2
The optimal values for Example 2

Model (i)	ψ	λ	β	$\beta+\lambda$	T	$T+\psi$	$T+\lambda$	p	Π_i
1	0.1833	0.2560	0.4396	0.6957	1.1757	1.359	1.4317	16.8683	876.19
2	0.2597	0.3700	0.5008	0.8708	1.2469	1.5066	1.6169	16.4814	999.11

Example 2. To see the effect of the unit cost on the net profit per unit time, let us assume that the unit cost is as below, and the rest parameters are the same as in Example 1.

$$v(t) = \begin{cases} \$9 + \mathrm{e}^{-(t-\beta)}, & 0 \leqslant t \leqslant \beta \\ \$10, & \beta \leqslant t \leqslant \beta + \lambda \\ \$9 + \mathrm{e}^{(t-\beta-\lambda)}, & \beta + \lambda \leqslant t \leqslant T + \lambda \end{cases} \quad /\text{unit}. \tag{45}$$

The graph of $v(t)$ is shown in Fig. 3. It is clear from Fig. 3 that

$$\int_0^{\beta} v(t) - v(t+\lambda)\,\mathrm{d}t + \int_{\psi}^{\lambda} v(t+T) - v(t)\,\mathrm{d}t > 0. \tag{46}$$

Consequently, we know from Theorem 2 that Model 2 has more net profit per unit time than Model 1, which is shown in Table 2.

5. Conclusions

If we omit the shortage cost and the cost of lost goodwill into the production-inventory model with many lost sales, then we alter the results, and overestimate the profits. In this paper, we not only extend Abad's (2003) model by adding the shortage cost and the cost of lost goodwill into his model, but also compare his modeling approach and Goyal and Giri's (2003) approach. We analytically prove that both models provide the same net profit per unit time if all parameters are constant. Otherwise, under certain conditions Abad's model has more net profit per unit time than Goyal and Giri's approach, and vice versa. In short, there is no dominant modeling approach.

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Appendix A. The proof of Theorem 1

Proof. If $v(t) = v$, $h(t) = h$, $c_1(t) = c_1$, and $c_2(t) = c_2$, then from Eqs. (16)–(21) and (34)–(39), we have

$$R_1 = R_2 = pDT + pR(\lambda - \psi), \quad SC_1 = SC_2 = K, \quad PC_1 = PC_2 = vR(\beta + \lambda - \psi),$$
$$HC_1 = HC_2 = \frac{h}{\sigma}[R\beta - DT],$$
$$BC_1 = BC_2 = \frac{Rc_1}{k_1}\left\{ \int_0^{\lambda} \ln\left(R - Dk_0 e^{-k_1\psi} + Dk_0 e^{k_1(t-\psi)}\right) dt - (\ln R)\lambda - \frac{k_1}{2}(\lambda - \psi)^2 \right\},$$

and

$$LC_1 = LC_2 = c_2[D\lambda - R(\lambda - \psi)], \quad \text{for any same}(p, T, \lambda). \tag{A.1}$$

Therefore, we have

$$\begin{aligned} \text{Max. } \Pi_2(p, T, \lambda) &= \Pi_2(p_2, T_2, \lambda_2) \\ &= \Pi_1(p_2, T_2, \lambda_2) \leqslant \text{Max. } \Pi_1(p, T, \lambda) = \Pi_1(p_1, T_1, \lambda_1). \end{aligned} \tag{A.2}$$

Similarly, we can easily obtain

$$\begin{aligned} \text{Max. } \Pi_1(p, T, \lambda) &= \Pi_1(p_1, T_1, \lambda_1) \\ &= \Pi_2(p_1, T_1, \lambda_1) \leqslant \text{Max. } \Pi_2(p, T, \lambda) = \Pi_2(p_2, T_2, \lambda_2) \end{aligned} \tag{A.3}$$

From (A.2) and (A.3), we have

$$\Pi_1(p_1, T_1, \lambda_1) = \Pi_2(p_2, T_2, \lambda_2), \tag{A.4}$$

which completes the proof of Part (a).

Similarly, if the holding cost is non-decreasing, and the other parameters are constants, then we have

$$R_1 = R_2, \quad SC_1 = SC_2, \quad PC_1 = PC_2, \quad BC_1 = BC_2, \quad LC_1 = LC_2$$

and

$$\begin{aligned} HC_1 - HC_2 = &\int_0^{\beta} [h(t) - h(t+\lambda)] \frac{(R-D)}{\sigma}\left(1 - e^{-\sigma t}\right) dt \\ &+ \int_{\beta}^{T} [h(t) - h(t+\lambda)] \frac{D}{\sigma}\left(e^{\sigma(T-t)} - 1\right) dt \leqslant 0, \quad \text{for any same } (p, T, \lambda). \end{aligned} \tag{A.5}$$

Therefore, we have

$$\begin{aligned} \text{Max. } \Pi_2(p, T, \lambda) &= \Pi_2(p_2, T_2, \lambda_2) \\ &\leqslant \Pi_1(p_2, T_2, \lambda_2) \leqslant \text{Max.} \Pi_1(p, T, \lambda) = \Pi_1(p_1, T_1, \lambda_1) \end{aligned} \tag{A.6}$$

and vice versa. This completes the proof of Part (b).

Next, if the shortage cost is non-increasing with t, and the other parameters are constants, then we have

$$R_1 = R_2, \quad SC_1 = SC_2, \quad PC_1 = PC_2, \quad HC_1 = HC_2, \quad LC_1 = LC_2$$

and

$$\mathrm{BC}_1-\mathrm{BC}_2=\frac{R}{k_1}\left\{\int_0^{\psi}[c_1(t+T)-c_1(t)]\ln\left[\frac{Dk_0\mathrm{e}^{k_1(t-\psi)}+R-Dk_0\mathrm{e}^{-k_1\psi}}{R}\right]\mathrm{d}t\right.$$
$$\left.+\int_{\psi}^{\lambda}[c_1(t+T)-c_1(t)]\ln\left[\frac{R\mathrm{e}^{-k_1(t-\psi)}-Dk_0\mathrm{e}^{-k_1t}+Dk_0}{R}\right]\mathrm{d}t\right\}\leqslant 0,\quad \text{for any same } (p,T,\lambda). \tag{A.7}$$

Therefore, we have

$$\text{Max. }\Pi_2(p,T,\lambda)=\Pi_2(p_2,T_2,\lambda_2)$$
$$\leqslant \Pi_1(p_2,T_2,\lambda_2)\leqslant \text{Max. }\Pi_1(p,T,\lambda)=\Pi_1(p_1,T_1,\lambda_1), \tag{A.8}$$

and vice versa. This completes the proof of Part (c).

Finally, if the cost of lost goodwill is non-increasing, and the other parameters are constants, then we have

$$\mathrm{R}_1=\mathrm{R}_2,\quad \mathrm{SC}_1=\mathrm{SC}_2,\quad \mathrm{PC}_1=\mathrm{PC}_2,\quad \mathrm{HC}_1=\mathrm{HC}_2,\quad \mathrm{BC}_1=\mathrm{BC}_2$$

and

$$\mathrm{LC}_1-\mathrm{LC}_2$$
$$=\int_0^{\lambda}[c_2(t+T)-c_2(t)]\left(1-\frac{Rk_0\mathrm{e}^{k_1(t-\psi)}}{R-Dk_0\mathrm{e}^{-k_1\psi}+Dk_0\mathrm{e}^{k_1(t-\psi)}}\right)D\,\mathrm{d}t\leqslant 0,\quad \text{for any same } (p,T,\lambda). \tag{A.9}$$

Consequently, we have

$$\text{Max. }\Pi_2(p,T,\lambda)=\Pi_2(p_2,T_2,\lambda_2)$$
$$\leqslant \Pi_1(p_2,T_2,\lambda_2)\leqslant \text{Max. }\Pi_1(p,T,\lambda)=\Pi_1(p_1,T_1,\lambda_1) \tag{A.10}$$

and vice versa. This completes the proof of Part (d). □

Appendix B. The proof of Theorem 2

Proof. If the unit cost $v(t)$ is varying with time, and the other parameters are constants, then we have

$$\int_0^{\beta}[v(t)-v(t+\lambda)]\,\mathrm{d}t+\int_{\psi}^{\lambda}[v(t+T)-v(t)]\,\mathrm{d}t\leqslant 0$$
$$\Leftrightarrow R\left[\int_{\psi}^{\lambda}v(t)-v(t+T)\,\mathrm{d}t+\int_0^{\beta}v(t+\lambda)-v(t)\,\mathrm{d}t\right]\geqslant 0$$
$$\Leftrightarrow\left[\int_{\psi}^{\lambda}v(t)R\,\mathrm{d}t+\int_{\lambda}^{\beta+\lambda}v(t)R\,\mathrm{d}t\right]-\left[\int_0^{\beta}v(t)R\,\mathrm{d}t+\int_{T+\psi}^{T+\lambda}v(t)Rdt\right]\geqslant 0$$
$$\Leftrightarrow PC_2-PC_1\geq 0\Leftrightarrow \Pi_1(p,T,\lambda)\geq \Pi_2(p,T,\lambda). \tag{B.1}$$

Consequently, we have

$$\text{Max. }\Pi_2(p,T,\lambda)=\Pi_2(p_2,T_2,\lambda_2)$$
$$\leqslant \Pi_1(p_2,T_2,\lambda_2)\leqslant \text{Max. }\Pi_1(p,T,\lambda)=\Pi_1(p_1,T_1,\lambda_1) \tag{B.2}$$

and vice versa. This completes the proof of Theorem 2. □

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