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On Sample Size in Using Central Limit Theorem for Gamma Distribution

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Abstract

A general criterion in using the central limit theorem is based on the sample size $n \geq 30$, no matter what the population is. Such only one generalized criterion may not be suitable for various shapes of probability distributions. This study is to check gamma distribution, one of asymmetric continuous distributions, how fit that criterion by computer simulation techniques, and finds out the least required sample sizes that satisfy the central limit theorem under different parameters of the gamma distribution.

Keywords: Central Limit Theorem, Sample Size, Sample Mean, Gamma Distribution, Shapiro-Wilk W Test, Normality Test.

1. Introduction

Based on the central limit theorem, when the sample size n is sufficiently large, the distribution of sample mean $\overline{X}_n$ is approximated to normal distribution. In practice, statisticians and researchers have accepted the criterion of the sample size $n \geq 30$ to assume the distribution of sample mean approximated to normal distribution. But various shapes of probability distributions exist, like single peak and multi-peak, symmetric and asymmetric, high skewness and low skewness, and also, the uniform distribution with symmetry but no peak and no skewness. Furthermore, there are distributions similar to the normal distribution while the others are vastly different. Chang et al. (2006) have carried out computer simulation techniques to study the practical choices of the least required sample sizes that satisfy the central limit theorem for Weibull distribution. The

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paper is to check that criterion for gamma distribution. The question is, how large should the sample size n be, the sample mean $\overline{X}_n$ from a gamma distribution can be assumed to be normally distributed?

2. Statistical Tests and Simulation Procedure

Based on the central limit theorem, if $\overline{X}_n$ is the mean of a random sample $X_1, X_2, \ldots, X_n$ of size n from a distribution with a finite mean μ and a finite positive variance σ^2, then the distribution of

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} \tag{2.1}$$

approaches $N(0,1)$ as $n \to \infty$.

Plane and Gordon(1982) have shown that if $n \geq 30$, we can assume the distribution of sample mean $\overline{X}_n$ to be a normal distribution. However, this is very rough. So, we try to use computer simulation techniques to show that for different shapes of probability distribution, the required sample sizes will differ in using the central limit theorem.

A random variable X has a gamma distribution if its probability density function of form is

$$p_X(x) = \frac{(x-\gamma)^{\alpha-1} e^{-(x-\gamma)/\beta}}{\beta^{\alpha}\Gamma(\alpha)}, \quad \alpha > 0, \quad \beta > 0; \quad x > \gamma. \tag{2.2}$$

This distribution, denoted gamma(α, β, γ), depends on three parameters. If $\gamma = 0$, the distribution is termed as a two-parameter gamma distribution, denoted by gamma(α, β). The standard form of gamma distribution is obtained by setting $\beta = 1$ and $\gamma = 0$. This gives

$$p_X(x) = \frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)}, \quad \alpha > 0, \quad x > 0 \tag{2.3}$$

denoted it by gamma(α). If $\alpha = 1$, we have an exponential distribution. And, if α is a positive integer, we have an Erlang distribution. With these properties, we use the standard form of gamma distribution as the population distribution in this study.

The W test statistic for normality proposed by Shapiro and Wilk (1965) is defined by

$$W = \left\{ \sum_{i=1}^{h} a_{in} \Big(x_{(n-i+1)} - x_{(i)} \Big) \right\}^2 \Big/ \sum_{i=1}^{n} (x_i - \overline{x})^2, \quad x_{(1)} \leq \cdots \leq x_{(n)} \tag{2.4}$$

where $h = \dfrac{1}{2}n$ or $h = \dfrac{1}{2}(n-1)$ depends on whether n be even or odd. Shapiro and Wilk (1965) have provided a table of coefficients a_{in}.

The Shapiro-Wilk W test is comparatively sensitive to a wide range of non-normality, even for small samples ($n < 20$) or with outliers (Shapiro and Wilk, 1965). Pearson (1977) have also noted that Shapiro-Wilk W test is very sensitive omnibus against skewed alternatives, and for many skewed alternatives it is clearly the most powerful. Therefore, we adapt the Shapiro-Wilk W test to test the normality of the sample mean $\overline{X}_n$.

We use the built-in random function of SAS to simulate the drawing of random samples. For each $\alpha \in \{1, 2, 3, \ldots, 60\}$, we generate 1000 random samples of size n from gamma distribution where $n = 2, 3, 4, \ldots, 150$ to obtain 1,000 sample means denoted by $\overline{X}_{n,1}, \overline{X}_{n,2}, \ldots, \overline{X}_{n,1000}$. Next, we use the Shapiro-Wilk W test under significant level 0.05 to test the normality of $\overline{X}_n$ by these 1,000 sample means sample. For each n, there will be a test outcome, either to accept the normality or to reject it . If the outcome is not accepted by the normality hypothesis, let it be denoted as a "success". For each sample size n, repeat the above process 100 times, and we will obtain 100 Bernoulli trial outcomes, which can be viewed as a new binomial sample with a sample size of 100. Using the number n as the X-axis and the number of "success" m as the Y-axis to draw the simulation results, we can employ the figure to find the relationship and estimate the function of them, which will be displayed as Figure 1. This simulation employed 67,944,000,000 ($= 1000 \times (2+3+4+\cdots+150) \times 60 \times 100$) random numbers from random number generator, and performed 894,000 times of normality tests ($= 149 \times 60 \times 100$).

3. Simulation Results

According to the above simulation procedure and statistical tests, we put $\alpha = 1, 2, 3, \ldots, 60$, $n = 2, 3, 4, \ldots, 150$ into the simulation, and the result is shown in Table 1.

As Table 1 shows, when the sample size is less than 17, there is nearly no chance to accept the normality under gamma(1). Even if the sample size is 150, there are also 18 out of 100 trials to be rejected. It can clearly be seen that this result is still not good enough. Therefore, we have to increase the sample size of gamma(1) to meeting the normality requirement.

When the sample size is fixed to 30, there are 78 out of 100 trials to be rejected under gamma(1). The outcome drops down to 51 times under gamma(2) and 22 times under gamma(3). As long as $\alpha > 10$, the rejected frequency drops down to between 3 and 11, and when $\alpha = 60$, the rejected frequency is only 6. Under gamma(60), while the sample size n is more than 10, the rejected frequency is not larger than 10.

Table 1. W test results of Gamma distribution as α and n varies. ($\alpha = 1, 2, 3, \ldots, 60$, $n = 2, 3, 4, \ldots, 150$; Numbers in the table are reject frequency of repeating 100 W tests)

Sample	α														
Size (n)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	100	100	100	100	100	100	100	100	99	97	95	95	93	83	86
3	100	100	100	100	100	100	97	90	87	88	87	78	67	68	74
4	100	100	100	99	98	92	83	81	80	65	62	58	58	54	44
5	100	100	100	97	92	82	82	68	60	58	58	51	41	39	41
6	100	100	98	88	82	79	71	60	56	50	46	28	37	36	35
7	100	100	96	82	78	71	57	58	53	40	37	34	34	25	24
8	100	100	96	82	67	66	42	49	46	29	38	28	30	23	26
9	100	99	86	72	62	61	42	40	34	34	31	29	26	14	23
10	100	99	82	70	55	49	41	34	38	24	27	22	24	25	20
11	100	94	78	62	53	37	42	34	29	20	22	20	15	13	23
12	100	95	71	57	48	36	33	29	35	21	21	16	18	18	9
13	99	92	75	57	47	31	35	33	22	21	21	25	18	16	8
14	99	87	69	46	48	32	34	32	23	21	9	18	14	14	14
15	99	81	59	36	44	30	31	26	18	20	23	17	9	11	11
16	100	76	62	52	34	29	21	31	16	17	14	14	21	7	9
17	99	84	63	38	35	26	30	22	16	14	19	20	20	9	10
18	97	73	52	53	26	18	27	15	18	16	14	15	12	15	18
19	98	74	52	29	31	28	23	19	18	14	13	18	8	8	9
20	97	60	58	36	37	20	17	15	9	16	18	14	12	13	9
21	96	66	48	27	32	17	26	13	21	11	13	8	8	12	13
22	93	55	34	24	25	24	18	11	12	10	17	17	8	8	5
23	98	57	40	25	20	18	11	11	12	8	11	9	8	9	16
24	95	60	32	31	25	20	16	23	15	16	12	6	12	8	7
25	91	56	42	35	20	18	15	13	13	21	13	7	7	3	5
26	95	50	32	34	22	12	15	12	9	13	8	14	7	11	7
27	88	54	40	19	20	20	19	8	12	14	9	11	8	13	10
28	91	49	29	23	21	15	19	12	11	9	15	6	6	10	7
29	87	52	31	26	17	9	11	11	12	5	12	9	16	6	12
30	78	51	22	23	21	11	13	12	11	7	13	8	13	11	11
31	82	49	24	24	17	13	15	10	12	4	6	5	13	5	8
32	83	55	27	30	12	15	10	17	11	10	6	11	6	13	6
33	77	36	28	21	17	14	10	11	7	10	12	8	8	7	7
34	78	41	28	27	18	18	15	14	7	10	12	9	7	9	5
35	67	38	25	21	11	13	15	12	11	15	12	8	9	8	7
36	77	36	29	22	17	16	12	15	13	10	4	9	11	6	10
37	73	37	23	17	14	12	14	21	8	9	11	6	10	8	6
38	64	37	20	17	9	14	12	9	9	9	12	4	9	8	8
39	69	37	17	13	18	15	14	7	10	10	8	7	8	10	5
40	67	34	23	15	12	10	12	11	10	10	14	9	9	9	11
41	63	41	18	15	16	15	13	9	3	14	11	4	10	9	10
42	67	37	27	18	15	11	11	10	8	9	13	10	8	8	9
43	65	32	28	18	12	13	8	11	13	14	10	7	6	8	11
44	64	30	18	15	7	12	13	4	8	12	7	7	9	7	7
45	69	39	23	20	11	14	11	10	15	6	5	13	7	7	4
46	66	29	17	20	13	5	13	14	6	6	7	12	12	6	8
47	67	27	15	14	12	11	6	13	5	9	13	6	6	4	6
48	61	29	23	11	22	17	11	10	11	10	9	5	3	9	5
49	58	30	17	17	14	9	10	8	10	4	4	4	8	5	5
50	56	33	26	16	13	13	6	10	6	3	8	8	9	10	6

Table 1. (Continued 1)
W test results of Gamma distribution as α and n varies.
($\alpha = 1, 2, 3, \ldots, 60$, $n = 2, 3, 4, \ldots, 150$; Numbers in the table are reject frequency of repeating 100 W tests)

Sample	α														
Size (n)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
51	54	27	23	14	12	13	8	11	7	17	7	9	5	5	6
52	56	28	17	7	7	11	12	10	6	11	14	5	9	7	9
53	60	33	17	17	13	15	14	5	5	7	5	8	7	5	8
54	48	25	16	6	17	13	15	10	12	11	9	11	6	4	5
55	41	30	21	15	6	5	4	14	8	3	5	11	4	8	5
56	55	29	12	17	12	11	11	8	3	1	11	7	6	2	6
57	50	23	16	14	13	12	4	15	7	11	5	6	3	4	4
58	53	21	19	16	12	4	7	5	11	5	6	12	5	6	2
59	51	30	19	12	12	14	17	9	8	5	8	6	9	6	5
60	57	18	15	12	7	7	9	10	5	5	6	11	10	7	6
61	51	27	11	9	13	9	9	6	12	6	6	10	5	10	11
62	43	24	21	17	11	13	7	13	4	10	6	3	8	7	3
63	43	15	14	13	7	9	12	9	7	7	10	7	4	8	3
64	38	21	15	6	8	9	5	9	2	9	10	1	8	7	11
65	43	20	16	12	7	10	8	9	7	12	5	6	10	4	6
66	52	24	17	16	11	7	6	2	4	8	5	4	8	5	6
67	49	27	20	12	9	6	3	11	8	5	6	8	1	6	3
68	37	24	12	10	13	10	6	11	10	5	6	11	7	3	5
69	47	19	17	7	11	16	8	12	9	6	6	7	4	6	7
70	47	18	10	7	12	13	11	11	6	4	7	10	7	9	9
71	39	24	15	8	9	10	11	2	5	7	9	3	3	8	3
72	29	18	16	12	13	5	9	8	11	9	7	9	6	5	8
73	28	15	14	9	9	8	12	10	4	8	8	11	8	3	9
74	33	20	10	8	14	14	7	11	6	5	5	19	3	15	5
75	39	18	11	7	6	7	7	6	5	4	8	6	5	9	4
76	34	20	12	8	8	5	6	6	13	7	8	5	4	7	10
77	35	16	14	9	7	5	12	6	5	6	4	7	14	6	8
78	32	22	8	16	13	10	8	8	6	5	9	6	6	10	5
79	26	17	13	7	13	4	7	8	4	8	6	7	4	6	5
80	39	16	11	14	10	7	5	3	5	5	5	7	9	7	6
81	34	21	7	10	9	8	11	7	4	5	10	8	7	7	0
82	33	18	8	7	11	12	10	10	6	5	6	7	5	6	3
83	32	17	12	15	8	5	8	8	2	2	6	12	7	4	5
84	38	23	10	12	14	11	10	4	7	9	7	6	6	10	10
85	27	15	7	8	9	4	12	8	7	4	6	10	6	5	6
86	38	18	16	13	11	5	4	7	8	9	9	8	6	9	9
87	30	22	14	14	10	8	6	9	5	3	5	6	4	4	5
88	31	20	13	15	5	13	9	12	4	6	5	7	4	5	4
89	24	12	17	15	7	6	5	7	10	10	6	7	5	7	7
90	27	17	8	10	8	8	6	4	2	10	11	4	5	7	3
91	33	16	15	8	11	7	8	9	5	6	10	8	4	9	9
92	33	10	13	12	8	9	4	9	8	3	6	5	5	6	3
93	30	21	12	11	7	9	4	6	8	5	5	3	3	3	4
94	32	13	13	15	9	10	8	10	6	4	2	6	4	6	6
95	28	8	9	12	8	9	1	8	8	4	7	2	7	6	12
96	28	16	13	12	9	11	7	7	7	7	8	4	6	4	8
97	31	9	9	14	5	6	4	7	7	10	7	5	7	10	1
98	20	17	12	7	6	8	8	8	9	7	3	1	6	10	2
99	19	15	19	10	8	5	6	5	8	3	9	8	7	3	8
100	24	15	16	8	11	8	7	7	9	6	5	8	7	6	1

Table 1. (Continued 2)
W test results of Gamma distribution as α and n varies.
($\alpha = 1, 2, 3, \ldots, 60$, $n = 2, 3, 4, \ldots, 150$; Numbers in the table are reject frequency of repeating 100 W tests)

Sample	α														
Size (n)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
101	30	7	7	8	6	7	9	9	11	4	5	6	6	9	4
102	24	15	8	7	7	7	5	4	4	5	6	5	9	8	8
103	31	11	14	7	4	11	4	8	4	9	7	5	4	4	8
104	27	15	11	7	5	5	5	6	6	8	5	5	9	7	8
105	24	14	13	9	12	5	8	7	8	9	9	14	4	6	9
106	30	16	10	8	5	6	9	3	6	7	6	12	6	4	4
107	25	16	16	5	9	6	8	8	5	4	8	7	7	6	6
108	25	12	14	9	14	8	4	3	9	8	3	11	2	6	4
109	30	23	9	11	6	4	9	14	5	7	4	6	8	7	8
110	21	14	10	6	2	9	7	6	10	11	4	6	7	2	11
111	26	17	10	4	9	7	4	7	5	5	3	6	1	4	7
112	25	11	12	6	12	5	4	3	5	8	5	6	5	9	4
113	26	14	10	16	9	7	10	3	5	10	3	4	5	7	3
114	17	9	12	9	11	7	6	4	6	7	7	8	8	4	6
115	23	13	8	12	11	7	6	4	8	2	3	6	6	6	7
116	31	11	5	11	7	10	7	4	6	5	8	4	3	5	9
117	21	13	13	5	6	9	7	4	12	3	10	6	5	7	5
118	20	13	9	9	5	4	6	7	3	5	8	7	7	8	4
119	19	12	12	11	7	3	8	3	8	10	5	3	6	9	4
120	24	8	9	7	8	2	9	3	7	8	5	3	6	9	6
121	28	17	8	9	7	10	2	6	7	7	3	5	3	4	3
122	24	15	10	9	5	7	4	3	8	7	7	5	7	7	4
123	23	17	10	9	11	4	9	1	4	4	6	8	4	2	8
124	16	11	8	12	6	8	7	11	4	8	6	5	8	6	5
125	21	16	7	8	4	7	5	7	6	9	5	7	4	7	6
126	21	8	9	12	4	6	4	4	9	5	6	3	7	5	5
127	19	11	8	6	5	7	6	8	7	8	6	3	3	2	7
128	26	12	7	6	4	7	2	8	3	2	7	9	5	4	6
129	22	14	5	5	6	11	6	7	5	4	5	3	6	2	5
130	19	11	9	4	5	6	3	7	5	6	3	10	6	8	7
131	16	9	9	14	8	4	5	6	3	4	4	11	6	7	4
132	18	5	8	15	7	7	9	5	9	7	6	8	8	7	4
133	23	9	5	10	7	8	8	5	9	3	7	8	6	3	8
134	24	18	5	8	8	10	8	3	5	2	7	6	8	6	10
135	24	11	11	11	3	4	6	4	6	6	9	4	7	5	5
136	21	11	11	10	10	9	3	9	8	3	8	4	6	3	5
137	20	7	5	7	9	6	7	11	10	10	9	3	3	4	7
138	15	11	11	8	9	7	8	3	7	2	5	5	8	5	5
139	14	12	7	8	10	3	8	4	7	8	7	1	6	2	4
140	18	7	8	13	8	6	7	4	5	6	8	5	5	4	6
141	19	10	8	6	2	4	4	10	4	5	8	4	2	5	4
142	18	9	7	8	9	13	2	4	10	7	8	8	6	10	7
143	16	15	11	8	5	2	8	4	14	4	7	6	5	6	4
144	24	10	12	6	13	5	7	9	6	9	3	6	9	5	5
145	19	4	9	4	5	5	6	5	4	5	6	7	5	5	4
146	17	11	10	6	11	2	9	5	3	1	9	5	8	7	10
147	22	11	12	11	11	8	7	4	6	8	7	3	10	8	8
148	24	16	6	9	7	8	5	4	8	12	5	8	7	1	3
149	23	9	14	9	6	2	5	6	4	7	1	6	6	3	5
150	18	13	7	2	6	4	4	2	3	2	5	5	3	2	6

Table 1. (Continued 3)
W test results of Gamma distribution as α and n varies.
($\alpha = 1, 2, 3, \ldots, 60$, $n = 2, 3, 4, \ldots, 150$; Numbers in the table are reject frequency of repeating 100 W tests)

Sample Size (n)	α														
	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
2	82	77	72	75	64	63	56	58	71	58	60	50	49	46	50
3	61	56	47	47	46	35	39	39	41	39	39	30	42	36	33
4	43	46	41	33	40	32	31	35	28	28	28	25	27	25	26
5	36	32	37	25	31	31	25	26	23	17	17	26	21	17	10
6	29	32	26	19	19	24	22	20	19	19	16	20	18	13	14
7	22	28	14	15	21	22	15	16	19	17	10	19	12	18	15
8	24	17	18	22	12	28	13	16	13	16	17	17	13	9	10
9	20	20	21	18	24	19	11	17	11	13	9	13	15	6	11
10	17	19	12	15	12	12	12	11	10	7	9	15	8	12	15
11	20	13	7	13	21	14	10	10	15	7	7	12	5	9	10
12	5	13	18	10	16	13	10	8	6	12	10	9	12	11	15
13	16	19	14	11	7	12	11	9	12	13	15	15	5	7	12
14	19	8	15	14	9	9	8	10	9	7	15	7	15	10	12
15	9	6	13	13	10	8	11	7	8	13	10	6	11	8	5
16	7	12	14	7	12	10	9	13	11	10	8	9	7	5	13
17	10	10	10	15	7	12	7	7	9	16	9	4	9	6	9
18	10	15	8	11	13	17	13	6	3	10	8	6	10	10	6
19	14	9	13	14	10	12	2	17	7	13	4	7	8	4	7
20	12	13	10	8	10	11	19	12	9	6	6	11	6	15	6
21	14	16	11	10	8	10	4	6	15	6	4	13	8	7	11
22	13	5	15	11	4	3	6	10	9	4	11	2	7	5	7
23	14	9	5	6	7	8	8	5	6	3	10	5	11	5	6
24	8	12	13	6	6	13	10	5	5	15	6	10	8	8	7
25	6	5	6	6	9	4	7	4	6	8	8	9	10	5	9
26	4	5	8	10	4	11	9	8	5	4	8	10	5	14	10
27	14	4	9	6	11	8	5	5	5	5	5	5	6	13	6
28	7	9	7	8	9	7	8	6	12	5	9	9	10	5	6
29	5	11	8	10	9	9	7	7	7	12	4	7	11	3	5
30	4	8	10	10	5	7	7	7	4	7	6	8	4	11	4
31	7	8	10	8	3	5	8	6	5	10	7	6	5	6	4
32	9	7	5	5	10	5	12	7	2	5	8	4	4	6	4
33	7	9	11	12	8	7	7	4	7	5	3	6	6	5	10
34	7	10	12	6	8	7	7	6	7	6	11	5	4	8	7
35	11	8	11	7	10	8	7	5	7	5	10	6	8	5	4
36	3	4	10	4	6	5	6	9	5	6	6	4	3	4	7
37	6	8	4	7	7	9	11	3	5	11	10	7	4	7	9
38	12	9	5	6	8	7	5	6	4	10	9	8	2	13	7
39	3	5	5	6	6	7	8	7	5	7	6	10	8	6	4
40	7	8	10	4	4	1	8	9	6	3	5	6	9	7	8
41	7	16	10	3	10	3	7	8	11	7	6	8	5	6	4
42	3	8	5	8	6	7	5	6	11	5	2	12	7	12	2
43	10	7	5	6	3	7	4	8	3	11	8	7	9	4	6
44	7	10	8	5	6	9	5	6	6	4	10	6	7	8	9
45	5	4	6	11	2	11	8	8	3	9	6	8	5	7	8
46	6	2	5	5	3	11	10	5	5	13	5	3	6	6	5
47	9	6	6	5	2	8	3	7	8	9	6	7	4	3	6
48	7	8	7	4	1	7	2	6	7	6	5	9	7	5	10
49	6	3	3	8	7	5	6	10	6	5	6	7	4	7	6
50	1	6	11	7	4	7	8	6	3	6	7	11	12	3	3

Table 1. (Continued 4)
W test results of Gamma distribution as α and n varies.
($\alpha = 1, 2, 3, \ldots, 60$, $n = 2, 3, 4, \ldots, 150$; Numbers in the table are reject frequency of repeating 100 W tests)

Sample	α														
Size (n)	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
51	7	10	4	10	6	8	8	5	10	6	8	6	8	6	8
52	5	6	12	9	3	7	10	7	5	6	7	6	6	3	5
53	11	4	10	9	9	5	3	5	3	12	7	6	7	7	6
54	4	8	8	8	7	5	7	6	4	6	7	8	5	6	7
55	6	7	12	4	2	8	6	5	5	4	5	6	8	6	6
56	2	7	4	7	2	4	4	4	1	7	4	5	8	8	4
57	2	9	9	6	8	7	9	3	6	4	5	5	5	6	9
58	3	5	3	8	6	6	8	6	6	2	7	3	9	5	4
59	4	7	7	5	9	3	3	8	7	4	7	2	5	4	4
60	2	8	8	4	6	5	5	7	2	10	3	4	8	4	5
61	4	9	7	3	4	8	6	4	9	6	1	10	3	5	10
62	11	9	4	9	6	4	6	8	4	10	6	2	8	7	5
63	2	6	5	4	8	5	2	6	6	6	10	7	4	7	2
64	9	10	9	9	2	5	5	4	9	5	10	6	3	4	7
65	7	4	4	9	8	4	5	4	6	7	6	7	5	7	1
66	11	6	5	6	6	3	6	6	4	6	10	8	4	3	6
67	9	5	6	7	2	6	8	9	5	5	4	3	5	5	6
68	7	4	9	10	7	5	8	8	7	4	7	2	12	3	3
69	7	5	8	13	4	5	6	2	6	6	8	10	3	10	3
70	6	5	6	7	11	2	7	8	1	5	3	9	9	5	5
71	8	7	5	4	5	7	8	5	3	2	7	7	4	1	5
72	6	6	7	8	5	7	5	6	8	6	5	5	4	6	7
73	6	8	3	4	5	5	4	5	7	11	6	8	4	2	2
74	5	3	5	6	6	3	5	4	7	6	9	7	6	6	5
75	5	5	4	5	4	7	3	4	11	5	4	4	5	10	3
76	4	3	7	8	6	7	8	8	1	6	7	5	6	8	4
77	7	8	6	9	4	6	4	9	9	3	3	5	8	6	4
78	10	4	4	8	3	4	5	5	6	4	3	4	9	5	7
79	2	5	2	3	3	4	4	6	5	6	1	2	4	5	7
80	7	5	8	5	9	4	8	4	7	10	10	2	5	3	5
81	5	4	0	4	5	7	5	4	7	7	3	6	2	7	6
82	6	8	4	3	4	5	9	8	8	7	7	7	3	4	6
83	8	2	5	6	7	7	6	7	5	5	5	4	7	3	8
84	7	8	6	9	7	2	7	5	9	6	8	5	5	5	3
85	4	8	7	8	2	10	4	2	6	6	6	4	4	6	11
86	3	7	4	7	6	5	4	9	3	6	5	4	8	3	8
87	8	7	8	3	5	4	2	8	5	4	4	6	7	3	4
88	6	3	10	8	4	5	6	4	2	5	5	4	2	5	3
89	8	4	9	6	5	8	6	8	10	9	7	6	10	4	3
90	6	11	7	6	8	6	5	6	7	10	3	6	2	6	4
91	7	7	5	8	5	6	3	5	10	5	9	5	6	7	9
92	6	5	7	6	7	2	4	8	4	6	8	4	2	5	5
93	8	4	10	6	4	7	7	7	4	7	5	8	2	4	7
94	3	7	2	4	11	5	9	9	8	6	7	4	3	3	6
95	6	5	7	8	6	6	4	8	8	7	9	6	7	4	4
96	6	9	7	5	2	6	5	11	6	4	8	6	1	3	8
97	5	3	6	2	6	8	8	1	3	9	3	4	3	7	6
98	3	7	5	5	8	5	6	6	9	7	2	5	4	6	7
99	6	4	2	5	5	6	10	7	5	8	8	7	3	3	6
100	5	6	1	10	3	6	5	6	8	3	9	9	4	6	4

Table 1. (Continued 5)
W test results of Gamma distribution as α and n varies.
($\alpha = 1, 2, 3, \ldots, 60$, $n = 2, 3, 4, \ldots, 150$; Numbers in the table are reject frequency of repeating 100 W tests)

Sample	α														
Size (n)	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
101	7	9	10	4	4	2	5	9	5	3	2	7	7	2	13
102	7	8	8	3	4	6	4	4	7	6	12	8	7	1	4
103	4	6	4	3	9	5	7	9	6	5	5	8	6	6	8
104	5	3	8	3	6	4	4	7	8	4	5	9	3	9	4
105	5	5	7	3	5	5	4	5	9	2	6	9	3	7	5
106	3	6	6	7	6	8	6	4	6	9	8	1	3	3	4
107	10	5	4	3	5	8	1	7	7	10	10	12	7	4	5
108	4	6	6	7	6	7	4	6	6	3	4	4	3	4	6
109	6	5	6	5	4	7	5	4	5	6	8	9	9	8	5
110	4	6	4	4	8	6	4	7	1	5	4	5	5	5	7
111	9	4	2	5	8	4	6	7	6	5	3	4	5	4	8
112	7	8	6	7	4	2	5	3	8	7	5	5	5	3	4
113	3	2	6	8	10	6	5	4	3	10	9	3	2	4	4
114	6	8	7	9	3	5	8	6	4	4	6	6	8	7	5
115	7	8	5	5	4	6	3	4	3	6	6	8	8	6	8
116	3	4	4	8	5	3	9	7	5	4	6	6	6	7	6
117	2	5	4	8	8	3	8	1	4	1	4	6	2	4	6
118	5	2	5	12	3	4	3	8	6	7	4	7	4	8	7
119	6	8	10	3	6	7	6	0	5	5	5	7	7	10	3
120	5	9	4	3	4	4	6	3	6	3	2	5	4	2	5
121	5	3	6	9	5	4	5	4	4	5	4	11	7	8	5
122	4	5	7	5	7	6	11	9	5	7	5	6	5	4	3
123	5	6	10	6	10	3	6	9	8	6	5	4	5	8	4
124	4	6	5	3	5	6	8	8	7	6	4	3	9	5	6
125	11	9	5	9	5	5	5	4	9	3	5	3	8	3	5
126	1	6	6	6	3	5	2	6	4	5	9	4	4	5	5
127	3	6	3	4	5	7	6	12	6	5	2	10	5	3	6
128	7	1	5	6	5	7	10	2	7	4	2	7	2	2	5
129	12	4	2	3	6	7	8	4	7	6	3	3	3	4	3
130	11	10	5	2	6	4	10	1	5	4	9	5	6	3	5
131	6	8	6	7	4	2	6	4	5	7	3	5	4	4	2
132	7	4	4	6	2	5	8	8	10	4	4	3	8	7	5
133	6	8	8	7	7	4	2	5	8	2	7	3	7	3	4
134	5	8	6	7	5	5	8	5	7	8	10	9	3	5	4
135	4	13	8	5	8	4	7	13	11	7	7	9	7	4	8
136	4	2	8	8	4	4	12	8	6	8	7	8	4	9	3
137	6	6	4	3	3	4	4	3	5	2	7	5	6	7	6
138	6	10	12	9	10	8	6	3	2	9	6	4	5	3	7
139	9	1	7	7	3	4	6	4	1	1	4	4	6	5	6
140	7	3	5	7	9	1	4	5	2	9	4	7	6	4	8
141	3	10	4	8	6	4	5	4	6	12	5	6	7	9	6
142	4	4	5	9	8	5	6	6	5	7	6	4	5	6	9
143	4	4	3	3	5	8	6	5	7	4	5	12	9	3	3
144	4	5	4	9	5	4	2	10	5	3	5	6	3	11	5
145	6	4	3	2	5	4	6	5	11	6	2	4	7	5	9
146	4	6	8	5	9	6	7	6	7	9	4	5	3	10	4
147	1	6	5	7	6	1	4	7	10	10	10	3	2	6	3
148	8	6	4	5	6	5	6	3	6	9	2	6	4	7	3
149	8	2	3	3	5	6	4	3	4	3	6	9	4	5	6
150	5	5	3	6	3	8	6	13	5	7	1	8	3	4	4

Table 1. (Continued 6)
W test results of Gamma distribution as α and n varies.
($\alpha = 1, 2, 3, \ldots, 60$, $n = 2, 3, 4, \ldots, 150$; Numbers in the table are reject frequency of repeating 100 W tests)

Sample Size (n)	α														
	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
2	42	42	41	48	38	31	44	30	31	30	34	32	33	29	25
3	29	24	37	24	25	27	23	29	18	20	13	17	21	23	22
4	26	24	24	22	22	15	10	15	26	15	18	19	16	14	22
5	18	22	23	10	18	17	13	18	19	12	18	12	14	23	14
6	15	11	14	16	19	17	20	11	11	13	17	5	8	15	12
7	10	20	12	10	18	18	15	13	10	12	15	13	12	10	10
8	16	11	15	13	10	10	9	6	9	5	7	12	9	13	3
9	14	9	18	12	10	5	19	13	12	11	12	15	10	5	7
10	6	11	9	19	10	6	13	13	13	10	11	5	14	11	7
11	6	6	8	14	5	6	6	6	8	4	12	7	7	9	13
12	6	8	7	12	8	13	13	5	9	10	11	11	6	2	9
13	12	9	10	9	9	5	9	6	13	6	8	3	8	6	7
14	7	8	5	13	6	11	4	7	12	5	9	8	4	8	9
15	9	6	8	9	7	2	7	11	7	11	7	5	6	6	9
16	14	8	8	7	8	6	8	7	13	6	8	8	4	8	14
17	8	8	7	4	8	7	7	6	4	8	2	7	4	8	6
18	9	4	8	6	8	9	8	11	7	4	7	10	6	6	5
19	6	12	10	5	5	6	4	10	4	3	4	3	4	8	7
20	6	7	6	10	8	9	6	3	14	9	6	10	2	7	8
21	4	9	4	8	8	10	6	7	6	7	7	2	7	5	4
22	9	2	9	6	6	6	5	9	9	6	6	2	11	6	6
23	7	5	9	4	10	4	2	8	6	4	6	6	7	6	11
24	7	6	8	9	10	6	7	9	4	5	5	10	7	5	13
25	6	5	8	5	4	4	7	5	6	5	6	9	2	2	8
26	11	7	10	8	5	10	7	7	6	4	4	9	5	3	7
27	8	5	7	8	6	7	10	2	12	3	7	9	8	9	10
28	6	8	4	9	5	5	11	6	5	7	4	11	2	6	6
29	6	4	3	6	11	5	6	9	6	4	6	10	4	7	6
30	10	10	7	7	6	3	5	8	7	3	10	9	5	9	7
31	7	9	8	6	4	5	6	8	6	4	6	5	1	4	5
32	3	3	7	14	10	5	9	7	14	12	5	3	4	6	4
33	3	8	8	6	9	5	3	5	7	6	3	5	4	5	4
34	7	6	4	8	3	6	14	6	4	9	7	5	5	7	5
35	9	6	5	7	12	5	9	4	6	3	6	8	7	5	6
36	4	3	6	4	5	6	10	4	8	5	8	6	7	6	5
37	4	7	7	3	6	7	4	9	2	5	7	4	4	11	7
38	6	5	3	3	6	4	7	5	6	2	5	9	6	6	4
39	6	4	7	7	4	3	8	4	6	5	10	4	7	10	7
40	4	1	5	4	5	4	3	5	5	6	4	4	3	9	9
41	6	2	5	9	8	4	6	8	4	6	4	7	7	6	9
42	7	12	8	6	8	4	5	3	8	3	7	9	7	3	8
43	3	3	5	4	3	4	2	1	4	8	7	3	2	2	2
44	8	5	8	5	5	8	7	6	3	4	7	4	1	6	7
45	8	8	8	9	7	9	7	7	4	3	3	3	7	0	3
46	3	7	5	11	9	3	5	4	2	5	6	4	5	2	3
47	6	7	7	1	9	4	6	2	7	7	14	5	2	6	4
48	4	6	4	7	5	6	11	5	4	10	7	5	6	7	8
49	6	10	7	9	6	5	3	3	2	4	6	7	3	4	7
50	5	4	7	6	7	8	8	3	3	4	6	7	4	5	2

Table 1. (Continued 7)
W test results of Gamma distribution as α and n varies.
($\alpha = 1, 2, 3, \ldots, 60$, $n = 2, 3, 4, \ldots, 150$; Numbers in the table are reject frequency of repeating 100 W tests)

Sample	α														
Size (n)	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
51	7	6	9	4	5	4	2	5	5	6	6	6	11	5	8
52	4	8	7	9	2	3	5	4	6	6	8	3	6	7	6
53	6	4	6	0	4	3	6	9	4	4	3	7	9	8	2
54	7	10	4	5	6	3	6	8	4	6	0	5	6	7	11
55	8	6	7	2	4	9	9	11	4	3	6	9	3	7	5
56	8	3	6	7	7	5	5	6	2	7	4	3	3	6	2
57	6	5	6	5	3	10	5	4	5	10	6	4	6	3	5
58	6	3	6	4	5	5	2	1	4	4	5	8	6	2	7
59	8	11	10	9	8	9	7	3	3	9	10	7	10	6	5
60	12	4	4	8	8	5	2	7	7	3	4	7	5	7	4
61	8	2	5	4	5	10	6	1	10	9	11	5	3	7	5
62	8	3	10	3	6	7	8	6	5	6	10	6	3	8	4
63	4	8	7	5	4	5	5	4	10	5	6	7	2	7	6
64	2	4	11	3	9	10	4	5	3	7	8	4	7	10	5
65	6	5	6	5	3	5	6	8	6	4	3	5	10	6	6
66	6	6	6	5	6	7	7	3	8	7	6	6	6	5	3
67	4	1	3	6	6	3	6	4	0	3	3	8	7	4	7
68	4	3	3	4	6	5	8	8	7	5	7	6	11	6	8
69	3	9	4	6	4	8	2	4	2	8	1	2	8	8	3
70	4	10	3	7	1	7	8	4	7	4	3	4	8	4	12
71	2	4	7	4	8	7	6	11	6	2	6	0	3	5	9
72	3	6	6	9	7	11	5	7	3	5	4	4	5	8	4
73	7	5	3	6	2	5	2	6	4	5	3	3	2	6	6
74	1	4	5	8	8	7	7	9	2	5	6	4	3	9	2
75	8	8	7	5	5	6	1	6	6	8	6	6	6	3	2
76	4	8	6	5	5	7	5	8	3	7	9	9	4	6	8
77	3	5	3	5	3	5	10	7	1	9	5	7	4	2	4
78	6	5	7	5	6	4	8	1	5	6	2	8	6	7	8
79	7	7	4	5	9	8	5	7	5	6	6	7	4	7	4
80	8	7	5	8	8	6	6	3	4	6	7	11	2	3	5
81	8	6	5	3	7	3	9	3	4	6	6	6	6	4	5
82	6	8	5	4	6	9	4	9	6	6	3	4	8	3	4
83	9	5	5	5	5	6	7	4	5	3	6	6	6	5	10
84	7	5	5	4	5	8	9	2	3	7	6	7	5	3	12
85	4	3	11	5	6	3	12	5	6	7	5	6	10	5	3
86	0	2	2	8	6	5	8	7	8	3	5	5	2	6	4
87	1	7	6	5	9	6	5	8	7	5	2	5	6	6	7
88	9	3	8	4	5	3	5	6	10	4	5	4	9	7	3
89	3	7	5	5	6	6	4	3	5	5	9	5	8	10	6
90	6	4	4	6	7	4	5	9	9	3	5	5	7	8	4
91	2	7	8	4	6	4	5	5	4	3	4	4	2	3	4
92	4	1	7	5	4	6	5	3	3	6	4	9	6	5	1
93	4	7	5	6	7	4	4	7	7	4	6	5	5	3	5
94	4	8	3	1	4	1	7	3	4	5	8	3	3	3	5
95	5	8	5	4	3	5	1	6	4	3	5	6	6	2	5
96	7	10	2	6	4	5	8	3	4	7	2	8	10	2	3
97	5	10	10	8	8	4	4	3	7	3	5	3	7	2	5
98	7	8	8	11	4	3	5	7	6	4	11	4	5	6	4
99	5	3	1	7	2	7	4	7	1	4	4	1	6	6	5
100	3	4	3	4	6	4	6	8	6	3	7	7	7	2	5

Table 1. (Continued 8)
W test results of Gamma distribution as α and n varies.
($\alpha = 1, 2, 3, \ldots, 60$, $n = 2, 3, 4, \ldots, 150$; Numbers in the table are reject frequency of repeating 100 W tests)

Sample	α														
Size (n)	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
101	5	2	4	1	5	6	7	7	9	3	8	8	10	4	6
102	4	6	6	3	8	4	3	3	3	7	8	6	6	4	7
103	10	7	11	1	6	3	7	6	6	11	5	7	2	5	2
104	3	4	2	5	7	7	6	4	1	6	5	9	10	5	9
105	4	5	5	7	7	6	4	6	4	6	1	1	7	6	4
106	6	1	2	5	6	6	7	10	7	3	3	6	8	4	6
107	4	4	5	0	2	4	4	3	5	8	3	2	5	5	6
108	5	6	8	7	7	3	3	10	3	2	5	4	3	9	5
109	2	3	5	5	3	7	3	8	6	6	3	4	5	2	3
110	5	4	7	9	1	3	5	5	2	4	8	8	6	5	5
111	3	5	0	4	2	4	6	2	1	7	5	6	7	4	5
112	7	2	6	5	3	5	5	5	5	6	5	6	5	6	5
113	3	3	5	4	5	3	7	4	2	3	7	3	5	2	4
114	5	3	5	6	8	10	5	4	4	5	9	5	5	5	4
115	4	2	5	3	2	7	6	5	5	3	4	4	4	2	3
116	0	5	6	5	7	3	4	10	4	4	4	3	3	6	3
117	5	7	6	5	4	4	3	9	4	1	4	7	3	3	4
118	4	5	2	9	8	4	3	2	10	10	6	10	6	7	6
119	8	9	7	3	4	4	6	4	6	7	11	7	4	9	1
120	6	4	7	6	4	4	5	3	8	6	11	5	5	4	3
121	3	7	5	8	9	2	2	2	7	4	5	3	8	4	2
122	5	6	6	7	7	6	4	4	2	2	4	3	4	6	5
123	8	4	3	10	6	5	10	4	4	6	2	4	3	4	8
124	5	10	6	6	3	6	4	7	5	7	3	6	4	7	7
125	5	4	8	7	7	3	6	5	5	8	5	4	1	4	5
126	6	5	7	2	4	4	2	5	8	5	4	7	6	3	11
127	4	6	3	3	6	7	7	7	6	5	8	3	7	4	6
128	5	10	7	5	5	4	2	5	3	5	7	6	1	2	5
129	5	1	5	4	7	1	5	4	5	7	3	6	8	6	2
130	5	7	6	6	9	7	4	6	2	11	3	6	4	8	4
131	3	6	8	5	4	5	2	7	3	7	5	9	7	6	5
132	4	6	9	3	3	3	5	4	4	5	2	8	6	5	7
133	7	5	5	4	5	5	8	8	7	4	6	3	4	3	6
134	6	4	2	5	3	2	4	3	1	4	11	3	6	7	6
135	4	4	8	7	4	8	7	4	5	2	9	5	5	4	6
136	2	6	6	8	5	6	3	2	3	5	7	6	5	2	6
137	10	4	2	5	7	4	6	5	5	2	5	7	6	5	3
138	7	5	7	3	3	6	2	8	6	5	5	4	8	4	3
139	4	6	4	5	7	2	3	3	1	6	14	5	1	3	6
140	5	2	3	2	10	2	4	2	7	4	9	5	5	4	5
141	9	3	7	9	7	5	6	5	2	3	2	5	3	4	4
142	5	5	2	7	3	8	1	6	6	4	6	4	5	4	7
143	4	3	7	6	8	5	3	4	6	7	8	2	7	7	4
144	8	5	6	7	3	3	8	6	3	8	7	10	2	3	1
145	7	6	3	9	3	4	7	9	6	3	5	5	7	3	4
146	8	7	5	3	5	2	4	2	9	11	5	4	2	6	6
147	1	6	8	7	5	4	6	5	5	8	5	8	7	4	9
148	0	1	4	6	6	4	10	4	6	3	3	5	6	0	7
149	9	4	4	7	3	4	6	4	5	4	1	2	4	5	7
150	2	8	6	4	3	8	4	3	5	3	4	4	1	11	8

Table 1. (Continued 9)
W test results of Gamma distribution as α and n varies.
($\alpha = 1, 2, 3, \ldots, 60$, $n = 2, 3, 4, \ldots, 150$; Numbers in the table are reject frequency of repeating 100 W tests)

Sample Size (n)	α														
	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
2	30	26	27	27	22	29	32	30	33	30	37	22	20	23	23
3	28	20	23	19	15	23	20	14	16	16	16	14	16	20	16
4	18	13	16	11	11	17	17	17	14	12	20	11	16	16	11
5	15	16	8	14	12	15	18	12	6	9	11	14	12	9	14
6	15	7	17	13	12	6	7	12	11	12	9	10	14	11	8
7	9	15	9	10	11	16	4	11	7	7	8	6	8	9	13
8	13	11	15	15	14	10	7	12	12	15	8	7	13	7	6
9	10	7	9	8	14	12	7	6	9	14	5	12	9	10	3
10	11	10	5	10	10	5	11	13	8	8	12	8	8	12	7
11	4	10	11	6	12	7	8	8	3	7	3	5	8	8	5
12	7	10	9	7	6	9	8	5	7	6	6	10	8	5	10
13	10	8	11	6	7	8	4	6	12	5	8	5	4	8	10
14	6	6	9	9	8	12	13	11	7	7	9	11	9	8	6
15	8	10	6	6	8	8	8	8	6	6	4	11	5	5	9
16	5	3	8	8	2	5	3	9	7	6	6	2	4	7	4
17	8	3	10	8	5	2	7	5	2	7	7	10	3	5	5
18	11	12	6	4	6	6	10	10	8	5	6	9	4	5	3
19	15	14	10	5	3	9	9	6	7	8	8	4	6	3	4
20	6	9	4	4	3	6	12	4	4	13	6	5	3	5	5
21	6	5	7	7	5	11	8	9	4	8	5	2	7	6	6
22	7	5	8	5	4	9	9	6	7	7	3	6	8	8	6
23	10	3	9	8	7	4	6	4	6	11	4	7	4	7	5
24	8	8	5	6	9	5	3	3	3	7	5	6	5	2	5
25	11	2	5	9	6	8	5	4	4	9	5	3	7	5	9
26	7	5	5	6	8	5	11	3	8	5	3	6	4	5	5
27	4	4	6	7	5	6	6	4	8	5	4	10	6	5	3
28	5	7	11	4	7	5	8	11	4	6	6	11	7	7	5
29	6	9	4	8	8	6	3	8	6	3	5	8	9	6	4
30	2	9	5	6	7	3	4	5	10	7	4	9	8	6	6
31	4	3	6	10	9	8	6	9	8	8	9	5	7	6	3
32	6	4	12	10	11	7	1	5	5	4	7	3	6	7	5
33	4	5	2	3	5	6	2	9	6	3	10	7	11	12	8
34	12	4	5	11	7	4	7	7	8	4	7	5	4	4	3
35	4	7	6	9	3	5	5	6	5	11	3	10	7	4	6
36	5	6	9	3	6	7	8	4	5	2	7	0	7	5	3
37	5	4	3	8	6	7	4	4	6	14	4	4	8	2	2
38	7	6	6	2	5	4	7	7	9	7	4	6	6	5	5
39	5	7	2	4	9	5	6	7	8	5	5	4	10	6	3
40	6	6	5	7	7	8	11	6	6	10	8	6	8	8	7
41	3	7	2	1	0	6	7	9	7	6	4	5	3	7	3
42	3	3	6	6	8	4	7	4	6	4	6	7	5	5	2
43	10	3	3	4	4	7	3	6	4	8	3	7	7	3	10
44	12	6	6	9	3	9	6	3	7	4	5	3	9	7	7
45	5	11	6	5	4	6	5	7	4	7	6	2	4	7	5
46	6	2	5	6	11	5	4	6	3	6	3	9	4	3	1
47	3	3	4	6	4	5	5	7	5	5	4	4	6	6	3
48	5	4	6	9	3	6	5	3	9	9	2	9	2	3	3
49	6	7	7	8	8	9	8	1	7	7	8	4	3	4	5
50	6	4	7	3	4	3	8	10	6	7	3	6	7	5	7

Table 1. (Continued 10)
W test results of Gamma distribution as α and n varies.
($\alpha = 1, 2, 3, \ldots, 60$, $n = 2, 3, 4, \ldots, 150$; Numbers in the table are reject frequency of repeating 100 W tests)

Sample	α														
Size (n)	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
51	7	8	6	6	8	7	8	4	6	6	6	6	1	4	8
52	8	7	9	8	10	3	2	1	8	8	6	6	5	7	3
53	3	5	6	4	4	6	7	3	13	9	5	3	8	5	5
54	9	6	1	4	5	7	3	5	4	1	4	9	5	3	4
55	6	6	3	9	3	5	7	8	9	4	10	6	6	9	5
56	6	9	6	6	6	8	5	7	4	8	5	5	5	4	7
57	4	6	5	5	3	7	10	4	6	8	5	6	5	8	6
58	5	6	3	7	6	3	4	3	5	6	2	5	5	5	7
59	6	6	11	4	7	3	7	5	9	5	3	4	6	3	6
60	2	1	11	4	5	7	4	6	10	3	5	6	7	6	2
61	4	3	5	11	8	5	4	7	10	4	4	3	3	6	5
62	5	8	4	8	4	8	3	9	9	5	8	2	3	5	6
63	8	5	4	6	9	8	5	3	5	4	6	2	5	5	6
64	2	3	5	4	7	6	9	5	5	9	8	5	4	8	4
65	6	4	3	2	5	5	3	3	8	3	9	9	9	8	4
66	15	3	5	5	8	5	6	4	7	7	7	4	8	6	15
67	6	4	4	6	4	8	7	4	7	3	7	6	5	4	8
68	4	7	6	9	3	6	3	6	5	2	4	4	6	3	6
69	6	5	6	5	3	4	2	8	5	5	6	7	5	6	6
70	7	4	2	6	4	4	5	4	8	8	8	8	5	5	4
71	5	9	6	3	10	2	7	9	5	4	8	2	5	5	15
72	3	7	6	4	5	6	4	6	8	3	5	2	6	3	8
73	4	5	5	4	6	10	7	4	6	6	3	2	5	6	7
74	7	6	5	2	5	2	2	4	4	3	8	6	3	9	4
75	5	5	7	5	6	4	6	3	3	7	7	4	8	6	6
76	7	4	3	5	5	4	6	4	6	7	8	4	8	6	5
77	4	2	3	4	8	10	9	6	5	8	8	6	14	4	11
78	4	4	10	8	8	9	4	4	2	4	4	4	7	7	3
79	5	5	5	6	3	4	6	5	4	8	1	6	8	2	3
80	3	1	2	5	5	5	6	4	7	3	4	7	6	5	5
81	4	4	4	6	3	6	7	2	7	5	4	6	7	6	3
82	6	9	3	6	2	7	3	5	4	4	3	6	2	5	5
83	2	6	6	5	4	5	4	2	3	4	6	6	0	9	5
84	6	8	9	2	6	3	6	5	3	3	5	9	4	7	5
85	9	5	6	7	9	6	9	2	4	1	5	7	6	4	5
86	4	6	5	7	11	6	5	6	6	7	10	6	5	5	3
87	5	6	7	5	5	9	7	6	5	7	7	7	4	6	9
88	11	4	9	3	3	3	5	7	6	5	7	6	6	9	0
89	6	6	11	1	5	6	8	6	10	3	6	6	5	2	7
90	7	4	1	7	5	9	4	5	5	7	7	6	4	2	5
91	2	3	5	4	2	4	6	4	4	6	7	4	6	4	9
92	6	2	4	6	7	3	3	3	4	3	4	3	7	6	4
93	4	3	9	5	7	6	3	8	8	6	6	7	5	4	6
94	8	4	3	5	7	3	6	3	4	2	2	6	2	5	6
95	10	8	10	3	8	3	9	8	7	7	10	4	6	8	6
96	5	4	6	6	3	7	4	4	4	8	3	4	7	6	6
97	9	7	9	3	4	5	4	7	3	3	5	5	9	4	4
98	9	8	3	9	5	6	8	6	3	3	5	2	2	6	9
99	4	2	4	7	7	6	5	7	6	1	9	2	9	5	3
100	9	6	10	2	1	4	5	6	8	7	7	7	8	4	3

Table 1. (Continued 11)
W test results of Gamma distribution as α and n varies.
($\alpha = 1, 2, 3, \ldots, 60$, $n = 2, 3, 4, \ldots, 150$; Numbers in the table are reject frequency of repeating 100 W tests)

Sample	α														
Size (n)	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
101	5	6	5	3	5	3	5	7	4	4	8	4	12	5	8
102	10	7	8	6	3	6	6	6	3	5	6	4	9	3	7
103	9	5	9	3	6	4	4	3	7	8	4	6	9	7	7
104	4	7	5	5	5	4	9	11	4	9	5	5	6	8	9
105	4	9	5	8	6	6	5	7	7	10	5	4	11	3	5
106	5	5	1	9	4	10	4	4	4	4	4	4	7	6	7
107	10	6	7	1	3	4	10	10	3	7	6	5	5	8	3
108	7	10	6	3	4	5	6	6	2	5	3	9	5	5	5
109	3	7	3	9	8	3	4	5	3	5	3	0	6	4	6
110	4	3	6	6	2	6	2	6	9	1	8	3	4	7	2
111	3	3	4	4	4	4	4	4	5	5	3	4	1	4	4
112	6	6	1	5	2	6	7	4	2	7	10	2	9	8	2
113	7	6	8	6	4	9	5	4	4	3	3	5	7	4	3
114	4	1	3	4	8	3	1	10	6	4	8	6	8	9	8
115	5	5	6	6	8	3	7	8	6	4	6	5	8	8	2
116	6	4	7	3	5	7	4	4	5	4	10	5	4	2	5
117	6	6	6	5	6	2	6	7	2	10	5	5	2	8	7
118	3	7	2	7	5	5	4	9	5	3	4	4	5	8	4
119	5	1	2	4	5	3	6	6	3	6	7	5	10	2	6
120	5	4	5	2	6	11	7	5	7	1	5	6	6	5	5
121	1	5	4	6	8	3	7	6	7	8	10	9	8	2	5
122	6	4	2	4	2	6	7	5	7	3	5	8	3	10	2
123	5	2	4	4	3	1	4	3	5	7	3	7	5	6	1
124	6	9	3	5	3	5	6	4	1	4	8	3	4	4	5
125	8	7	7	6	4	5	3	8	5	4	3	6	4	4	8
126	5	8	4	4	7	5	7	7	7	5	5	7	6	3	5
127	6	3	3	1	3	4	5	8	3	4	4	5	1	2	2
128	6	4	6	4	9	4	3	7	1	5	3	3	8	3	2
129	3	7	4	6	8	5	6	5	5	2	2	4	3	3	4
130	4	3	5	5	5	4	2	3	6	7	2	6	2	5	2
131	7	5	6	1	5	7	3	2	5	4	7	4	3	3	4
132	8	4	6	5	4	6	4	4	7	3	6	6	8	5	6
133	7	7	3	6	8	2	9	6	7	3	5	5	4	6	3
134	6	7	4	4	5	10	4	7	7	5	6	3	4	5	5
135	4	6	5	5	7	4	6	5	8	6	4	4	8	3	9
136	2	6	3	7	7	8	8	5	4	4	5	4	1	3	3
137	7	3	4	4	4	4	10	8	10	8	5	9	3	6	6
138	8	5	5	7	5	9	3	7	4	7	11	6	10	7	8
139	9	4	6	4	4	6	4	7	6	1	5	6	4	4	6
140	11	10	8	9	11	8	3	7	10	5	3	3	5	9	10
141	3	7	2	4	7	2	7	6	2	7	6	4	6	5	4
142	5	2	1	5	4	4	6	5	5	4	5	5	5	8	0
143	5	1	3	6	1	6	10	6	4	3	7	7	1	4	5
144	3	5	0	3	7	4	4	5	6	3	7	4	4	6	3
145	5	6	5	3	6	4	5	1	3	2	4	4	4	5	8
146	10	3	4	8	5	4	3	5	5	3	3	7	6	2	5
147	2	2	7	8	6	5	7	1	9	4	8	8	5	6	5
148	5	9	5	6	3	2	9	7	11	4	8	6	6	6	3
149	3	7	5	5	3	4	2	5	5	3	1	2	6	4	6
150	8	3	8	7	8	2	5	6	2	6	3	5	4	2	5

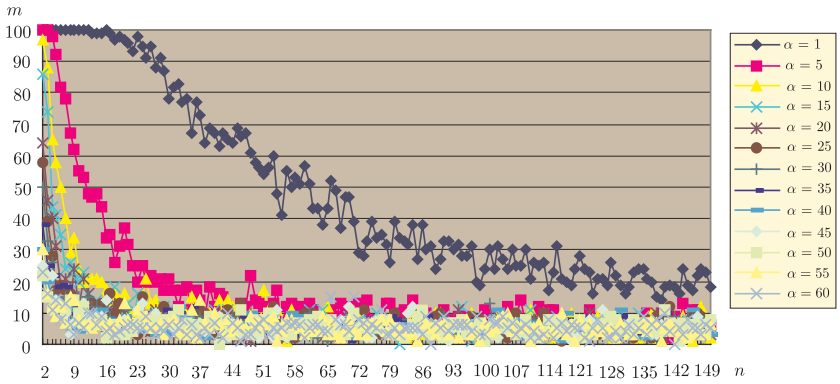


Figure 1. Relation between rejected frequency and sample sizes (m is reject frequency of repeating 100 W tests, n is the sample size, α is a parameter of gamma distribution).

So far, we can tell that when the sample size is larger, the approximation to normality is better; when the value of α is larger, the approximation to normality is faster. Thus, it can be seen that using the only one criterion is dangerous.

4. Discussion

Figure 1 shows the frequency m, as a result of the normality hypothesis being rejected under sample size n for each α by using the W test results in Section 3.

Except in situation $\alpha = 1$, when the sample size n is between 10 and 25, the frequency m, which rejects the normality hypothesis, drops down to about 10 and becomes steady. From Figure 1, we can reach the same conclusion as in Section 3, which shows when the sample size is larger, the approximation to normality is better. And, when the parameter α is larger, the approximation to normality is faster.

From Figure 1, we can see a trend between the rejected frequency m and the sample size n for each α. Here, we will try to find the trend by a regression model. Using the inverse regression model ($m = b_0 + b_1/n$) to fit the relationship between the rejected frequency m and sample size n for each α of gamma distribution, the result is shown in Table 2.

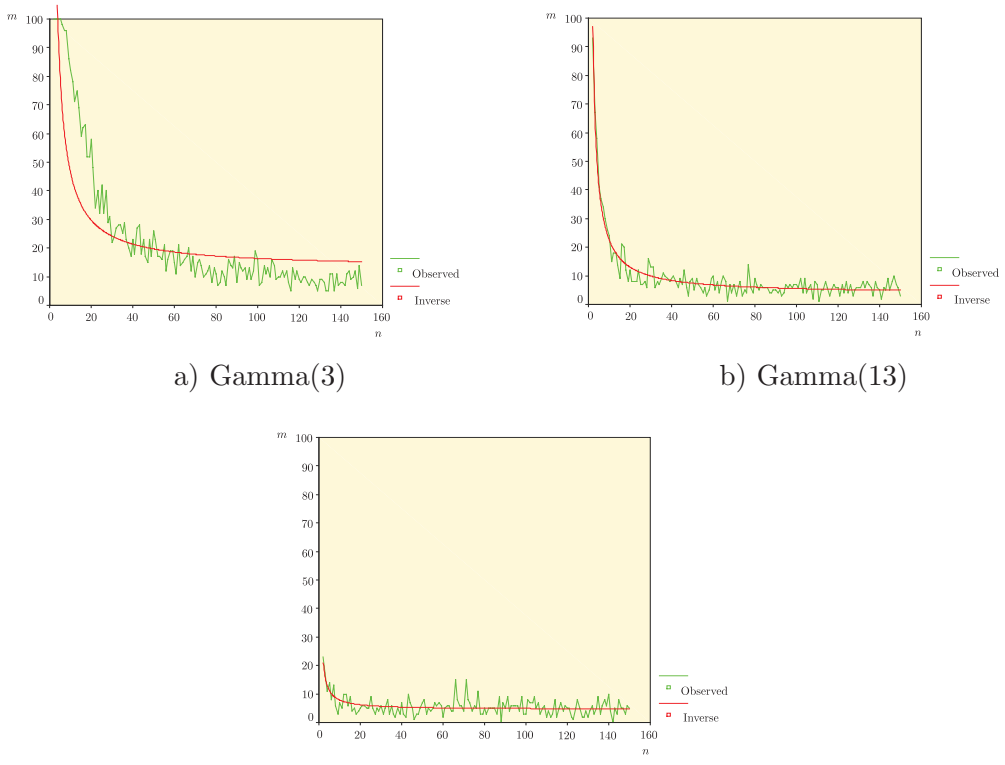
Table 2 shows that there is a significant relationship between the parameters α of gamma distribution and sample sizes n. The R^2 value climbs fast from 0.347 at $\alpha = 1$, reaching the acceptable standard of 0.667 at $\alpha = 3$, and then to the top $R^2 = 0.944$ at $\alpha = 13$. After that, the R^2 falls because the tail of the function gets longer and flatter, as Figure 2.

Table 2. The estimates of parameters of the inverse regression model for each parameter α of gamma distributions.

α	R-Square	F Value	p-value	b_0	b_1
1	0.347	78.02	0.0000	39.185	293.277
2	0.545	176.32	0.0000	20.425	336.230
3	0.667	293.80	0.0000	13.066	331.123
4	0.756	454.47	0.0000	9.765	310.179
5	0.811	630.68	0.0000	7.520	299.397
6	0.851	839.11	0.0000	5.847	283.772
7	0.885	1136.01	0.0000	5.258	266.934
8	0.899	1302.75	0.0000	4.844	251.633
9	0.913	1548.93	0.0000	4.192	239.677
10	0.927	1865.64	0.0000	3.970	219.442
11	0.938	2215.53	0.0000	4.148	214.743
12	0.925	1818.92	0.0000	4.065	196.085
13	0.944	2478.25	0.0000	3.703	186.486
14	0.922	1744.68	0.0000	3.752	168.346
15	0.915	1582.50	0.0000	3.518	172.037
16	0.910	1483.06	0.0000	3.599	155.686
17	0.906	1411.96	0.0000	3.979	147.481
18	0.889	1179.14	0.0000	4.204	132.341
19	0.884	1120.19	0.0000	4.258	124.715
20	0.875	1030.49	0.0000	3.836	121.196
21	0.892	1217.97	0.0000	4.003	114.700
22	0.840	771.82	0.0000	4.445	97.955
23	0.836	750.67	0.0000	4.337	102.330
24	0.826	698.19	0.0000	4.017	110.521
25	0.797	575.68	0.0000	4.596	95.457
26	0.801	593.08	0.0000	4.250	95.577
27	0.798	581.57	0.0000	4.733	85.545
28	0.814	644.36	0.0000	4.054	90.168
29	0.760	465.91	0.0000	4.228	79.289
30	0.794	565.33	0.0000	4.318	80.936
31	0.759	463.08	0.0000	4.195	72.966
32	0.714	367.42	0.0000	4.287	69.109
33	0.789	548.22	0.0000	4.428	76.943
34	0.725	386.84	0.0000	4.421	72.360
35	0.750	441.06	0.0000	4.552	64.863
36	0.664	290.91	0.0000	4.344	55.499
37	0.652	274.94	0.0000	4.480	64.143
38	0.618	238.30	0.0000	4.509	54.058
39	0.624	244.12	0.0000	4.296	56.387
40	0.569	194.07	0.0000	4.553	45.005
41	0.548	178.45	0.0000	4.877	50.388
42	0.552	181.23	0.0000	4.762	46.497
43	0.566	191.35	0.0000	4.428	48.615
44	0.620	240.04	0.0000	4.350	51.387
45	0.541	173.36	0.0000	4.752	45.732
46	0.591	212.43	0.0000	4.972	52.736
47	0.528	164.55	0.0000	4.650	42.958
48	0.554	182.67	0.0000	4.573	47.181
49	0.565	190.95	0.0000	4.706	42.215
50	0.424	108.14	0.0000	4.994	34.381
51	0.616	236.12	0.0000	4.637	48.377
52	0.556	184.19	0.0000	4.752	46.809
53	0.552	181.39	0.0000	4.816	42.578
54	0.489	140.71	0.0000	4.897	41.030
55	0.508	151.54	0.0000	4.617	42.710
56	0.553	181.60	0.0000	4.579	47.423
57	0.425	108.75	0.0000	4.774	32.164
58	0.375	88.15	0.0000	5.135	32.424
59	0.535	169.03	0.0000	4.670	37.268
60	0.362	83.52	0.0000	4.627	32.571

As a reminder from Section 2, m is the rejected frequency of normality test, that is, the number of "success" of the 100 Bernoulli trials. In Figure 3, each cell in the diagrams represents the result of 100 Bernoulli trials. The shadow cells represent those having rejected frequency more than our required levels m' in Bernoulli trials, the others represent those having rejected frequency no more than our required levels in Bernoulli trials. For example, our required level in Figure 3(a) is $m' = 5$, i.e. $m \leq 5$; in Figure 3(b) is $m' = 10$, i.e. $m \leq 10$; and in Figure 3(c) is $m' = 20$, i.e. $m \leq 20$.

When the required level m' changes, there is a trend that can be seen from Figure 3. Especially in Figure 3(c), there exists a curve that separates the plane into two zones, the black and the white. We select sample size n of which the first m has met the required level m' under each parameter α of gamma distribution, and look for the regression model of them. Using the inverse regression model ($n = b_0 + b_1/\alpha$) to estimate the relation between the parameter α of the gamma distribution and sample size n, the result we obtain is shown in Table 3 and Figure 4. For example, For $m' = 5$, the relation between

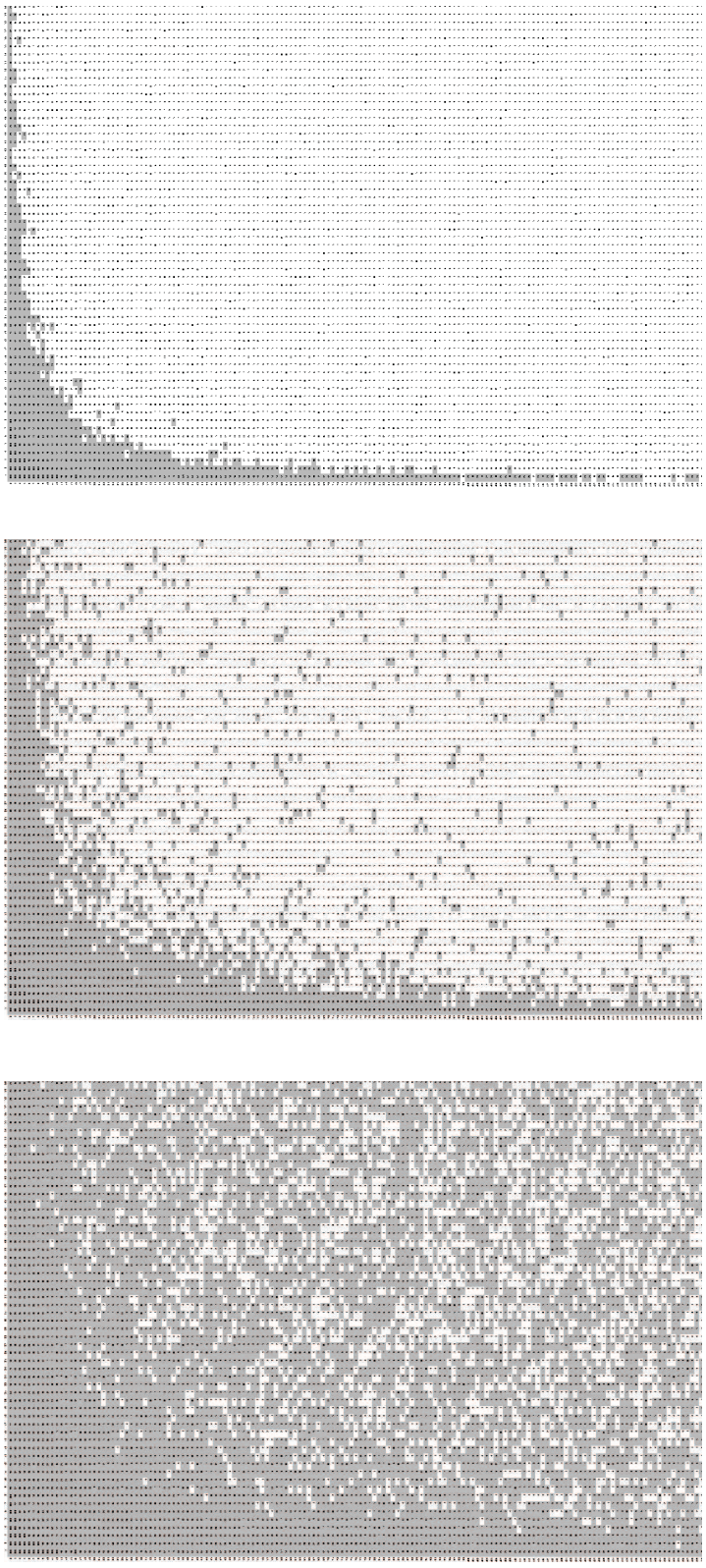


a) Gamma(3)

b) Gamma(13)

c) Gamma(60)

Figure 2. The approximate inverse regression curves under different parameter α of the gamma distribution.



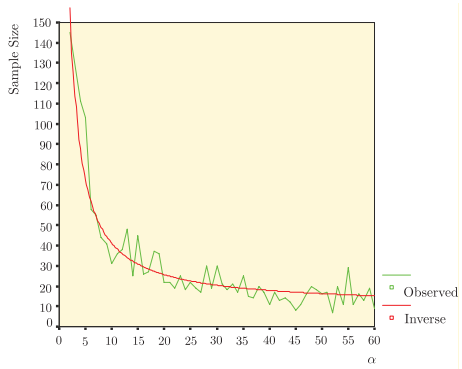
a) $m \leq 5$ b) $m \leq 10$ c) $m \leq 20$

Figure 3. The diagrams of the binominal trial results under different required level m, where the horizontal axis is α, the vertical axis is n.

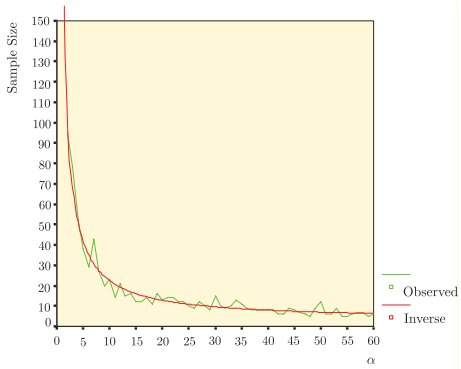
Table 3. The inverse regression model $n = b_0 + b_1/\alpha$ under required levels.

$m' = 5, 10, 20$

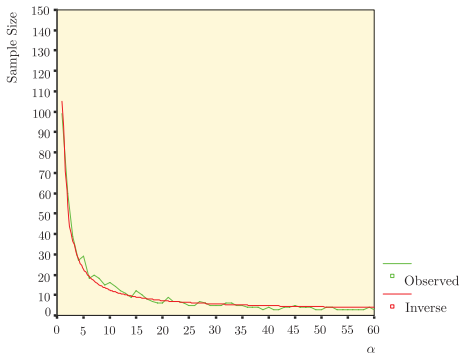
α	R-Square	F Value	p-value	b_0	b_1
5	0.894	473.89	0.0000	10.0918	310.673
10	0.962	1437.37	0.0000	3.1720	192.901
20	0.984	3573.37	0.0000	2.2114	102.848



a) $m \leq 5$



b) $m \leq 10$



c) $m \leq 20$

Figure 4. The approximate inverse regression model curves by using the first m with each parameter α of the gamma distribution.

α and n is estimated by $n = 10.0918 + 310.673/\alpha$; for $m' = 10$, it is estimated by $n = 3.1720 + 192.901/\alpha$; and for $m' = 20$, it is estimated by $n = 202114 + 102.848/\alpha$.

Using the result above, we can produce the least required sample sizes n as Table 4 for each parameter α of gamma distribution at different required level m' for using central limit theorem.

From Table 4, we can clearly see that the larger the parameter α is, the better it is to use the central limit theorem, because when α is larger, the smaller the required sample size for fitting normality of sample mean will be. As for the commonly used sample size

of 30, which is not suitable for smaller parameter α, especially for $\alpha \leq 15$ under $m \leq 5$ (which is almost equivalent to significant level 0.05 in normality test), for $\alpha \leq 7$ under $m \leq 10$ (which is almost equivalent to significant level 0.10 in normality test), and for $\alpha \leq 3$ under $m \leq 20$ (which is almost equivalent to significant level 0.20 in normality test). Indeed, the larger the parameter α, the smaller the least required sample size for fitting normality of sample mean. For example, under $m \leq 5$ and $\alpha = 30$, the least required sample size is only 21; under $m \leq 10$ and $\alpha = 30$, the least required sample size is only 10, other cases can be seen in Table 4.

Table 4. The least required sample sizes n for each parameter α of gamma distribution at different required level m' for using central limit theorem.

α	m'			α	m'		
	5	**10**	**20**		**5**	**10**	**20**
1	321	197	106	**31**	21	10	6
2	166	100	54	**32**	20	10	6
3	114	68	37	**33**	20	10	6
4	88	52	28	**34**	20	9	6
5	73	42	23	**35**	19	9	6
6	62	36	20	**36**	19	9	6
7	55	31	17	**37**	19	9	5
8	49	28	16	**38**	19	9	5
9	45	25	14	**39**	19	9	5
10	42	23	13	**40**	18	8	5
11	39	21	12	**41**	18	8	5
12	36	20	11	**42**	18	8	5
13	34	19	11	**43**	18	8	5
14	33	17	10	**44**	18	8	5
15	31	17	10	**45**	17	8	5
16	30	16	9	**46**	17	8	5
17	29	15	9	**47**	17	8	5
18	28	14	8	**48**	17	8	5
19	27	14	8	**49**	17	8	5
20	26	13	8	**50**	17	8	5
21	25	13	8	**51**	17	7	5
22	25	12	7	**52**	17	7	5
23	24	12	7	**53**	16	7	5
24	24	12	7	**54**	16	7	5
25	23	11	7	**55**	16	7	5
26	23	11	7	**56**	16	7	5
27	22	11	7	**57**	16	7	5
28	22	11	6	**58**	16	7	4
29	21	10	6	**59**	16	7	4
30	21	10	6	**60**	16	7	4

5. Conclusion Remarks

In practice, we can employ central limit theorem as long as $n \leq 30$. But, the shapes of probability distributions may vary. There are distributions similar to normal distribution, and others are vary greatly. Therefore, the approximated speeds toward normal

distribution of sample means are different, as well. This means the least required sample sizes for normality are different; they could be more than 30 or less than 30. Some of them have good normality when the sample size is less than 30, but some of them may not result in ideal normality, even if the sample size is much larger than 30. For example, in a gamma distribution with the parameter $\alpha = 1$, it needs more than 321 samples for the sample mean to accept in normality. In order for researchers to apply the central limit theorem correctly, this study suggests using simulation techniques to analyze the situation on gamma distribution.

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