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Locally optimal tests for exponential distributions with type-I censoring

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Abstract

This article studies a locally optimal test φ^* for testing $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$ for the lifetime parameter θ in an exponential distribution based on type-I censored data. Certain properties associated with φ^* are addressed. We compare the performance of φ^* with that of Spurrier and Wei's [Spurrier, J.D., Wei, L.J., 1980. A test of the parameter of the exponential distribution in the type-I censoring case. J. Amer. Statist. Assoc. 75, 405–409] test φ_{SW}, which is based on the MLE $\hat{\theta}_{ML}$ of θ. The exact powers and asymptotic powers of φ^* and φ_{SW} are computed. The numerical results indicate that the power of φ^* is better than that of φ_{SW} when $\theta (0 < \theta < \theta_0)$ is close to θ_0.

1. Introduction

Consider a life testing experiment where m lifetime components are independently and simultaneously put on test at the outset and are not replaced on failure. Let X_i denote the lifetime of component i, $i = 1, \ldots, m$. Suppose that X_i follows an exponential distribution with mean lifetime θ. That is, X_i has a probability density $f(x|\theta) = \frac{1}{\theta} e^{\frac{-x}{\theta}}$, $x > 0, \theta > 0$.

Due to practical reasons or the time restriction, the experiment terminates at a pre-specified time T. Therefore, the failure time of a component is observable if it fails before time T and not observable if it fails after T. If the latter occurs, the component then is said to be censored at time T. This type of time censoring is known as type-I censoring. Type-I censoring scheme has received much attention in the statistical literature. See Bartholomew (1963), Yang and Sirvanci (1977), Spurrier and Wei (1980), Gupta and Liang (1993), Huang and Chen (1992), Lam (1994) and Lin et al. (2002), among many others.

According to the time censoring scheme, we only observe $\min(X_i, T)$. Let $I(A)$ denote the indicator function which takes value 1 if event A occurs and 0, otherwise. Thus, $N \equiv N_T = \sum_{i=1}^{m} I(X_i < T)$ is the number of uncensored observations of the m components put on life testing up to time T. Let $Y_1 \leq \cdots \leq Y_{N_T}$ denote the ordered

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values of the N_T observable failure times. Thus, $Y = \sum_{j=1}^{N_T} Y_j + (m - N_T)T$ is the total lifetime of the m components put on life testing up to time T.

Let θ_0 be a known, positive value. We consider the problem of testing the hypotheses

$$H_0 : \theta \geq \theta_0 \quad \text{versus } H_1 : \theta < \theta_0. \tag{1.1}$$

In the literature, Epstein (1954) proposed a test φ_E based on the statistic N_T. If $N_T \geq r(\alpha, m)$, H_0 is rejected; otherwise, H_0 is accepted. The value $r(\alpha, m)$ is the upper α point of the binomial distribution $B(m, p)$ with $p = 1 - \exp(-T/\theta_0)$. According to the test φ_E, if the $r(\alpha, m)$-th failure occurs before time T, the test can be terminated immediately with H_0 being rejected. However, no early termination is allowed in the case in which the $r(\alpha, m)$-th failure does not occur by the time T, and hence, H_0 is accepted.

Spurrier and Wei (1980) investigated a test φ_{SW} based on the maximum likelihood estimator of θ, say $\hat{\theta}_{ML} = \frac{Y}{N_T}$ if $N_T > 0$; and $\hat{\theta}_{ML} = \infty$ if $N_T = 0$. The test φ_{SW} rejects H_0 if $\hat{\theta}_{ML} < c_{SW}(m, \alpha, \theta_0, T)$; and otherwise accepts H_0. The value $c_{SW}(m, \alpha, \theta_0, T)$ is determined so that the significance level of the test φ_{SW} is α. The test φ_{SW} allows for early termination under either case of acceptance or rejection of H_0. Spurrier and Wei (1980) has shown that the test φ_{SW} is more powerful and has a smaller average sampling time than that of φ_E by a numerical study.

We notice that the test φ_{SW} is not a uniformly most powerful test for the testing hypotheses (1.1) with type-I censoring. Indeed, there does not exist a uniformly most powerful test under such situation. Thus, one may be interested in finding a test possessing certain local optimality. Let φ be a test for the testing hypotheses (1.1) based on $(N_T, Y_1, \ldots, Y_{N_T})$, defined as: $\varphi(n, y_1, \ldots, y_n) =$ the probability of rejecting H_0 when $(n, y_1, \ldots, y_n)$ is observed. Thus, $E_{\theta_0}\varphi(N_T, Y_1, \ldots, Y_{N_T})$ denotes the level of significance of the test φ at $\theta = \theta_0$, and $E_{\theta_0-\Delta}\varphi(N_T, Y_1, \ldots, Y_{N_T})$, $0 < \Delta < \theta_0$, its power at $\theta = \theta_0 - \Delta$. Let $C(\alpha)$ be the class of all tests having level of significance α. We seek a test φ^* which may maximize $\frac{\partial}{\partial\Delta}[E_{\theta_0-\Delta}\varphi(N_T, Y_1, \ldots, Y_{N_T})]|_{\Delta=0}$ among all tests in $C(\alpha)$. That is, we desire to find a test φ^* in $C(\alpha)$ for which, $\frac{\partial}{\partial\Delta}[E_{\theta_0-\Delta}\varphi^*(N_T, Y_1, \ldots, Y_{N_T})]|_{\Delta=0} = \max_{\varphi \in C(\alpha)} \frac{\partial}{\partial\Delta}[E_{\theta_0-\Delta}[\varphi(N_T, Y_1, \ldots, Y_{N_T})]|_{\Delta=0}]$. A test possessing such a property is called a locally optimal test.

2. A locally optimal test

According to the time censoring scheme, the observable random variables are $(N_T, Y_1, \ldots, Y_{N_T})$ that have a joint probability density

$$f(n, y_1, \ldots, y_n|\theta) = \frac{m!}{(m-n)!}\frac{1}{\theta^n}\exp\left(\frac{-y(m, T, n)}{\theta}\right), \tag{2.1}$$

where $0 < y_1 < \cdots < y_n < T$, $y(m, T, n) = \sum_{j=1}^{n} y_j + (m-n)T$, and $n = 0, 1, \ldots, m$. For a test φ, the power of φ at $\theta = \theta_0 - \Delta$ is given by

$$\begin{aligned} E_{\theta_0-\Delta}\varphi(N_T, Y_1, \ldots, Y_{N_T}) &= \sum_n \int_{y_1} \cdots \int_{y_n} \varphi(n, y_1, \ldots, y_n) f(n, y_1, \ldots, y_n|\theta_0 - \Delta)\, \mathrm{d}y_1 \cdots \mathrm{d}y_n \\ &= \sum_n \int_{y_1} \cdots \int_{y_n} \varphi(n, y_1, \ldots, y_n) \frac{m!}{(m-n)!} \\ &\quad \times \frac{1}{(\theta_0 - \Delta)^n} \exp\left(\frac{-y(m, T, n)}{\theta_0 - \Delta}\right) \mathrm{d}y_1 \cdots \mathrm{d}y_n. \end{aligned} \tag{2.2}$$

Thus,

$$\begin{aligned} &\frac{\partial}{\partial\Delta}\left[E_{\theta_0-\Delta}\varphi(N_T, Y_1, \ldots, Y_{N_T})\right]|_{\Delta=0} \\ &= \sum_n \int_{y_1} \cdots \int_{y_n} \varphi(n, y_1, \ldots, y_n) f^{(1)}(n, y_1, \ldots, y_n|\theta_0 - \Delta)\, \mathrm{d}y_1 \cdots \mathrm{d}y_n|_{\Delta=0} \\ &= \sum_n \int \cdots \int \varphi(n, y_1, \ldots, y_n) \frac{f^{(1)}(n, y_1, \ldots, y_n|\theta_0)}{f(n, y_1, \ldots, y_n|\theta_0)} f(n, y_1, \ldots, y_n|\theta_0)\, \mathrm{d}y_1 \cdots \mathrm{d}y_n. \end{aligned} \tag{2.3}$$

Here, we denote $f^{(1)}(n, y_1, \ldots, y_n|\theta_0)$ the derivative of $f(n, y_1, \ldots, y_n|\theta_0 - \Delta)$ with respect to Δ and taking $\Delta = 0$.

In order to maximize $\frac{\partial}{\partial \Delta}\left[E_{\theta_0-\Delta}\varphi(N_T, Y_1, \ldots, Y_{N_T})\right]|_{\Delta=0}$ among all tests in $C(\alpha)$, we consider a test φ^* defined by

$$\varphi^*(n, y_1, \ldots, y_n) = \begin{cases} 1 & \text{if } \dfrac{f^{(1)}(n, y_1, \ldots, y_n|\theta_0)}{f(n, y_1, \ldots, y_n|\theta_0)} \geq k(m, \alpha, \theta_0, T); \\ 0 & \text{otherwise;} \end{cases} \tag{2.4}$$

where $k(m, \alpha, \theta_0, T)$ is a value to be determined so that

$$E_{\theta_0}\varphi^*(N_T, Y_1, \ldots, Y_{N_T}) = \alpha. \tag{2.5}$$

According to (2.4) and (2.5), φ^* is a test in $C(\alpha)$ which maximizes $\frac{\partial}{\partial \Delta}\left[E_{\theta_0-\Delta}\varphi(N_T, Y_1, \ldots, Y_{N_T})\right]|_{\Delta=0}$ among all tests in $C(\alpha)$.

From (2.1), (2.2) and (2.4), the test φ^* can be presented as:

$$\varphi^*(n, y_1, \ldots, y_n) = \begin{cases} 1 & \text{if } y(m, T, n) - n\theta_0 \leq -\theta_0^2 k(m, \alpha, \theta_0, T); \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

According to (2.6), the test φ^* depends on $(N_T, Y_1, \ldots, Y_{N_T})$ only through $(N_T, Y(m, T, N_T))$.

2.1. *Justification for being a likelihood ratio test*

For $0 < \theta_1 < \theta_0$, consider the testing hypotheses $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$. The likelihood ratio test φ_L rejects H_0 if $\frac{f(n, y_1, \ldots, y_n|\theta_1)}{f(n, y_1, \ldots, y_n|\theta_0)} = \left(\frac{\theta_0}{\theta_1}\right)^n \exp((\frac{1}{\theta_0} - \frac{1}{\theta_1})y(m, T, n)) > c$, which is equivalent to rejecting H_0 if

$$y(m, T, n) - n\frac{\theta_0\theta_1}{\theta_0 - \theta_1}\ln\left(\frac{\theta_0}{\theta_1}\right) < \frac{\theta_0\theta_1}{\theta_0 - \theta_1}\ln c, \tag{2.7}$$

where c is a value so determined that the probability of type-I error of the likelihood ratio test φ_L is α. Note that c depends on the value of θ_1 which is in H_1, so, the likelihood ratio test φ_L depends on the value of θ_1. Thus, φ_L is not a uniformly most powerful test for the testing hypotheses (1.1). Therefore, there exists no uniformly most powerful test for the testing hypotheses based on type-I censored data.

In (2.7), if we let θ_1 increase to θ_0, then $\frac{\theta_0\theta_1}{\theta_0-\theta_1}\ln\left(\frac{\theta_0}{\theta_1}\right) \longrightarrow \theta_0$. Thus, $y(m, T, n) - n\frac{\theta_0\theta_1}{\theta_0-\theta_1}\ln\left(\frac{\theta_0}{\theta_1}\right) \longrightarrow y(m, T, n) - n\theta_0$. Recall in (2.6) that $\varphi^*(n, y_1, \ldots, y_n) = 1$ if $y(m, T, n) - n\theta_0 \leq -\theta_0^2 k(m, \alpha, \theta_0, T)$. Hence, the locally optimal test φ^* can be viewed as a likelihood ratio test for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ where $0 < \theta_1 < \theta_0$ and θ_1 is very close to θ_0.

2.2. *Unbiasedness of the test φ^**

Let $(n, y_1, \ldots, y_n)$ and $(n^*, y_1^*, \ldots, y_n^*)$ be two points in the sample space of the random variables $(N_T, Y_1, \ldots, Y_{N_T})$. Let $y = y(m, T, n) = \sum_{j=1}^{n} y_j + (m - n)T$, and $y^* = y^*(m, T, n^*) = \sum_{j=1}^{n^*} y_j^* + (m - n^*)T$. Suppose that $n \leq n^*$ and $y \geq y^*$. Then, the test φ^* has the monotonicity property in the sense that

$$\varphi^*(n, y) \leq \varphi^*(n^*, y^*). \tag{2.8}$$

Consider the likelihood ratio function

$$\ell\left(\theta|(n, y), (n^*, y^*)\right) = \frac{f(n, y|\theta)}{f(n^*, y^*|\theta)} = \frac{(m - n^*)!}{(m - n)!}\theta^{n^*-n}\exp\left(\frac{-1}{\theta}\left(y - y^*\right)\right). \tag{2.9}$$

Since $n^* \geq n$ and $y \geq y^*$, $\ell(\theta|(n, y), (n^*, y^*))$ has the monotone likelihood ratio in the sense that

$$\ell\left(\theta|(n, y), (n^*, y^*)\right) \text{ is increasing in } \theta \text{ for } \theta > 0. \tag{2.10}$$

Let $\beta(\theta, \alpha, T) = E_\theta \varphi^*(N_T, Y_1, \ldots, Y_{N_T})$, the probability of rejecting H_0 when θ is the true state of nature. When $0 < \theta < \theta_0$, $\beta(\theta, \alpha, T)$ is the power of the test φ^*; and when $\theta \geq \theta_0$, $\beta(\theta, \alpha, T)$ is the probability of committing type-I error.

Combining (2.8)–(2.10), we see that: $\beta(\theta, \alpha, T)$ is decreasing in θ for $\theta > 0$. Thus, $\beta(\theta, \alpha, T) \geq \beta(\theta_0, \alpha, T) = E_{\theta_0}\varphi^*(N_T, Y_1, \ldots, Y_{N_T}) = \alpha$ for $\theta < \theta_0$ and $\beta(\theta, \alpha, T) \leq \beta(\theta_0, \alpha, T) = \alpha$ for $\theta > \theta_0$. Therefore, φ^* is an unbiased test.

2.3. *Justification for being a Bayes test*

Consider the hypotheses testing (1.1). For each $i = 0, 1$, let i denote the action deciding in favor of H_i. For each $\theta > 0$ and action i, we consider a loss as following.

$$L(\theta, i) = i(\theta - \theta_0) I(\theta - \theta_0) + (1 - i)(\theta_0 - \theta) I(\theta_0 - \theta), \tag{2.11}$$

where $I(x) = 1$ if $x > 0$, and 0 otherwise. Note that in the RHS of (2.11), the first term is the loss due to rejecting $H_0 : \theta \geq \theta_0$ when H_0 is true; and the second term is the loss of accepting H_0 when $H_1 : \theta < \theta_0$ is true.

Consider a Bayes framework. Suppose the mean lifetime is a random variable (denoted Θ) following an inverse gamma prior distribution $I\Gamma(\alpha, \gamma)$ with probability density $g(\theta|\alpha, \gamma) = \frac{\gamma^\alpha}{\Gamma(\alpha)\theta^{\alpha+1}} e^{\frac{-\gamma}{\theta}}, \theta > 0$.

Consider a test φ defined by $\varphi(n, y_1, \ldots, y_n) = P\{\text{accepting } H_1|(n, y_1, \ldots, y_n)\}$. Through straightforward computation, a Bayes test φ_B, which minimizes the Bayes risks among all tests, can be obtained as follows:

$$\varphi_B(n, y_1, \ldots, y_n) = \begin{cases} 1 & \text{if } E[\Theta|n, y_1, \ldots, y_n] < \theta_0, \\ 0 & \text{otherwise,} \end{cases} \tag{2.12}$$

where $E[\Theta|n, y_1, \ldots, y_n]$ is the posterior expectation of Θ when $(N_T, Y_1, \ldots, Y_{N_T}) = (n, y_1, \ldots, y_n)$ is observed.

Since Θ follows $I\Gamma(\alpha, \gamma)$, thus, the marginal pdf of $(N_T, Y_1, \ldots, Y_{N_T})$ is given by

$$f(n, y_1, \ldots, y_n) = \int f(n, y_1, \ldots, y_n|\theta) g(\theta|\alpha, \gamma)\, d\theta = \frac{m!\gamma^\alpha}{(m-n)!\Gamma(\alpha)} \times \frac{\Gamma(n+\alpha)}{[y(m, T, n) + \gamma]^{n+\alpha}}.$$

The posterior pdf of θ given $(n, y_1, \ldots, y_n)$ can be computed as:

$$g(\theta|n, y_1, \ldots, y_n) = \frac{f(n, y_1, \ldots, y_n|\theta) g(\theta|\alpha, \gamma)}{f(n, y_1, \ldots, y_n)} = \frac{[y(m, T, n) + \gamma]^{n+\alpha}}{\Gamma(n+\alpha)} \times \frac{e^{-(y(m,T,n)+\gamma)/\theta}}{\theta^{n+\alpha+1}}.$$

Therefore,

$$E[\Theta|n, y_1, \ldots, y_n] = \int \theta g(\theta|n, y_1, \ldots, y_n)\, d\theta = \frac{y(m, T, n) + \gamma}{n + \alpha - 1}. \tag{2.13}$$

Combining (2.12) and (2.13), we see that the Bayes test φ_B rejects H_0 if, and only if,

$$y(m, T, n) - n\theta_0 \leq (\alpha - 1)\theta_0 - \gamma. \tag{2.14}$$

Comparing the forms of (2.6) and (2.14), we see that the test φ^* has the same form as that of the Bayes test φ_B for the hypotheses testing (1.1) under the loss $L(\theta, i)$ when θ follows an $I\Gamma(\alpha, \gamma)$ prior distribution.

3. Computation of critical values

In order to determine the critical value for a level of significance α test, according to (2.6), we need to find the value $k(m, \alpha, \theta_0, T)$ such that

$$P_{\theta_0}\left\{Y(m, T, N_T) - N_T\theta_0 \leq -\theta_0^2 k(m, \alpha, \theta_0, T)\right\} = \alpha. \tag{3.1}$$

For this purpose, in the following, we first consider the case where $\theta_0 = 1$, and then extend the result to general $\theta_0 > 0$.

Let $Z_1, \ldots, Z_m$ be *iid* exponentially distributed random variables with mean 1, and let $Z_{(1)} < \cdots < Z_{(m)}$ be the associated order statistics. Let R be a censoring time. Define

$$N_R = \max\left\{i \,|\, Z_{(i)} < R\right\} \quad \text{and} \quad Z(m, R, N_R) = \sum_{i=1}^{N_R} Z_{(i)} + (m - N_R)R.$$

Note that $P\{Z_i > R\} = \mathrm{e}^{-R}$ is the probability that a component is censored by time R. For each α, $0 < \alpha < 1 - (\mathrm{e}^{-R})^m$, let $c(m, \alpha, R)$ be the number such that

$$P\{Z(m, R, N_R) - N_R \le c(m, \alpha, R)\} = \alpha. \tag{3.2}$$

Since $-m < Z(m, R, N_R) - N_R \le mR$, thus, if $c \le -m$, $P\{Z(m, R, N_R) - N_R \le c\} = 0 < \alpha$, and if $c \ge mR$, $P\{Z(m, R, N_R) - N_R \le c\} \ge 1 - \mathrm{e}^{-mR} > \alpha$. Therefore, we must have: $-m < c(m, \alpha, R) < mR$. Note that as $N_R = 0$, $Z(m, R, 0) = mR > c(m, \alpha, R)$.

For each $n = 1, 2, \ldots, m$ and values c and R with $-m < c < mR$, set

$$\Delta_n(a, R) = \left\{(z_1, \ldots, z_n) \,|\, 0 < z_1 < \cdots < z_n < R, \sum_{i=1}^{n} z_i \le a\right\}.$$

Define

$$P(c|m, R) = P\{Z(m, R, N_R) - N_R \le c\}. \tag{3.3}$$

Let $x(n) = c + n - (m - n)R$. A straightforward computation leads to:

$$\begin{aligned} P(c|m, R) &= \sum_{n=1}^{m} \frac{m!}{(m-n)!} \exp(-(m-n)R) \int_{\Delta_n(x(n), R)} \exp\left(-\sum_{i=1}^{n} z_i\right) \mathrm{d}z_1 \cdots \mathrm{d}z_n \\ &= \sum_{n=1}^{m} \frac{m!}{(m-n)!} \exp(-(m-n)R) G_n(x(n), R), \end{aligned} \tag{3.4}$$

where

$$G_n(x(n), R) = \begin{cases} 0 & \text{if } x(n) \le 0, \\ \dfrac{1}{n!(n-1)!} \displaystyle\sum_{j=0}^{\left[\frac{x(n)}{R}\right]} \int_0^{x(n)-jR} \binom{n}{j} (-1)^j u^{n-1} \exp\left(-(u + jR)\right) \mathrm{d}u & \text{if } 0 < x(n) \le nR, \\ \dfrac{1}{n!} \left(1 - \mathrm{e}^{-R}\right)^n & \text{if } x(n) > nR. \end{cases} \tag{3.5}$$

In (3.5), the equality is obtained from Lam (1994).

For $\alpha = 0.01, 0.025$ and 0.05 and $R = \ln 2, \ln 4, \ln 10$, the software *Fortran 6.5* is used for computing the value $c(m, \alpha, R)$ such that $P(c(m, \alpha, R)|m, R) = \alpha$. These values are computed through (3.4) and are tabulated on the upper entry, the asymptotic critical values are tabulated on the lower entry in Table 1. Accuracy of the true sizes of test φ^* using these critical values has the following numerical properties.

For $R = \ln 2, \ln 4$ and $\ln 10$ cases, and $m = 10(5)50$,

$$|P\,(c(m, \alpha, R)|m, R) - \alpha| \le 10^{-8}.$$

3.1. Determination of $k\ (m, \alpha, \theta_0, T)$

Let $X_1, \ldots, X_m$ be a sample of size m from an exponential distribution having mean θ. Note that $X_i \overset{id}{\equiv} \theta Z_i$, where Z_i is an exponentially distributed random variable with mean 1. Taking $T = \theta_0 R$, we have

$$N_T = \sum_{i=1}^{m} I(X_i < T) = \sum_{i=1}^{m} I(\theta Z_i < \theta_0 R) = \sum_{i=1}^{m} I\left(\frac{\theta}{\theta_0} Z_i < R\right), \tag{3.6}$$

Table 1
Exact and asymptotic values of $c(m, \alpha, R)$, for $\alpha = 0.01, 0.025, 0.05$

m	$R = \ln 2$			$R = \ln 4$			$R = \ln 10$		
	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$
	$c(m, \alpha, R)$	$c(m, \alpha, R)$	$c(m, \alpha, R)$	$c(m, \alpha, R)$	$c(m, \alpha, R)$	$c(m, \alpha, R)$	$c(m, \alpha, R)$	$c(m, \alpha, R)$	$c(m, \alpha, R)$
10	−5.134663	−4.414645	−3.691131	−5.778897	−5.023238	−4.326138	−5.868387	−5.196448	−4.551192
	−5.201872	−4.382613	−3.678005	−6.370966	−5.367582	−4.504617	−6.979044	−5.879892	−4.934561
15	−6.355422	−5.381881	−4.530547	−7.235587	−6.242804	−5.342011	−7.494627	−6.549373	−5.670202
	−6.370966	−5.367582	−4.504617	−7.802808	−6.573919	−5.517007	−8.547548	−7.201368	−6.043578
20	−7.348012	−6.214758	−5.228337	−8.461066	−7.266052	−6.196011	−8.843551	−7.675929	−6.609709
	−7.356558	−6.197950	−5.201484	−9.009907	−7.590908	−6.370491	−9.869858	−8.315423	−6.978523
25	−8.221944	−6.949095	−5.842532	−9.536460	−8.166249	−6.948129	−10.024502	−8.665199	−7.436292
	−8.224882	−6.929519	−5.815436	−10.073382	−8.486893	−7.122425	−11.034837	−9.296925	−7.802226
30	−9.012005	−7.612219	−6.397850	−10.506398	−8.979307	−7.628040	−11.088811	−9.557908	−8.182935
	−9.009907	−7.590908	−6.370491	−11.034837	−9.296925	−7.802226	−12.088058	−10.184272	−8.546910
35	−9.738026	−8.221600	−6.908170	−11.396975	−9.726503	−8.253256	−12.065606	−10.377878	−8.869175
	−9.731811	−8.199118	−6.880916	−11.918986	−10.041827	−8.427367	−13.056595	−11.000271	−9.231718
40	−10.413315	−8.788618	−7.382945	−12.225004	−10.421655	−8.835175	−12.973548	−11.140476	−9.507676
	−10.403744	−8.765225	−7.356009	−12.741932	−10.735165	−9.009234	−13.958087	−11.759784	−9.869122
45	−11.047195	−9.321110	−7.828854	−13.002078	−11.074330	−9.381713	−13.825460	−11.856307	−10.107211
	−11.034837	−9.296925	−7.802226	−13.514860	−11.386362	−9.555736	−14.804787	−12.473134	−10.467784
50	−11.646479	−9.824344	−8.250493	−13.736594	−11.691480	−9.898634	−14.630612	−12.533060	−10.674153
	−11.631739	−9.799820	−8.224268	−14.245913	−12.002279	−10.072630	−15.605616	−13.147838	−11.034014

Upper entry: exact value; lower entry: asymptotic value.

$$Y(m, T, N_T) = \sum_{i=1}^{N_T} Y_i + (m - N_T)T = \sum_{i=1}^{N_T} \theta Z_{(i)} + (m - N_T)\theta_0 R$$
$$= \left[\sum_{i=1}^{N_T} \frac{\theta}{\theta_0} Z_{(i)} + (m - N_T)R \right] \theta_0. \tag{3.7}$$

As $\theta = \theta_0$, from (3.6), $N_T = N_R$. From (2.6),

$$\alpha = E_{\theta_0} \varphi^*(N_T, Y_1, \ldots, Y_{N_T})$$
$$= P_{\theta_0} \left\{ Y(m, T, N_T) - N_T \theta_0 \le -\theta_0^2 k(m, \alpha, \theta_0, T) \right\}$$
$$= P_{\theta_0} \left\{ \left[\sum_{i=1}^{N_R} Z_{(i)} + (m - N_R)R \right] \theta_0 - N_R \theta_0 \le -\theta_0^2 k(m, \alpha, \theta_0, T) \right\}$$
$$= P_{\theta_0} \left\{ \sum_{i=1}^{N_R} Z_{(i)} + (m - N_R)R - N_R \le -\theta_0 k(m, \alpha, \theta_0, T) \right\}. \tag{3.8}$$

By (3.2) and (3.8), it follows that: $-\theta_0 k(m, \alpha, \theta_0, T) = c(m, \alpha, R)$, i.e.

$$k(m, \alpha, \theta_0, T) = -c(m, \alpha, R)/\theta_0. \tag{3.9}$$

3.2. Asymptotic critical values

Let $\delta_j = I(X_j < T)$ and $V_j = (X_j - \theta_0)\delta_j + (1 - \delta_j)T$, $j = 1, \ldots, m$. Then V_j, $j = 1, \ldots, m$, are *iid*, and

$$E_\theta[V_j] = (\theta - \theta_0)\left(1 - e^{-\frac{T}{\theta}}\right) \equiv \eta(\theta), \text{ say.}$$

$$\mathrm{Var}_\theta(V_j) = \theta^2 \left(1 - e^{-\frac{T}{\theta}}\right) + (\theta - \theta_0)^2 e^{-\frac{T}{\theta}} - 2(\theta - \theta_0) T e^{-\frac{T}{\theta}} - (\theta - \theta_0)^2 e^{-\frac{2T}{\theta}}$$
$$\equiv \sigma^2(\theta).$$

Note that

$$Y(m,T,N_T) - N_T\theta_0 = \sum_{j=1}^{N_T} X_j + (m - N_T)T - N_T\theta_0$$
$$= \sum_{j=1}^{m}\left[(X_j - \theta_0)\delta_j + (1-\delta_j)T\right] = \sum_{j=1}^{m} V_j.$$

Therefore,

$$E_\theta[Y(m,T,N_T) - N_T\theta_0] = m(\theta-\theta_0)\left(1 - \mathrm{e}^{-\frac{T}{\theta}}\right) \equiv m\eta(\theta), \quad \text{and}$$
$$\mathrm{Var}_\theta(Y(m,T,N_T)) = m\sigma^2(\theta). \tag{3.10}$$

As $\theta = \theta_0$,

$$E_{\theta_0}[Y(m,T,N_T) - N_T\theta_0] = 0, \quad \text{and}$$
$$\mathrm{Var}_{\theta_0}(Y(m,T,N_T) - N_T\theta_0) = m\theta_0^2\left(1 - \mathrm{e}^{-\frac{T}{\theta_0}}\right). \tag{3.11}$$

When m is sufficiently large, by Central Limit theorem,

$$\alpha = P_{\theta_0}\left\{Y(m,T,N_T) - N_T\theta_0 \le -\theta_0^2 k\,(m,\alpha,\theta_0,T)\right\}$$
$$= P_{\theta_0}\left\{\frac{Y(m,T,N_T) - N_T\theta_0}{\sqrt{m\theta_0^2\left(1-\mathrm{e}^{-\frac{T}{\theta_0}}\right)}} \le \frac{-\theta_0^2 k\,(m,\alpha,\theta_0,T)}{\sqrt{m\theta_0^2\left(1-\mathrm{e}^{-\frac{T}{\theta_0}}\right)}}\right\}$$
$$\approx \Phi\left(\frac{-\theta_0 k\,(m,\alpha,\theta_0,T)}{\sqrt{m\left(1-\mathrm{e}^{-\frac{T}{\theta_0}}\right)}}\right). \tag{3.12}$$

From (3.9), it follows that

$$k\,(m,\alpha,\theta_0,T) \approx z_\alpha\sqrt{m(1-\mathrm{e}^{-\frac{T}{\theta_0}})}/\theta_0, \tag{3.13}$$

where z_α is the $1-\alpha$-quantile of a standard normal density.

By (3.9) and (3.13), and noting that $T = \theta_0 R$, we have:

$$c(m,\alpha,R) = -\theta_0 k\,(m,\alpha,\theta_0,T) \approx -z_\alpha\sqrt{m(1-\mathrm{e}^{-R})}. \tag{3.14}$$

The asymptotic critical values for the test φ^* have been computed and they are tabulated on the lower entry in Table 1.

As can be seen, the asymptotic critical values are generally speaking inaccurate though they are easy to be obtained.

4. Comparison of powers

4.1. Small sample sizes case

For each $n = 1, \ldots, m$, let

$$w(n,\theta_0,T) = -\theta_0^2 k\,(m,\alpha,\theta_0,T) + n\theta_0 - (m-n)T.$$

For $0 < \theta < \theta_0$, the power of the test φ^* at θ is:

$$\begin{aligned}\beta\left(\theta|\varphi^*,\alpha,T\right) &= P_\theta\left\{Y(m,T,N_T) - N_T\theta_0 \le -\theta_0^2 k\left(m,\alpha,\theta_0,T\right)\right\}\\ &= \sum_{n=1}^{m}\frac{m!}{(m-n)!}\exp\left(\frac{-(m-n)T}{\theta}\right)\int_{\Delta_n(w(n,\theta_0,T),T)}\frac{1}{\theta^n}\exp\left(\frac{-1}{\theta}\sum_{j=1}^{n}y_j\right)\mathrm{d}y_1\cdots\mathrm{d}y_n\\ &= \sum_{n=1}^{m}\frac{m!}{(m-n)!}\exp\left(\frac{-(m-n)T}{\theta}\right)H_n(w(n,\theta_0,T),T,\theta),\end{aligned}\tag{4.1}$$

where

$$\begin{aligned}H_n\left(x,T,\theta\right) &= \int_{\Delta_n(x,T)}\frac{1}{\theta^n}\exp\left(\frac{-1}{\theta}\sum_{j=1}^{n}y_j\right)\mathrm{d}y_1\cdots\mathrm{d}y_n\\ &= \begin{cases}0 & \text{if } x\le 0,\\ \dfrac{1}{n!(n-1)!\theta^n}\displaystyle\sum_{j=0}^{\left[\frac{x}{T}\right]}\int_0^{x-jT}\binom{n}{j}(-1)^j u^{n-1}\exp\left(\frac{-1}{\theta}\left(u+jT\right)\right)\mathrm{d}u & \text{if } 0<x\le nT,\\ \dfrac{1}{n!}\left(1-\mathrm{e}^{-\frac{T}{\theta}}\right)^n & \text{if } x>nT.\end{cases}\end{aligned}\tag{4.2}$$

Let $c_{SW}\left(m,\alpha,\theta_0,T\right)$ be the critical value of the test φ_{SW} for which $P_{\theta_0}\{\hat{\theta}_{ML} \le \theta_0\, c\left(m,\alpha,\theta_0,T\right)\} = \alpha$. For $R = \ln 2$, $\ln 4$ and $\ln 10$, and $m = 10(5)55$, the values of $c_{SW}\left(m,\alpha,\theta_0,T\right)$ $(T = \theta_0 R)$ can be obtained from Table 1 of Spurrier and Wei (1980). Let

$$v\left(n,\theta_0,T\right) = \theta_0 n c_{SW}\left(m,\alpha,\theta_0,T\right) - (m-n)T.$$

For $0 < \theta < \theta_0$, the power of the test φ_{SW} at θ is:

$$\begin{aligned}\beta\left(\theta|\varphi_{SW},\alpha,T\right) &= P_\theta\left\{\hat{\theta}_{ML} \le \theta_0 c_{SW}\left(m,\alpha,\theta_0,T\right)\right\}\\ &= P_\theta\left\{Y(m,T,N_T) \le N_T\theta_0 c_{SW}\left(m,\alpha,\theta_0,T\right)\right\}\\ &= \sum_{n=1}^{m}\frac{m!}{(m-n)!}\exp\left(\frac{-(m-n)T}{\theta}\right)\int_{\Delta_n(v(n,\theta_0,T),T)}\frac{1}{\theta^n}\exp\left(\frac{-1}{\theta}\sum_{j=1}^{n}y_j\right)\mathrm{d}y_1\cdots\mathrm{d}y_n\\ &= \sum_{n=1}^{m}\frac{m!}{(m-n)!}\exp\left(\frac{-(m-n)T}{\theta}\right)H_n(v(n,\theta_0,T),T,\theta).\end{aligned}\tag{4.3}$$

In the following comparison, we let $\theta_0 = 1$. For $\alpha = 0.05$, $R = \ln 2$, $\ln 4$ and $\ln 10$, and $m = 10(10)50$, we compute the powers of φ^* and φ_{SW} at $\theta = 0.50(0.05)0.95$ and $\theta = 0.95(0.01)1$. An efficiency of φ^* with respect to φ_{SW} is defined by $r(\varphi^*,\varphi_{SW}) = (\beta(\varphi^*) - \beta(\varphi_{SW}))/\beta(\varphi_{SW})$.

The critical values of c_{SW} are re-calculated to the sixth decimal for its accuracy of computing the power. Power $\beta\left(\theta|\varphi^*, 0.05, R\right)$ and $\beta\left(\theta|\varphi_{SW}, 0.05, R\right)$ are computed through (4.1)–(4.3) and partial numerical results are tabulated in Table 2.

From Table 2, we see that: When $0.75 < \theta < \theta_0 = 1$, $\alpha = 0.05$, $m = 10(10)50$, $r(\varphi^*,\varphi_{SW}) > 0$, i.e. $\beta\left(\theta|\varphi^*,\alpha,R\right) > \beta\left(\theta|\varphi_{SW},\alpha,R\right)$. In fact, it is also true for other values of α. The values of $r(\varphi^*,\varphi_{SW})$ may be negative when θ is in $(0.5, 0.75)$, depending on α, m and R. This phenomenon agrees and confirms that the test φ^* behaves like a "likelihood ratio" test for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ where $\theta_1 < \theta_0$ and is close to θ_0. However, for heavy censoring ($\zeta \equiv F_{\theta_0}(T) < 0.5$ noting that $R = -\ln(1-\zeta)$), φ_{SW} is superior to φ^* except for instance, $F_{\theta_0}(T) = 0.49$ or 0.48, $\theta \ge 0.75$ and $m \ge 20$. When $F_{\theta_0}(T)$ is smaller, say 0.46, φ^* can be better only when $m \ge 20$ and $\theta \ge 0.90$. Based on this fact, we recommend φ_{SW} over φ^* when $F_{\theta_0}(T) < 0.5$.

Table 2
Comparison of powers of φ^* with respect to φ_{SW} for $m = 10(10)50$, $\alpha = 0.05$ and $\theta_0 = 1$

θ	$R = \ln 2$			$R = \ln 4$			$R = \ln 10$		
	$\beta(\varphi^*)$	$\beta(\varphi_{SW})$	$r(\varphi^*, \varphi_{SW})$	$\beta(\varphi^*)$	$\beta(\varphi_{SW})$	$r(\varphi^*, \varphi_{SW})$	$\beta(\varphi^*)$	$\beta(\varphi_{SW})$	$r(\varphi^*, \varphi_{SW})$
$m = 10$									
0.999	0.050231	0.050229	0.000040	0.050261	0.050259	0.000040	0.050268	0.050268	0
0.99	0.052358	0.052338	0.000382	0.052673	0.052657	0.000304	0.052753	0.052750	0.000057
0.98	0.054836	0.054794	0.000767	0.055494	0.055463	0.000559	0.055663	0.055657	0.000108
0.97	0.057440	0.057376	0.001115	0.058474	0.058426	0.000822	0.058739	0.058730	0.000153
0.96	0.060177	0.060089	0.001464	0.061621	0.061554	0.001088	0.061992	0.061979	0.000210
0.95	0.063054	0.062941	0.001795	0.064943	0.064857	0.001326	0.065431	0.065415	0.000245
0.90	0.079813	0.079555	0.003243	0.084554	0.084357	0.002335	0.085801	0.085764	0.000431
0.85	0.101351	0.100921	0.004261	0.110248	0.109918	0.003002	0.112632	0.112568	0.000569
0.80	0.129024	0.128404	0.004829	0.143783	0.143309	0.003308	0.147806	0.147712	0.000636
0.75	0.164498	0.163694	0.004912	0.187246	0.186648	0.003204	0.193545	0.193423	0.000631
0.70	0.209739	0.208803	0.004483	0.242948	0.242296	0.002691	0.252267	0.252128	0.000551
0.65	0.266916	0.265975	0.003538	0.313126	0.312569	0.001782	0.326227	0.326094	0.000408
0.60	0.338115	0.337406	0.002101	0.399330	0.399119	0.000529	0.416778	0.416698	0.000192
0.55	0.424763	0.424656	0.000252	0.501369	0.501845	−0.000949	0.523164	0.523203	−0.000075
0.50	0.526590	0.527554	−0.001827	0.615801	0.617308	−0.002441	0.640863	0.641092	−0.000357
$m = 20$									
0.999	0.050327	0.050326	0.000020	0.050380	0.050379	0.000020	0.050400	0.050399	0.000020
0.99	0.053370	0.053359	0.000206	0.053931	0.053918	0.000241	0.054142	0.054135	0.000129
0.98	0.056967	0.056943	0.000421	0.058160	0.058133	0.000464	0.058611	0.058598	0.000222
0.97	0.060803	0.060766	0.000609	0.062707	0.062666	0.000654	0.063430	0.063410	0.000315
0.96	0.064894	0.064844	0.000771	0.067593	0.067536	0.000844	0.068624	0.068596	0.000408
0.95	0.069257	0.069191	0.000954	0.072842	0.072768	0.001017	0.074218	0.074181	0.000499
0.90	0.095737	0.095586	0.001580	0.105371	0.105197	0.001654	0.109153	0.109066	0.000798
0.85	0.131799	0.131548	0.001908	0.150854	0.150567	0.001906	0.158454	0.158314	0.000884
0.80	0.180207	0.179866	0.001896	0.212865	0.212490	0.001765	0.225991	0.225814	0.000784
0.75	0.243875	0.243497	0.001552	0.294566	0.294188	0.001285	0.314880	0.314721	0.000505
0.70	0.325244	0.324956	0.000886	0.397342	0.397128	0.000539	0.425754	0.425712	0.000099
0.65	0.425156	0.425192	−0.000085	0.518803	0.518985	−0.000351	0.554431	0.554631	−0.000361
0.60	0.541178	0.541921	−0.001371	0.650707	0.651488	−0.001199	0.689932	0.690464	−0.000770
0.55	0.665720	0.667781	−0.003086	0.778372	0.779750	−0.001767	0.814975	0.815790	−0.000999
0.50	0.785045	0.789544	−0.005698	0.883843	0.885479	−0.001848	0.911257	0.912124	−0.000951
$m = 30$									
0.999	0.050401	0.050401	0	0.050471	0.050470	0.000020	0.050500	0.050500	0
	0.050408	0.050172		0.050439	0.050281		0.050433	0.050346	
0.99	0.054156	0.054147	0.000166	0.054905	0.054894	0.000200	0.055215	0.055208	0.000127
	0.054220	0.051763		0.054557	0.052908		0.054499	0.053586	
0.98	0.058642	0.058624	0.000307	0.060256	0.060233	0.000382	0.060928	0.060915	0.000213
	0.058769	0.053636		0.059503	0.056035		0.059387	0.057469	
0.97	0.063481	0.063452	0.000457	0.066086	0.066051	0.000530	0.067179	0.067159	0.000298
	0.063671	0.055629		0.064868	0.059401		0.064695	0.061674	
0.96	0.068697	0.068657	0.000583	0.072433	0.072383	0.000691	0.074011	0.073982	0.000392
	0.068949	0.057749		0.070683	0.063026		0.070457	0.066230	
0.95	0.074316	0.074263	0.000714	0.079333	0.079268	0.000820	0.081466	0.081428	0.000467
	0.074627	0.060008		0.07698	0.066931		0.076707	0.071166	
0.90	0.109425	0.109302	0.001125	0.123540	0.123387	0.001240	0.129698	0.129609	0.000687
	0.109952	0.073788		0.116955	0.091569		0.116648	0.102784	
0.85	0.159004	0.158806	0.001247	0.187732	0.187494	0.001269	0.200435	0.200301	0.000669
	0.159473	0.093126		0.174684	0.127754		0.175128	0.149898	
0.80	0.226940	0.226694	0.001085	0.276494	0.276225	0.000974	0.298368	0.298231	0.000459
	0.226901	0.120766		0.255219	0.180962		0.258097	0.219106	
0.75	0.316327	0.316121	0.000652	0.391592	0.391426	0.000424	0.424117	0.424070	0.000111
	0.315318	0.160882		0.36235	0.258284		0.370375	0.317531	
0.70	0.427693	0.427708	−0.000035	0.528760	0.528887	−0.000240	0.570442	0.570604	−0.000284
	0.425646	0.219603		0.495638	0.367028		0.511804	0.449439	

(*continued on next page*)

Table 2 (*continued*)

θ	$R = \ln 2$			$R = \ln 4$			$R = \ln 10$		
	$\beta(\varphi^*)$	$\beta(\varphi_{SW})$	$r(\varphi^*, \varphi_{SW})$	$\beta(\varphi^*)$	$\beta(\varphi_{SW})$	$r(\varphi^*, \varphi_{SW})$	$\beta(\varphi^*)$	$\beta(\varphi_{SW})$	$r(\varphi^*, \varphi_{SW})$
0.65	0.556587	0.557157	−0.001023	0.675010	0.675578	−0.000841	0.720052	0.720488	−0.000605
	0.554429	0.305123		0.646239	0.510207		0.671396	0.609394	
0.60	0.691465	0.693231	−0.002547	0.809545	0.810500	−0.001178	0.848983	0.849607	−0.000734
	0.691543	0.425981		0.793814	0.677681		0.823206	0.774431	
0.55	0.813827	0.818310	−0.005478	0.911141	0.912166	−0.001124	0.937459	0.938037	−0.000616
	0.819663	0.584439		0.910266	0.837901		0.933639	0.906384	
0.50	0.902449	0.913915	−0.012546	0.969921	0.970636	−0.000737	0.982137	0.982476	−0.000345
	0.918536	0.762499		0.975740	0.947950		0.986184	0.977223	
$m = 40$									
0.999	0.050464	0.050463	0.000020	0.050548	0.050547	0.000020	0.050584	0.050584	0
	0.050470	0.050234		0.050515	0.050358		0.050517	0.050429	
0.99	0.054825	0.054818	0.000128	0.055735	0.055725	0.000179	0.056129	0.056123	0.000107
	0.054886	0.052410		0.055379	0.053714		0.055400	0.054478	
0.98	0.060083	0.060067	0.000266	0.062063	0.062043	0.000322	0.062930	0.062917	0.000207
	0.060206	0.054992		0.061288	0.057754		0.061345	0.059387	
0.97	0.065806	0.065781	0.000380	0.069036	0.069004	0.000464	0.070460	0.070440	0.000284
	0.065991	0.057762		0.067770	0.062151		0.067883	0.064771	
0.96	0.072029	0.071994	0.000486	0.076704	0.076659	0.000587	0.078782	0.078754	0.000356
	0.072276	0.060735		0.074873	0.066937		0.075066	0.070672	
0.95	0.078789	0.078744	0.000571	0.085123	0.085064	0.000694	0.087959	0.087922	0.000421
	0.079095	0.063928		0.082645	0.072148		0.082946	0.077139	
0.90	0.121993	0.121886	0.000878	0.140432	0.140296	0.000969	0.148918	0.148833	0.000571
	0.122509	0.083924		0.133363	0.106070		0.134882	0.119914	
0.85	0.184573	0.184404	0.000916	0.222702	0.222503	0.000894	0.240411	0.240294	0.000487
	0.185008	0.113122		0.208808	0.157937		0.213466	0.186088	
0.80	0.271081	0.270890	0.000705	0.336368	0.336188	0.000535	0.366287	0.366203	0.000229
	0.270994	0.156271		0.314729	0.235915		0.325619	0.284533	
0.75	0.383529	0.383416	0.000295	0.479104	0.479099	0.000010	0.521096	0.521154	−0.000111
	0.382608	0.220209		0.452201	0.348492		0.472712	0.421208	
0.70	0.518284	0.518431	−0.000284	0.637450	0.637775	−0.000510	0.685798	0.686084	−0.000417
	0.516869	0.313610		0.612214	0.498970		0.643143	0.590655	
0.65	0.663230	0.663944	−0.001075	0.786906	0.787566	−0.000838	0.830949	0.831418	−0.000564
	0.662759	0.444230		0.771805	0.675070		0.807360	0.766492	
0.60	0.797897	0.799870	−0.002467	0.901023	0.901785	−0.000845	0.930952	0.931410	−0.000492
	0.800701	0.610613		0.898943	0.840705		0.927181	0.905289	
0.55	0.899179	0.904642	−0.006039	0.966855	0.967403	−0.000566	0.980751	0.981023	−0.000277
	0.907973	0.788161		0.971373	0.950598		0.984408	0.977614	
0.50	0.950349	0.966849	−0.017066	0.992972	0.993201	−0.000231	0.996850	0.996937	−0.000087
	0.970818	0.926216		0.996074	0.992772		0.998673	0.997762	
$m = 50$									
0.999	0.050520	0.050518	0.000040	0.050616	0.050615	0.000020	0.050658	0.050658	0
	0.050525	0.050288		0.050583	0.050425		0.050591	0.050503	
0.99	0.055412	0.055412	0	0.056473	0.056464	0.000159	0.056943	0.056938	0.000088
	0.055479	0.052985		0.056112	0.054433		0.056204	0.055273	
0.98	0.061388	0.061361	0.000440	0.063689	0.063670	0.000298	0.064733	0.064721	0.000185
	0.061494	0.056209		0.062895	0.059303		0.063112	0.061119	
0.97	0.067924	0.067884	0.000589	0.071714	0.071685	0.000405	0.073447	0.073428	0.000259
	0.068089	0.059695		0.070410	0.064655		0.070792	0.067597	
0.96	0.075062	0.075034	0.000373	0.080619	0.080577	0.000521	0.083169	0.083142	0.000325
	0.075308	0.063464		0.078720	0.070536		0.079314	0.074771	
0.95	0.082879	0.082852	0.000326	0.090474	0.090420	0.000597	0.093982	0.093946	0.000383
	0.083197	0.067544		0.087891	0.076999		0.088753	0.082710	
0.90	0.133866	0.133770	0.000718	0.156539	0.156415	0.000793	0.167324	0.167244	0.000478
	0.134378	0.093691		0.149075	0.120162		0.152492	0.136595	
0.85	0.209108	0.208936	0.000823	0.256328	0.256162	0.000648	0.278885	0.278787	0.000352
	0.209492	0.133129		0.241897	0.188124		0.250973	0.222096	

Table 2 (*continued*)

θ	$R = \ln 2$			$R = \ln 4$			$R = \ln 10$		
	$\beta(\varphi^*)$	$\beta(\varphi_{SW})$	$r(\varphi^*, \varphi_{SW})$	$\beta(\varphi^*)$	$\beta(\varphi_{SW})$	$r(\varphi^*, \varphi_{SW})$	$\beta(\varphi^*)$	$\beta(\varphi_{SW})$	$r(\varphi^*, \varphi_{SW})$
0.80	0.313196	0.313036	0.000511	0.392799	0.392695	0.000265	0.429808	0.429773	0.000081
	0.313094	0.192745		0.371559	0.291100		0.390381	0.348963	
0.75	0.445760	0.445773	−0.000029	0.557095	0.557206	−0.000199	0.605564	0.605700	−0.000225
	0.445072	0.281554		0.533616	0.436050		0.564634	0.517623	
0.70	0.597206	0.597624	−0.000699	0.724940	0.725355	−0.000572	0.774443	0.774775	−0.000429
	0.596666	0.408267		0.707433	0.615795		0.746662	0.706485	
0.65	0.746866	0.748311	−0.001931	0.863592	0.864192	−0.000694	0.901080	0.901477	−0.000440
	0.748398	0.573485		0.857790	0.797113		0.892909	0.869377	
0.60	0.867808	0.872200	−0.005036	0.950447	0.950955	−0.000534	0.969912	0.970187	−0.000283
	0.874332	0.755790		0.953243	0.929179		0.972559	0.964429	
0.55	0.937313	0.951530	−0.014941	0.988287	0.988535	−0.000251	0.994470	0.994574	−0.000105
	0.954808	0.906086		0.991664	0.987337		0.996819	0.995511	
0.50	0.942342	0.987726	−0.045948	0.998476	0.998538	−0.000062	0.999494	0.999513	−0.000019
	0.990134	0.981900		0.999447	0.999234		0.999897	0.999832	

Note: $r(\varphi^*, \varphi_{SW}) = (\beta(\varphi^*) - \beta(\varphi_{SW}))/\beta(\varphi_{SW})$.
Upper entry: exact value; lower entry: asymptotic value.

4.2. Large sample sizes case

When m is sufficiently large, the power of φ^* at θ, $(0 < \theta < \theta_0)$ is:

$$\begin{aligned}
\beta\left(\theta|\varphi^*, \alpha, T\right) &= P_\theta\left\{Y(m, T, N_T) - N_T\theta_0 \le -\theta_0^2 k\,(m, \alpha, \theta_0, T)\right\} \\
&= P_\theta\left\{\frac{Y(m, T, N_T) - N_T\theta_0 - m\eta(\theta)}{\sqrt{m}\sigma(\theta)} \le \frac{-\theta_0^2 k\,(m, \alpha, \theta_0, T) - m\eta(\theta)}{\sqrt{m}\sigma(\theta)}\right\} \\
&\approx \Phi\left(\frac{-\theta_0^2 k\,(m, \alpha, \theta_0, T) - m(\theta - \theta_0)(1 - e^{-\frac{T}{\theta}})}{\sqrt{m}\sigma(\theta)}\right) \\
&\approx \Phi\left(\frac{m(\theta_0 - \theta)(1 - e^{-\frac{T}{\theta}}) - z_\alpha\theta_0\sqrt{m\left(1 - e^{-\frac{T}{\theta_0}}\right)}}{\sqrt{m}\sigma(\theta)}\right) \\
&= \Phi\left(\frac{\sqrt{m}(\theta_0 - \theta) - z_\alpha\theta_0\sqrt{1 - e^{-\frac{T}{\theta_0}}}/(1 - e^{-\frac{T}{\theta}})}{\sqrt{\frac{\theta^2}{1 - e^{-\frac{T}{\theta}}} + \frac{(\theta - \theta_0)^2 e^{-\frac{T}{\theta}}}{1 - e^{-\frac{T}{\theta}}} + \frac{2(\theta_0 - \theta)T e^{-\frac{T}{\theta}}}{(1 - e^{-\frac{T}{\theta}})^2}}}\right).
\end{aligned} \tag{4.4}$$

Recall that $\eta(\theta)$ and $\sigma(\theta)$ are defined in (3.10) and $k\,(m, \alpha, \theta_0, T) \approx z_\alpha\sqrt{m\left(1 - e^{-\frac{T}{\theta_0}}\right)}/\theta_0$, see (3.13).

From Spurrier and Wei (1980), the power of φ_{SW} at $\theta(0 < \theta < \theta_0)$ is, approximately,

$$\begin{aligned}
\beta\,(\theta|\varphi_{SW}, \alpha, T) &= P_\theta\,\{Y(m, T, N_T) - N_T\theta_0 c_{SW}\,(m, \alpha, \theta_0, T) \le 0\} \\
&\approx \Phi\left(\frac{\sqrt{m}(\theta_0 - \theta) - z_\alpha\theta_0/\sqrt{1 - e^{-\frac{T}{\theta_0}}}}{\theta/\sqrt{1 - e^{-\frac{T}{\theta}}}}\right).
\end{aligned} \tag{4.5}$$

Table 3
Data from Nelson (1982)

Group 1	Group 2	Group 3	Group 4	Group 5	Group 6
1.89	1.30	1.99	1.17	8.11*	2.12
4.03	2.75	0.64	3.87	3.17	3.97
1.54	0.00	2.15	2.80	5.55*	1.56
0.31	2.17	1.08	0.70	0.80	1.34
0.66	0.66	2.57	3.82	0.20	1.49
1.70	0.55	0.93	0.02	1.13	8.71*
2.17	0.18	4.75	0.50	6.63*	2.10
1.82	10.60*	0.82	3.72	1.08	7.21*
9.99*	1.63	2.06	0.06	2.44	3.83
2.24	0.71	0.49	3.57	0.78	5.13
$n_1 = 9$	$n_2 = 9$	$n_3 = 10$	$n_4 = 10$	$n_5 = 7$	$n_6 = 8$
$y_1 = 21.9052$	$y_2 = 15.4952$	$y_3 = 17.48$	$y_4 = 20.23$	$y_5 = 26.2355$	$y_6 = 32.6304$

For $\alpha = 0.05$, $\alpha = 0.025$, $\alpha = 0.01$, $R = \ln 2, \ln 4, \ln 10$, and $m = 30, 40, 50$, the asymptotic powers of φ^* and φ_{SW} are computed based on (4.4) and (4.5), respectively. In Table 2, for given $m = 30(10)50$, α, R and θ, the upper entry is the exact power and lower entry is the asymptotic power for both cases of φ^* and φ_{SW}.

5. Examples

Example 1. Suppose 20 lifetime components are put on life testing experiment with censoring time $R = \ln 2$. One wants to test the hypotheses $H_0 : \theta \geq 1$ against $H_1 : \theta < 1$. At the end of the experiment, the MLE $\overset{\wedge}{\theta}_{ML} = 0.660$ is computed. Should one reject H_0 at level of significance $\alpha = 0.05$?

Suppose the test φ_{SW} is applied. For a recomputed value $c_{SW}(m, \alpha, \theta_0, R) = 0.614412$. Since $\overset{\wedge}{\theta}_{ML} = 0.660 > c_{SW}(m, \alpha, \theta_0, R) = 0.614412$, thus, the test φ_{SW} does not reject H_0.

Does the test φ^* reject H_0 at level of significance $\alpha = 0.05$? In order to answer this question, we also need to know the observed value of N_R, the number of failures before censoring time R. Note that $Z(m, R, N_R) - N_R = \overset{\wedge}{\theta}_{ML} \times N_R - N_R$. Also, from Table 1, $c(m, \alpha, R) = -5.228337$.

For $N_R = 15$, $Z(m, R, N_R) - N_R = -5.1$, and for $N_R = 16$, $Z(m, R, N_R) - N_R = -5.44$.

Therefore, φ^* accepts H_0 for $N_R = 15$; however, φ^* rejects H_0 for $N_R = 16$.

Example 2. We use the insulating fluid example (See Table 4.1, page 462 of Nelson (1982)) for illustration.

There are six groups of insulating fluid. Ten items from each group are put on a life testing experiment which is subject to high voltage stress. The record of the times to breakdown in minutes is shown in Table 3. The result of Nelson (1982) indicates that the data in each group follows an exponential distribution.

For group i, let θ_i denote the associated expected lifetime. Let $\theta_0 = 4$. We want to test the hypotheses $H_0 : \theta_i \geq 4$ against $H_1 : \theta_i < 4$ at level of significance $\alpha = 0.05$, for each $i = 1, \ldots, 6$.

Suppose type-I censoring is applied with censoring time $T = \theta_0 R$ with $R = \ln 4$. Thus, $T = 4 \ln 4 = 5.54518$.

In Table 3, those values marked with "$*$" should be censored data according to this censoring scheme. For group i, let n_i denote the number of failures before the censoring time T and y_i be the total lifetime of the 10 components put on test up to time T. Thus, for each $i = 1, 2, \ldots, 6$, $y_i - n_i\theta_0$ can be computed and they are respectively given by -14.0948, -20.5048, -22.52, -19.77, -1.7645 and 0.6304.

The critical value for the test φ^* is: $-\theta_0^2 k(m, \alpha, \theta_0, T) = \theta_0 c(m, \alpha, R) = 4 \times (-4.326138) = -17.30455$.

Therefore, we reject $H_0 : \theta_i \geq 4$ for $i = 2, 3, 4$.

On the other hand, the ML estimator of θ_i is given by $\overset{\wedge}{\theta}_{MLi} = \frac{y_i}{n_i}$, $i = 1, 2, \ldots, 6$. They are respectively 2.43391, 1.72169, 1.748, 2.023, 3.74793 and 4.0788.

For a recomputed value $c_{SW}(m, \alpha, \theta_0, R) = 0.537102$, the test φ_{SW} rejects $H_0 : \theta_i \geq 4$ if $\overset{\wedge}{\theta}_{MLi} < \theta_0 c_{SW}(m, \alpha, \theta_0, R) = 4 \times 0.537102 = 2.148408$. According to the preceding numerical values, the test φ_{SW} rejects $H_0 : \theta_i \geq 4$ for $i = 2, 3$ and 4.

In this numerical example, the conclusions of the two tests φ^* and φ_{SW} are the same.

6. Conclusions

In this paper, for the test of hypotheses $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$, where θ_0 is known, we propose a local optimal test φ^* defined by (2.4) which is based on type-I censored data from an exponential density. The proposed test φ^*, being a likelihood ratio test, is unbiased and it behaves as a Bayes test. Exact critical values are derived and for some type-I errors, both exact and asymptotic critical values are tabulated in Table 1. It is easy to see that asymptotic critical values are rather inaccurate. Comparisons of powers between φ^* and the test φ_{SW} proposed by Spurrier and Wei (1980) have been studied for both small and large sample size. Some numerical power comparisons for $R = \ln 2$, $\ln 4$ and $\ln 10$ are tabulated in Table 2. The numerical results indicate that the power of φ^* is better than that of φ_{SW} when $\theta(0 < \theta < \theta_0)$ is close to θ_0. As can be seen from Table 2, when $\theta > 0.75$, φ^* is always superior to φ_{SW}; however, when θ decreases and is close to 0.5, φ_{SW} may behave better and it depends on the sample size m and R. When data is heavily censored ($R = -\ln(1 - \zeta)$ with $\zeta < 0.5$), we recommend φ_{SW} over φ^*.

For practical applications, two examples are illustrated.

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