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Comment on ‘An ordering policy for deteriorating items with allowable shortage and permissible delay in payment’ by Jamal, Sarker and Wang

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$$V(N) = \min\left\{ \min_{1 \leqslant j \leqslant N} \left[C_I + \Pr(X_j = 0) \left(G(j) + C_S \sum_{k=j+1}^{N} \Pr(Y_k = 1 | X_j = 0) \right) + \Pr(X_j = 1) \left(V(N-j) + C_P \sum_{i=1}^{j} \Pr(Y_i = 0 | X_j = 1) \right) \right], V^0(N) \right\} \quad (14)$$

$$G(N) = \min\left\{ \min_{1 \leqslant j \leqslant N-1} \left[C_I + \Pr(X_j = 0 | X_N = 0) \left(G(j) + C_s \sum_{k=j+1}^{N} \Pr(Y_k = 1 | X_j = 0, X_N = 0) \right) + \Pr(X_j = 1 | X_N = 0) \left(G(N-j) + C_p \sum_{i=1}^{j} \Pr(Y_i = 0 | X_j = 1, X_N = 0) \right) \right], G^0(N) \right\} \quad (15)$$

The solution algorithm in the original paper remains valid.

Conclusion

In our original paper, we endeavoured to extend Raz *et al*'s (2000) off-line inspection model to consider inspection errors. But due to the neglect pointed out by Chun, our mathematical formulation went only halfway and did not allow inspection errors for the last unit in the batch. We are grateful to Chun for his insight and are happy to have derived a more general model which can be used by researchers in the future. We are also grateful to the editors for giving us the opportunity to reply to Chun's *Viewpoint*.

References

Raz T, Herer YT and Grosfeld-Nir A (2000). Economic optimization of off-line inspection. *IIE Trans* **32**: 205–217.

Sheu SH, Chen YC, Wang WY and Shin NH (2003). Economic optimization of off-line inspection with inspection errors. *J Opl Res Soc* **54**: 888–895.

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In an article published in *JORS*, Aggarwal and Jaggi (1995) considered the inventory model with an exponential deterioration rate under the condition of permissible delay in payments. Later on, Jamal *et al* (1997) extended Aggarwal and Jaggi's model to allow for shortages. The main objective in their paper is to find the optimal replenishment policy which minimizes the total variable cost per unit, $TC(T_1, T)$, for each case. Their numerical results also show that inventory backlogging is beneficial from economics viewpoint. However, the uniqueness of the optimal solution for each case in their model has remained for future research. Also, they did not provide a procedure to find the global minimum. Therefore, their numerical results showed the situations with no feasible solutions in Table 4.

In this viewpoint, we complement the shortcomings of Jamal *et al* (1997). First, we show that the optimal solution for each case not only exists but is unique under specific circumstance. Then, we provide a procedure for finding the optimal solution and show in a rigorous way that the solution is indeed global minimum. In a specific circumstance, without the extremely high backorder cost, the model reduces to the case with no shortage and obtains the negative minimum cost.

For easy tractability with Jamal *et al* (1997), we use the same notations and assumptions as they did. The total variable costs per unit time for these two cases constructed by Jamal *et al* (1997) are reviewed as follows:

$$TC_1(T_1, T) = \frac{A}{T} + \frac{cD(\theta + i)}{\theta^2 T}(e^{\theta T_1} - 1 - \theta T_1) + \frac{cI_p D}{\theta^2 T}[e^{\theta(T_1 - M)} - 1 - \theta(T_1 - M)] - \frac{cI_e D T_1^2}{2T} + \frac{C_b D(T - T_1)^2}{2T}, \qquad T_1 \geqslant M \quad (1)$$

and

$$TC_2(T_1, T) = \frac{A}{T} + \frac{cD(\theta + i)}{\theta^2 T}(e^{\theta T_1} - 1 - \theta T_1) - \frac{cI_e D T_1(2M - T_1)}{2T} + \frac{C_b D(T - T_1)^2}{2T}, \qquad T_1 < M \quad (2)$$

respectively. Now, we want to prove the optimal solution for each case not only exists but is unique.

Case 1: $T_1 \geqslant M$

In this case, we let T_1^* and T^* denote the optimal values of T_1 and T, respectively. Then the optimal solution (T_1^*, T^*)

must satisfy $\partial TC_1(T_1, T)/\partial T_1 = 0$ and $\partial TC_1(T_1, T)/\partial T = 0$, simultaneously, which implies

$$\frac{c(\theta+i)}{\theta}(e^{\theta T_1}-1)+\frac{cI_p}{\theta}[e^{\theta(T_1-M)}-1]-cI_eT_1 = C_b(T-T_1) \quad (3)$$

and

$$-A-\frac{cD(\theta+i)}{\theta^2}(e^{\theta T_1}-1-\theta T_1) + \frac{cI_pD}{\theta^2}[e^{\theta(T_1-M)}-1-\theta(T_1-M)] - \frac{cI_eDT_1^2}{2}+\frac{C_bD(T^2-T_1^2)}{2}=0 \quad (4)$$

If we assume $I_e \leqslant \theta + i$, then the LHS of (3) is always positive. Hence, it can be seen that T is a function of T_1 from (3) and $T > T_1$. Next, by taking implicit differentiation on (3) with respect to T_1, we have $\mathrm{d}T/\mathrm{d}T_1 - 1 > 0$. Now, from (4), we set

$$G_1(T_1) = -A-\frac{cD(\theta+i)}{\theta^2}(e^{\theta T_1}-1-\theta T_1)-\frac{cI_pD}{\theta^2} \times [e^{\theta(T_1-M)}-1-\theta(T_1-M)]+\frac{cI_eDT_1^2}{2} + \frac{C_bD}{2}(T^2-T_1^2) \quad (5)$$

Owing to the relations shown in (3) and $\mathrm{d}T/\mathrm{d}T_1 - 1 > 0$, we get $\mathrm{d}G_1(T_1)/\mathrm{d}T_1 = C_bDT(\mathrm{d}T/\mathrm{d}T_1 - 1) > 0$. Therefore, $G_1(T_1)$ is a strictly increasing function of T_1. Furthermore, we have $T \to \infty$ as $T_1 \to \infty$ and $\lim_{T_1\to\infty} G_1(T_1) = \infty$.

For notational convenience, let $\Delta = M^2 + 2A/(C_bD) + 2c(\theta+i)(e^{\theta M} - 1 - \theta M)/(C_b\theta^2) - cI_eM^2/C_b$. Then from above arguments, we have the following results.

Proposition 1 *If $I_e \leqslant \theta+i$ and $G_1(M) > 0$, then the optimal solution is $(T_1^*, T^*) = (M, \sqrt{\Delta})$.*

Proof If $I_e \leqslant \theta + i$ and $G_1(M) > 0$, then we have $\partial TC_1(T_1, T)/\partial T = G_1(T_1)/T^2 \geqslant G_1(M)/T^2 > 0$, which implies that for any $T \in (T_1, \infty)$, a smaller value of T causes a smaller value of $TC_1(T_1, T)$. By using the fact that $\mathrm{d}T/\mathrm{d}T_1 - 1 > 0$, we know that the minimum value of $TC_1(T_1, T)$ occurs at the boundary point $T_1^* = M$. By putting $T_1^* = M$ into (4), we can obtain $T^* = \sqrt{\Delta}$, where $T^* > M$ under the condition $I_e \leqslant \theta + i$. □

Proposition 2 *If $I_e \leqslant \theta+i$ and $G_1(M) \leqslant 0$, then the optimal value (T_1^*, T^*) can be found by solving (3) and (4) simultaneously, and it not only exists but is unique.*

Proof The Intermediate Value Theorem implies that there exists a unique value T^* such that $G(T_1^*) = 0$ if $I_e \leqslant \theta + i$ and $G_1(M) \leqslant 0$. Consequently, the point (T_1^*, T^*) satisfying (3) and (4) simultaneously not only exists but is unique. Next, it can be easy to show the sufficient conditions to minimize $TC_1(T_1, T)$ are satisfied at the point (T_1^*, T^*). □

Case 2: $T_1 < M$

In this case, we let T_1^{**} and T^{**} denote the optimal values of T_1 and T, respectively. Then the optimal solution (T_1^{**}, T^{**}) must satisfy $\partial TC_2(T_1, T)/\partial T_1 = 0$ and $\partial TC_2(T_1, T)/\partial T = 0$, simultaneously, which implies

$$\frac{c(\theta+i)}{\theta}(e^{\theta T_1}-1)-cI_e(M-T_1)=C_b(T-T_1) \quad (6)$$

and

$$-A-\frac{cD(\theta+i)}{\theta^2}(e^{\theta T_1}-1-\theta T_1)+\frac{cI_eD(2M-T_1)T_1}{2} + \frac{C_bD(T^2-T_1^2)}{2}=0 \quad (7)$$

From (6), we set that T is a function of T_1. Now, we let the LHS of (6) as

$$F(T_1)=\frac{c(\theta+i)}{\theta}(e^{\theta T_1}-1)-cI_e(M-T_1)$$

which implies $F(T_1)$ is strictly increasing with respect to $T_1 > 0$. Because $\lim_{T_1\to M^-} F(T_1) = c(\theta+i)(e^{\theta M}-1)\theta > 0$ and $\lim_{T_1\to 0^+} F(T_1) = -cI_eM < 0$, there exists a unique value $\widehat{T}_1 \in (0, M)$ such that $F(\widehat{T}_1) = 0$. Since the property that the RHS of (6) is negative for $T_1 \in [0, \widehat{T}_1)$ implies $T < T_1$, the interval $T_1 \in [0, \widehat{T}_1)$ can be excluded from consideration.

Furthermore, by taking implicit differentiation on (6) with respect to T_1, we have $\mathrm{d}T/\mathrm{d}T_1 - 1 > 0$. Next, from (7), we set

$$G_2(T_1) = -A-\frac{cD(\theta+i)}{\theta^2}(e^{\theta T_1}-1-\theta T_1) + \frac{cI_eD(2M-T_1)T_1}{2}+\frac{c_bD(T^2-T_1^2)}{2} \quad (8)$$

Owing to the relations shown in (6) and $\mathrm{d}T/\mathrm{d}T_1 - 1 > 0$, we get $\mathrm{d}G_2(T_1)/\mathrm{d}T_1 = C_bDT(\mathrm{d}T/\mathrm{d}T_1 - 1) > 0$. Therefore, $G_2(T_1)$ is a strictly increasing function of T_1.

From the analysis carried out so far, we have the following results:

Proposition 3 *If $G_2(\widehat{T}_1) > 0$, then the model reduces to the model without shortages and the optimal solution is $(T_1^{**}, T^{**}) = (T_1^{\#}, T_1^{\#})$, where $T_1^{\#} \in (0, \widehat{T}_1)$.*

Proof In the interval $[\widehat{T}_1, M)$, if $G_2(\widehat{T}_1) > 0$, then we have $\partial TC_2(T_1, T)/\partial T = G_2(T_1)/T^2 \geqslant G_2(\widehat{T}_1)/T^2 > 0$ for any $\widehat{T}_1 \leqslant T_1 < T < \infty$, which implies the minimum value of $TC_2(T_1, T)$ occurs at $T = T_1 = \widehat{T}_1$. Thus the model reduces to the model without shortages.

When shortages are not allowed, the objective function can be obtained from (2) by letting $T = T_1$. Then we have $TC_2(T_1, T_1) \equiv TC_2(T_1)$ as follows:

$$TC_2(T_1) = \frac{A}{T_1} + \frac{cD(\theta + i)}{\theta^2 T_1}(e^{\theta T_1} - 1 - \theta T_1) - \frac{cI_e D(2M - T_1)}{2}, \qquad 0 \leqslant T_1 < M \tag{9}$$

Taking the first-order and the second-order derivatives of $TC_2(T)$ with respect to T_1 respectively, we obtain

$$\frac{\mathrm{d}TC_2(T_1)}{\mathrm{d}T_1} = -\frac{A}{T_1^2} + \frac{cD(\theta + i)}{\theta^2 T_1^2}(\theta T_1 e^{\theta T_1} - e^{\theta T_1} + 1) + \frac{cI_e D}{2}$$

and

$$\frac{\mathrm{d}^2 TC_2(T_1)}{\mathrm{d}T_1^2} = \frac{1}{T_1^3}\left[2A + \frac{cD(\theta + i)}{\theta^2}(\theta^2 T_1^2 e^{\theta T_1} - 2\theta T_1 e^{\theta T_1} + 2e^{\theta T_1} - 2)\right] > 0$$

by using the fact that $\theta^2 T_1^2 e^{\theta T_1} - 2\theta T_1 e^{\theta T_1} + 2e^{\theta T_1} - 2 \geqslant 0$ for all $T_1 \geqslant 0$.

Owing to the relations $F(\widehat{T}_1) = 0$ and $G_2(\widehat{T}_1) > 0$, it can be shown that there exists a unique value of $T_1 \in (0, \widehat{T}_1)$ (denoted by $T_1^{\#}$) such that $TC_2(T_1)$ is minimum. In particular, the optimal $TC_2(T_1)$ found from (9) is $TC_2(T_1^{\#}) = -cD(\theta + i)(1 - e^{\theta T_1^{\#}})/\theta - cI_e D(M - T_1^{\#}) < 0$. □

Proposition 4 *If $G_2(M) \leqslant 0$, then the optimal solution is $(T_1^{**}, T^{**}) = (M, \sqrt{\Delta})$.*

Proof If $G_2(M) \leqslant 0$, then we have $\partial TC_2(T_1, T)/\partial T = G_2(T_1)/T^2 < G_2(M)/T^2 \leqslant 0$ for any $T_1 \in [\widehat{T}_1, M)$ and $T \in (T_1, \infty)$, which implies the minimum value of $TC_2(T_1, T)$ occurs at the boundary point $T_1^{**} = M$. By putting $T_1^{**} = M$ into (7), we obtain $T^{**} = \sqrt{\Delta}$, where $T^{**} > M$ under the condition $G_2(M) \leqslant 0$. □

Proposition 5 *If $G_2(M) > 0$ and $G_2(\widehat{T}_1) \leqslant 0$, then the optimal value (T_1^{**}, T^{**}) can be found by solving* (6) *and* (7) *simultaneously, and it not only exists but is unique.*

Proof The proof can be obtained by the similar arguments above, here we omit it. □

The objective of this problem is to determine the optimal value $(T_1^{(0)}, T^{(0)})$ of (T_1, T) so that $TC(T_1^{(0)}, T^{(0)})$ is minimum. Let $T_1 = M$, from (1), (2), (5) and (8), we have $TC_1(M, T) = TC_2(M, T)$ and $G_1(M) < G_2(M)$. Using these results and Propositions 1–5 with the condition $I_e \leqslant \theta + i$, we can obtain the following proposition:

Proposition 6

(a) *If $G_2(M) \leqslant 0$, then the optimal solution is $(T_1^{(0)}, T^{(0)}) = (T_1^*, T^*)$, where (T_1^*, T^*) can be found by solving* (3) *and* (4) *simultaneously.*

(b) *If $G_1(M) > 0$ and $G_2(\widehat{T}_1) \leqslant 0$, then the optimal solution is $(T_1^{(0)}, T^{(0)}) = (T_1^{**}, T^{**})$, where (T_1^{**}, T^{**}) can be found by solving* (6) *and* (7) *simultaneously.*

(c) *If $G_1(M) > 0$ and $G_2(\widehat{T}_1) > 0$, then the optimal solution is $(T_1^{(0)}, T^{(0)}) = (T_1^{\#}, T^{\#})$, where $T_1^{\#} \in (0, \widehat{T}_1)$ and satisfies $\mathrm{d}TC_2(T_1)/\mathrm{d}T_1 = 0$.*

(d) *If $G_2(\widehat{T}_1) \leqslant 0$, $G_2(M) > 0$ and $G_1(M) \leqslant 0$, then the optimal solution is*

$$TC(T_1^{(0)}, T^{(0)}) = \min\{TC_1(T_1^*, T^*), TC_2(T_1^{**}, T^{**})\}$$

where (T_1^*, T^*) can be found by solving (3) *and* (4) *simultaneously, and (T_1^{**}, T^{**}) can be obtained by solving* (6) *and* (7) *simultaneously.*

Proof (a) If $G_2(M) \leqslant 0$, from Proposition 4, we have $TC_2(T_1^{**}, T^{**}) = TC_2(M, \sqrt{\Delta})$. On the other hand, since $G_1(M) < G_2(M) \leqslant 0$, then from Proposition 2, there exists a unique solution (T_1^*, T^*) which can be found by solving (3) and (4) simultaneously such that $TC_1(T_1^*, T^*)$ is minimum, which implies that $TC_1(T_1^*, T^*) < TC_1(M, \sqrt{\Delta}) = TC_2(T_1^{**}, T^{**})$. Therefore, $TC(T_1^{(0)}, T^{(0)}) = \min\{TC_1(T_1^*, T^*), TC_2(T_1^{**}, T^{**})\} = TC_1(T_1^*, T^*)$. Thus, the optimal solution is $(T_1^{(0)}, T^{(0)}) = (T_1^*, T^*)$.

(b) If $G_1(M) > 0$, from Proposition 1, we have $TC_1(T_1^*, T^*) = TC_1(M, \sqrt{\Delta})$. Furthermore, since $G_2(\widehat{T}_1) \leqslant 0$ and $G_2(M) > G_1(M) > 0$, by Proposition 5, there exists a unique solution (T_1^{**}, T^{**}) which can be obtained by solving (6) and (7) simultaneously such that $TC_2(T_1^{**}, T^{**})$ is minimum. Hence, we have $TC_2(T_1^{**}, T^{**}) < TC_2(M, \sqrt{\Delta}) = TC_1(T_1^*, T^*)$. Therefore, $TC(T_1^{(0)}, T^{(0)}) = \min\{TC_1(T_1^*, T^*), TC_2(T_1^{**}, T^{**})\} = TC_2(T_1^{**}, T^{**})$. Thus, the optimal solution is $(T_1^{(0)}, T^{(0)}) = (T_1^{**}, T^{**})$.

(c) If $G_1(M) > 0$, then from Proposition 1, we know that $TC_1(T_1^*, T^*) = TC_1(M, \sqrt{\Delta})$. Furthermore, since $G_2(\widehat{T}_1) > 0$, by Proposition 3, we have

$$TC_2(T_1^{**}, T^{**}) = TC_2(T_1^{\#}, T_1^{\#}) < TC_2(\widehat{T}_1, \widehat{T}_1) < TC_2(M, \sqrt{\Delta}) = TC_1(T_1^*, T^*)$$

Therefore, $TC(T_1^{(0)}, T^{(0)}) = \min\{TC_1(T_1^*, T^*), TC_2(T_1^{**}, T^{**})\} = TC_2(T_1^{**}, T^{**})$. Thus, the optimal solution is $(T_1^{(0)}, T^{(0)}) = (T_1^{**}, T^{**}) = (T_1^{\#}, T_1^{\#})$.

(d) If $G_1(M) \leqslant 0$, then, from Proposition 2, there exists a unique solution (T_1^*, T^*) which can be obtained by solving (3) and (4) simultaneously such that $TC_1(T_1^*, T^*)$ is minimum. Furthermore, since $G_2(\widehat{T}_1) \leqslant 0$ and $G_2(M) > 0$, then from Proposition 5, there exists a unique solution (T_1^{**}, T^{**}) which can be obtained by solving (6) and (7) simultaneously such that $TC_2(T_1^{**}, T^{**})$ is minimum. Then, the optimal solution of (T_1, T) can be determined by comparing the values of $TC_1(T_1^*, T^*)$ and $TC_2(T_1^{**}, T^{**})$. That is, if $(T_1^{(0)}, T^{(0)})$ is the optimal solution, then $TC(T_1^{(0)}, T^{(0)}) = \min\{TC_1(T_1^*, T^*), TC_2(T_1^{**}, T^{**})\}$. □

Authors' response

The authors were approached but did not wish to give a reply to this Viewpoint.

References

Aggarwal SP and Jaggi CK (1995). Ordering policies of deteriorating items under permissible delay in payments. *J Opl Res Soc* **46**: 658–662.

Jamal AM, Sarker BR and Wang S (1997). An ordering policy for deteriorating items with allowable shortage and permissible delay in payment. *J Opl Res Soc* **48**: 826–833.

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