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Determining optimal lot size for a two-warehouse system with deterioration and shortages using net present value

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Abstract

In this paper, we develop a deterministic inventory model for deteriorating items with two warehouses by minimizing the net present value of the total cost. Deterioration rates of items in the two warehouses may be different. In addition, we allow for shortages and complete backlogging. We then prove that the optimal replenishment policy not only exists but also is unique under some condition. Further, the result reveals that the reorder interval based on the average total cost, if it exists, must be longer than that derived using net present value. Finally, we use Yang's [H.L. Yang, European Journal of Operational Research 157 (2004) 344–356] numerical example to illustrate the model and conclude the paper with suggestions for possible future research.

Keywords: Deterioration; Shortages; Two-warehouse; Net present value; Discounted cash flow

1. Introduction

The classical inventory models usually assume the available warehouse has unlimited capacity. In many practical situations, there exists many factors like temporary price discounts making retailers buy a capacity of goods exceeding their own warehouse (OW). In this case, retailers will either rent other warehouses or rebuild a new warehouse. However, from economical point of views, they usually choose to rent other warehouses. Hence, an additional storages space known as rented warehouses (RW) is often required due to limited capacity of showroom facility. In addition, for certain types of commodities, such as medicine, volatile liquids, blood bank, foodstuffs, deterioration is usually observed during their normal storage period. By assuming constant demand rate, Sarma [7] developed a deterministic inventory model for a single deteriorating item with shortages and two levels of storage. Pakkala and Achary [5] extended the two-warehouse inventory model for deteriorating items with finite replenishment rate and shortages. Besides, the ideas of time-varying demand

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for deteriorating items with two storage facilities were considered by Benkherouf [1] and Bhunia and Maiti [2]. Recently, Goyal and Giri [3] presented a review of deteriorating inventory literature since the early 1990s. They mentioned other research articles dealing with both deterioration and two-warehouse inventory problem. We suggest the reader to Goyal and Giri's [3] article and the references given there for more details.

The RW usually results in additional cost of maintenance, material handling, etc. In these models mentioned above, it is generally assumed that the holding cost in RW are higher than one in OW. Again, since the deterioration depends on preserving facilities and environmental conditions available in a warehouse, different warehouses may have different deterioration rates. As deterioration phenomenon is taken into account, a unit of inventory stored incurs holding cost and deterioration cost. In a recent paper, Yang [9] proposed an inventory model for determining the optimal replenishment cycle for the two-warehouse problem under inflation, in which the inventory deteriorates at a constant rate over times and shortages are allowed. The author assumed that inventory costs (including holding cost and deterioration cost) in RW are higher than those in OW. However, unlike Benkherouf [1], Yang [9] supposes that the deterioration rate in RW is larger than one in OW. Obviously, Yang's assumption of deterioration rate contradicts the situation that the RW like "Central Warehousing Facility" generally provides better preserving facility than the OW resulting in a lower deterioration rate for the goods.

By assuming that the inventory system will operate for a long time, Yang [9] determined the optimal values of the decision variables by minimizing the average total cost. However, an alternative is to determine the decision variables by minimizing the discounted value of all future costs (i.e. the net present value (NPV) of total cost). Hadley [4] compared the optimal order quantities determined by minimizing these two different objective functions. When the discount rate is excessive, he obtained the optimal reorder intervals with significant differences for these two models. Rachamadugu [6] developed error bounds for EOQ model by minimizing net present value approximately. Further, Sun and Queyranne [8] investigated the general multiproduct, production and inventory model using the NPV of the total cost as the objective function. They pointed out that the reorder interval based on the average total cost could be much longer than that derived using net present value.

In this paper, we develop a deterministic inventory model for deteriorating items with two warehouses. As Yang [9], we allow for shortages and complete backlogging, and assume that the inventory costs (including holding cost and deterioration cost) in RW is higher than that in OW. The firm stores goods in OW before RW, but clears the stocks in RW before OW. However, we minimize the NPV of the total cost as proposed by Sun and Queyranne [8]. For generality, the deterioration rate in RW is different from one in OW. Due to consideration towards the effect of the discount rate, which relates to the purchasing power of money, purchasing cost must be included. We complement the shortcoming of Yang's model by computing the purchase cost instead of the deterioration cost. Then, we obtain the condition which guarantees the unique solution exists and develop the criterion to find the optimal replenishment policy. Next, we will compare the decision using the NPV with one using the average total cost. The result reveals that the reorder interval based on the average total cost, if it exists, must be longer than that derived using NPV. In the last two sections, a numerical example is discussed to illustrate the proposed model and concluding remarks are provided.

2. Notation and assumptions

2.1. Notation

To develop the mathematical model of inventory replenishment policy with two warehouses, the notation adopted in this paper is as below:

D the demand rate per unit time
A the replenishment cost per order
c the purchasing cost per unit
r discount rate
c_{ho} the holding cost per unit per unit time in OW
c_{hr} the holding cost per unit per unit time in RW
c_{s} the backorder cost per unit per unit time

α the deterioration rate in OW, where $0 \leqslant \alpha < 1$
β the deterioration rate in RW, where $0 \leqslant \beta < 1$
W the capacity of the own warehouse
Q the ordering quantity per cycle
B the maximum inventory level per cycle
t_r the length of period during which the inventory level reaches zero in RW
t_o the length of period during which the inventory level reaches zero in OW
t_s the length of period during which shortages are allowed
T the length of the inventory cycle
$I_r(t)$ the level of positive inventory in RW at time t
$I_o(t)$ the level of positive inventory in OW at time t
$I_s(t)$ the level of negative inventory at time t
$\mathrm{TC}(t_r,t_s)$ the net present value of cash flows for the first cycle
$\mathrm{NPV}(t_r,t_s)$ the net present value of total cost
$\mathrm{ATC}(t_r,t_s)$ the average total cost

2.2. Assumptions

In addition, the following assumptions are imposed:

1. Replenishment rate is infinite, and lead time is zero.
2. The time horizon of the inventory system is infinite.
3. The own warehouse (OW) has a fixed capacity of W units; the rented warehouse (RW) has unlimited capacity.
4. The goods of OW are consumed only after consuming the goods kept in RW, hence $T = t_o + t_s$.
5. The unit inventory costs (including holding cost and deterioration cost) per unit time in RW are higher than those in OW; that is, $c_{hr} + \beta c > c_{ho} + \alpha c$.
6. Shortages are allowed and completely backlogged.

3. Mathematical formulation

Using above assumptions, the inventory level follows the pattern depicted in Fig. 1. The ordering quantity over the replenishment cycle can be determined as

$$Q = I_r(0) + I_o(0) - I_s(t_o + t_s) = \frac{D}{\beta}(e^{\beta t_r} - 1) + W + Dt_s, \tag{1}$$

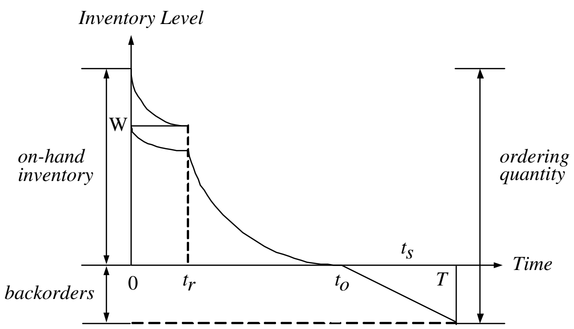

Fig. 1. Graphical representation of a two-warehouse inventory system.

and the maximum inventory level per cycle is

$$B = I_r(0) + I_o(0) = \frac{D}{\beta}(e^{\beta t_r} - 1) + W. \tag{2}$$

With an instantaneous cash transactions during sales, the present value of purchase cost for the first cycle can be obtained as

$$c[I_r(0) + I_o(0) - e^{-r(t_o+t_s)} I_s(t_o + t_s)] = c\left[\frac{D}{\beta}(e^{\beta t_r} - 1) + W + Dt_s e^{-r(t_o+t_s)}\right].$$

Hence, the present value of cash flows for the first cycle which comprises the present values of the replenishment cost, purchase cost, holding cost and backorder cost, and is given as follows:

$$\begin{aligned}\mathrm{TC}(t_r, t_s) = {} & A + c\left[\frac{D}{\beta}(e^{\beta t_r} - 1) + W + Dt_s e^{-r(t_o+t_s)}\right] + \frac{c_{hr}D}{r\beta(r+\beta)}[re^{\beta t_r} + \beta e^{-rt_r} - (r+\beta)] + \frac{c_{ho}W}{r+\alpha} \\ & + \frac{c_{ho}D}{r(r+\alpha)}(e^{-rt_o} - e^{-rt_r}) + \frac{c_s D}{r^2} e^{-r(t_o+t_s)}(e^{rt_s} - rt_s - 1),\end{aligned} \tag{3}$$

where t_o is a function of t_r and is defined as

$$t_o = t_r + \frac{1}{\alpha}\ln\left(1 + \frac{\alpha W e^{-\alpha t_r}}{D}\right). \tag{4}$$

Let $\mathrm{NPV}(t_r,t_s)$ be the net present value of total cost over horizon $[0,\infty)$. Then we have

$$\mathrm{NPV}(t_r, t_s) = \sum_{n=0}^{\infty} \mathrm{TC}(t_r, t_s) e^{-nr(t_o+t_s)} = \mathrm{TC}(t_r, t_s) \sum_{n=0}^{\infty} e^{-nr(t_o+t_s)} = \frac{\mathrm{TC}(t_r, t_s)}{1 - e^{-r(t_o+t_s)}}. \tag{5}$$

The problem is to determine t_r and t_s such that $\mathrm{NPV}(t_r,t_s)$ is minimized. Taking the first derivative of $\mathrm{NPV}(t_r,t_s)$ with respect to t_r and t_s, respectively, we obtain

$$\begin{aligned}\frac{\partial \mathrm{NPV}(t_r, t_s)}{\partial t_r} & = \frac{-re^{-r(t_o+t_s)}}{[1 - e^{-r(t_o+t_s)}]^2} \mathrm{TC}(t_r, t_s) \frac{dt_o}{dt_r} + \frac{1}{1 - e^{-r(t_o+t_s)}} \frac{\partial \mathrm{TC}(t_r, t_s)}{\partial t_r} \\ & = \frac{dt_o}{dt_r}\left\{\frac{-re^{-r(t_o+t_s)}}{[1 - e^{-r(t_o+t_s)}]^2} \mathrm{TC}(t_r, t_s) + \frac{1}{1 - e^{-r(t_o+t_s)}} \frac{1}{dt_o/dt_r} \frac{\partial \mathrm{TC}(t_r, t_s)}{\partial t_r}\right\},\end{aligned} \tag{6}$$

and

$$\frac{\partial \mathrm{NPV}(t_r, t_s)}{\partial t_s} = \frac{-re^{-r(t_o+t_s)}}{[1 - e^{-r(t_o+t_s)}]^2} \mathrm{TC}(t_r, t_s) + \frac{1}{1 - e^{-r(t_o+t_s)}} \frac{\partial \mathrm{TC}(t_r, t_s)}{\partial t_s}, \tag{7}$$

where

$$\frac{\partial TC(t_r, t_s)}{\partial t_r} = \frac{D}{1 + \alpha W e^{-\alpha t_r}/D}\{c[e^{-rt_o} - rt_s e^{-r(t_o+t_s)}] + e^{-rt_o} K(t_r) - \frac{c_s}{r} e^{-r(t_o+t_s)}(e^{rt_s} - rt_s - 1)\}, \tag{8}$$

$$\frac{\partial TC(t_r, t_s)}{\partial t_s} = De^{-r(t_o+t_s)}[c + (c_s - rc)t_s], \tag{9}$$

$$\begin{aligned}K(t_r) = {} & \frac{c_{hr} + (r+\beta)c}{r+\beta}[e^{\beta t_r + rt_o} - e^{r(t_o - t_r)}]\left(1 + \frac{\alpha W e^{-\alpha t_r}}{D}\right) + \frac{c_{ho} + (r+\alpha)c}{r+\alpha}[e^{r(t_o-t_r)} - 1] \\ & + \frac{c_{ho} + (r+\alpha)c}{r+\alpha} e^{r(t_o-t_r)} \frac{\alpha W e^{-\alpha t_r}}{D},\end{aligned} \tag{10}$$

and dt_o/dt_r is defined as

$$\frac{dt_o}{dt_r} = \frac{1}{1 + \alpha W e^{-\alpha t_r}/D}.$$

The optimal solution of (t_r, t_s) must satisfy the equations $\partial \mathrm{NPV}(t_r,t_s)/\partial t_r = 0$ and $\partial \mathrm{NPV}(t_r,t_s)/\partial t_s = 0$, simultaneously, which implies

$$r e^{-r(t_o+t_s)} \mathrm{TC}(t_r, t_s) = \frac{1 - e^{-r(t_o+t_s)}}{dt_o/dt_r} \frac{\partial \mathrm{TC}(t_r, t_s)}{\partial t_r}, \tag{11}$$

and

$$r e^{-r(t_o+t_s)} \mathrm{TC}(t_r, t_s) = [1 - e^{-r(t_o+t_s)}] \frac{\partial \mathrm{TC}(t_r, t_s)}{\partial t_s}, \tag{12}$$

respectively. Because both the left hand sides in Eqs. (11) and (12) are the same, hence the right hand sides in these equations are equal. After some algebraic simplification, Eqs. (11) and (12) reduce to the following

$$\frac{c_s - rc}{r}(1 - e^{-rt_s}) = K(t_r). \tag{13}$$

On the other hand, we substitute $\mathrm{TC}(t_r,t_s)$ of Eq. (3) and $\partial \mathrm{TC}(t_r,t_s)/\partial t_s = 0$ of Eq. (9) into Eq. (12) and obtain

$$\begin{aligned} D[1 - e^{-r(t_o+t_s)}][c + (c_s - rc)t_s] = r\Bigg\{ & A + c\left[\frac{D}{\beta}(e^{\beta t_r} - 1) + W + Dt_s e^{-r(t_o+t_s)}\right] \\ & + \frac{c_{hr}D}{r\beta(r+\beta)}[re^{\beta t_r} + \beta e^{-rt_r} - (r+\beta)] + \frac{c_{ho}W}{r+\alpha} + \frac{c_{ho}D}{r(r+\alpha)}(e^{-rt_o} - e^{-rt_r}) \\ & + \frac{c_s D}{r^2} e^{-r(t_o+t_s)}(e^{rt_s} - rt_s - 1)\Bigg\}. \end{aligned} \tag{14}$$

Now, we want to investigate the property of function $K(t_r)$ and we have

Lemma 1. *If $D > \alpha W$, then $K(t_r)$ is a continuous and strictly increasing function of $t_r \in [0,\infty]$, and its range is* $[\frac{c_{ho}+(r+\alpha)c}{r+\alpha}[(1+\frac{\alpha W}{D})^{\frac{r}{\alpha}+1} - 1], \infty]$.

Proof. See Appendix A. □

From Lemma 1, to guarantee the optimal solution exists, we assume that the demand rate D is larger than the maximum deteriorating quantity for the items in OW, αW; that is, $D > \alpha W$. This result is obvious because for the case $D \leqslant \alpha W$, we store the items in the own warehouse is inadequate. Thus, from now on, we assume that $D > \alpha W$ in this article.

Lemma 2. *If $K(0) \geqslant (c_s - rc)/r$, then the nonnegative solution of (t_r,t_s) which satisfies Eq. (13) does not exist.*

Proof. See Appendix B. □

From Lemma 2, we see that the optimal solution exists only if $(c_s - rc)/r > K(0)$. When the inequality $(c_s - rc)/r > K(0)$ holds, Eq. (13) implies that t_s is a function of $t_r \in [0,\infty)$. Taking the partial derivative of both sides in Eq. (13) with respect to t_r, it gives

$$(c_s - rc)e^{-rt_s} \frac{dt_s}{dt_r} = \frac{dK(t_r)}{dt_r} > 0. \tag{15}$$

Thus we obtain $dt_s/dt_r > 0$. From Lemma 1, $K(t_r)$ is a continuous and strictly increasing function of $t_r \in [0,\infty)$, thus we can find a unique value $\hat{t}_r \in (0,\infty)$ such that $K(\hat{t}_r) = (c_s - rc)/r$. Furthermore, because both t_r and t_s must be nonnegative, the feasible solution for t_r which satisfies in Eq. (13) should be chosen in the interval $[0, \hat{t}_r)$. Therefore, we can obtain the following result: once we get the optimal value $t_r^* \in [0, \hat{t}_r)$, the optimal solutions of t_o and t_s (denoted by t_o^* and t_s^*, respectively) can be uniquely determined by Eqs. (4) and (13), respectively, and given as follows:

$$t_o^* = t_r^* + \frac{1}{\alpha}\ln\left(1 + \frac{\alpha W e^{-\alpha t_r^*}}{D}\right), \quad t_s^* = \frac{1}{r}\ln\frac{c_s - rc}{c_s - rc - rK(t_r^*)}. \tag{16}$$

Now, we are ready to derive the optimal value t_r^*. Motivated by Eq. (14), we let

$$G(t_r) = D[1 - e^{-r(t_o+t_s)}][c + (c_s - rc)t_s] - r\left\{A + c\left[\frac{D}{\beta}(e^{\beta t_r} - 1) + W + Dt_s e^{-r(t_o+t_s)}\right]\right.$$
$$+ \frac{c_{hr}D}{r\beta(r+\beta)}[re^{\beta t_r} + \beta e^{-rt_r} - (r+\beta)] + \frac{c_{ho}D}{r(r+\alpha)}(e^{-rt_o} - e^{-rt_r}) + \frac{c_{ho}W}{r+\alpha}$$
$$\left. + \frac{c_s D}{r^2} e^{-r(t_o+t_s)}(e^{rt_s} - rt_s - 1)\right\}, \quad t_r \in [0, \hat{t}_r). \tag{17}$$

After assembling Eqs. (13) and (15), the first derivative of $G(t_r)$ with respect to $t_r \in (0, \hat{t}_r)$ becomes

$$\frac{dG(t_r)}{dt_r} = Dr[c + (c_s - rc)t_s]e^{-r(t_o+t_s)}\left(\frac{dt_o}{dt_r} + \frac{dt_s}{dt_r}\right) + D(c_s - rc)[1 - e^{-r(t_o+t_s)}]\frac{dt_s}{dt_r}$$
$$- Dr[c + (c_s - rc)t_s]e^{-r(t_o+t_s)}\left(\frac{dt_o}{dt_r} + \frac{dt_s}{dt_r}\right)$$
$$= D(c_s - rc)[1 - e^{-r(t_o+t_s)}]\frac{dt_s}{dt_r} > 0.$$

Therefore, $G(t_r)$ is a strictly increasing function in the interval $[0, \hat{t}_r)$. Because $\lim_{t_r \to \hat{t}_r^-} t_o = \hat{t}_o = \hat{t}_r + \frac{1}{\alpha}\ln(1 + \alpha W e^{-\alpha \hat{t}_r}/D) < \infty$ and $\lim_{t_r \to \hat{t}_r^-} t_s = \infty$, it yields

$$\lim_{t_r \to \hat{t}_r^-} G(t_r) = \lim_{t_s \to \infty} D[c + (c_s - rc)t_s] - r\left\{A + c\left[\frac{D}{\beta}(e^{\beta \hat{t}_r} - 1) + W\right] + \frac{c_{hr}D}{r\beta(r+\beta)}[re^{\beta \hat{t}_r} + \beta e^{-r\hat{t}_r} - (r+\beta)]\right.$$
$$\left. + \frac{c_{ho}W}{r+\alpha} + \frac{c_{ho}D}{r(r+\alpha)}(e^{-r\hat{t}_o} - e^{-r\hat{t}_r}) + \frac{c_s D e^{-r\hat{t}_o}}{r^2}\right\} = \infty.$$

Then we have the following results.

Lemma 3. *For any given $(c_s - rc)/r > K(0)$, we have*:

(*a*) *If $G(0) \leqslant 0$, then the solution $t_r^* \in [0, \hat{t}_r)$ which satisfies Eq. (14) not only exists but also is unique.*
(*b*) *If $G(0) > 0$, then the solution $t_r^* \in [0, \hat{t}_r)$ which satisfies Eq. (14) does not exist.*

Proof. See Appendix C. □

Theorem 1. *For any given $(c_s - rc)/r > K(0)$, we have*:

(*a*) *If $G(0) < 0$, then the point (t_r^*, t_s^*) which satisfies the Eqs. (13) and (14) simultaneously and $t_r^* \in (0, \hat{t}_r)$ is the global minimum point of the net present value of total cost.*
(*b*) *If $G(0) \geqslant 0$, then optimal $t_r^* = 0$. In this case, the inventory system reduces to the one-warehouse problem.*

Proof. See Appendix D. □

From Theorem 1(a), once the optimal solution (t_r^*, t_s^*) is obtained, we substitute (t_r^*, t_s^*) into Eqs. (1) and (5), the optimal ordering quantity per cycle, Q^*, and the minimum net present value of total cost $NPV(t_r^*, t_s^*)$ are as follows:

$$Q^* = \frac{D}{\beta}(e^{\beta t_r^*} - 1) + W + Dt_s e^{-r(t_o^*+t_s^*)},$$

and

$$\mathrm{NPV}(t_r^*, t_s^*) = \frac{D}{r}[c + (c_s - rc)t_s^*]. \tag{18}$$

For the special circumstance that $t_r^* = 0$ in Theorem 1(b), the model reduces to the one-warehouse inventory problem. Let $c_{hr} = c_{ho}$, $\beta = \alpha$ and $W = 0$, we can obtain the objective function from Eq. (3). Then, the optimal solution of the one-warehouse inventory problem can be solved by using the similar arguments.

Next, we want to compare the decision using the net present value with one using the average total cost. Let ATC(t_r,t_s) be the average total cost, then we have

$$\mathrm{ATC}(t_r, t_s) = \frac{\mathrm{TC}(t_r, t_s)}{t_o + t_s}. \tag{19}$$

Solving the necessary conditions: $\partial\mathrm{ATC}(t_r,t_s)/\partial t_r = 0$ and $\partial\mathrm{ATC}(t_r,t_s)/\partial t_s = 0$ for the minimum value of ATC(t_r,t_s), we get

$$\frac{c_s - rc}{r}(1 - e^{-rt_s}) = K(t_r), \tag{20}$$

and

$$\begin{aligned} D(t_o + t_s)e^{-r(t_o+t_s)}[c + (c_s - rc)t_s] = {} & A + c\left[\frac{D}{\beta}(e^{\beta t_r} - 1) + W + Dt_s e^{-r(t_o+t_s)}\right] \\ & + \frac{c_{hr}D}{r\beta(r+\beta)}\left[re^{\beta t_r} + \beta e^{-rt_r} - (r+\beta)\right] + \frac{c_{ho}W}{r+\alpha} + \frac{c_{ho}D}{r(r+\alpha)}(e^{-rt_o} - e^{-rt_r}) \\ & + \frac{c_s D}{r^2}e^{-r(t_o+t_s)}(e^{rt_s} - rt_s - 1). \end{aligned} \tag{21}$$

It is obvious that Eq. (20) is the same as Eq. (13). By using the similar arguments as the previous section, if Eq. (20) holds, then we have $(c_s - rc)/r > K(0)$ and t_s is a function of t_r, where $\hat{t}_r \in (0, \infty)$ and satisfies $K(\hat{t}_r) = (c_s - rc)/r$. Once we get the optimal value $t_r^{**} \in [0, \hat{t}_r)$, the optimal solutions of t_o and t_s (denoted by t_o^{**} and t_s^{**}, respectively) can be uniquely determined. Next, motivated by Eq. (21), we let

$$\begin{aligned} Z(t_r) = {} & D(t_o + t_s)e^{-r(t_o+t_s)}[c + (c_s - rc)t_s] - \left\{A + c\left[\frac{D}{\beta}(e^{\beta t_r} - 1) + W + Dt_s e^{-r(t_o+t_s)}\right]\right. \\ & + \frac{c_{hr}D}{r\beta(r+\beta)}\left[re^{\beta t_r} + \beta e^{-rt_r} - (r+\beta)\right] + \frac{c_{ho}D}{r(r+\alpha)}(e^{-rt_o} - e^{-rt_r}) \\ & \left. + \frac{c_{ho}W}{r+\alpha} + \frac{c_s D}{r^2}e^{-r(t_o+t_s)}(e^{rt_s} - rt_s - 1)\right\}. \end{aligned} \tag{22}$$

Because $1 - e^{-r(t_o+t_s)} = [e^{r(t_o+t_s)} - 1]e^{-r(t_o+t_s)} > r(t_o + t_s)e^{-r(t_o+t_s)}$, we obtain that $G(t_r) > rZ(t_r)$ for all $t_r \geq 0$. Therefore, if there exists a value t_r^{**} such that $Z(t_r^{**}) = 0$, t_r^{**} must be larger than the value t_r^* such $G(t_r^*) = 0$. Summarize the above arguments, we have the following result.

Proposition 1. *If the solution of $t_r^{**} \in [0, \hat{t}_r)$ which satisfies $Z(t_r) = 0$ exists, then $t_r^{**} > t_r^*$.*

From Proposition 1, if t_r^{**} exists, then it is easy to see $t_o^{**} > t_o^*$ and $t_s^{**} > t_s^*$. That is, the length of the inventory cycle based on the average cost is longer than one based on NPV.

4. Numerical example

To illustrate the above results, we consider the same example in Yang [9]: $D = 400$, $W = 100$, $A = 100$, $c = 10$, $c_{ho} = 0.2$, $c_{hr} = 0.5$, $c_s = 2$, $\alpha = 0.02$, $\beta = 0.05$, $r = 0.06$ in appropriate units. We first portray the functions $G(t_r)$ and $rZ(t_r)$ in Fig. 2. We can find that $G(t_r) > rZ(t_r)$ for $t_r \geqslant 0$. Besides, the value t_r^{**} such that $Z(t_r^{**}) = 0$ does not exist uniquely and all $t_r^{**} > t_r^*$. Therefore, it is important to take right choices of different initial values when we search the root of the equation $Z(t_r) = 0$ by using Newton–Raphson Method. Then, the numerical results for NPV(t_r,t_s) and ATC(t_r,t_s) are shown in Table 1. It is clear that the reorder interval based on the average total cost is longer than that derived using NPV, i.e. $t_o^{**} + t_s^{**} > t_o^* + t_s^*$.

By using the key parameters as Yang's [9, Table 3], we perform a sensitivity analysis on NPV(t_r^*, t_s^*) with respect to each of the parameters r, W, c_s, A, c_{ho}, c_{hr}, c, α and β by assuming the rest are fixed. We first let

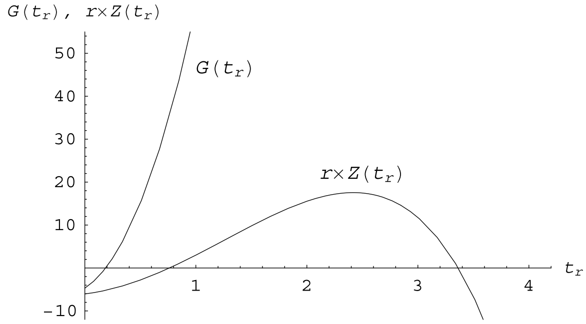

Fig. 2. Graphical representation of $G(t_r)$ and $rZ(t_r)$.

Table 1
Numerical results for $NPV(t_r,t_s)$ and $ATC(t_r,t_s)$

$NPV(t_r,t_s)$	t_r^*	t_o^*	t_s^*	Q^*	$NPV(t_r^*,t_s^*)$	
	0.1875	0.4359	0.4052	337.4	70447.6	
$ATC(t_r,t_s)$	t_r^{**}	t_o^{**}	t_s^{**}	Q^{**}	$ATC(t_r^{**},t_s^{**})$	
	0.7619	1.0075	1.1417	867.3	4078.1	
	3.3637	3.5969	5.8132	3890.6	4125.3	(saddle point)

Table 2
Sensitivity analysis on $NPV(t_r^*,t_s^*)$

r	Δ	$G(0)$	$NPV(t_r^*,t_s^*)$	W	Δ	$G(0)$	$NPV(t_r^*,t_s^*)$	c_s	Δ	$G(0)$	$NPV(t_r^*,t_s^*)$
0.02	89.85	−1.80	210632.0	20	23.28	−5.95	70706.1	0.2	−6.92	–	–
0.04	39.80	−3.40	105551.0	30	23.26	−5.88	70665.3	0.5	−1.92	–	–
0.08	14.70	−5.55	52838.6	40	23.23	−5.79	70627.0	1	6.41	−3.29	69244.9
0.10	9.65	−5.67	42224.7	50	23.21	−5.68	70591.1	2	23.08	−4.70	70447.6
A				c_{ho}				c_{hr}			
60	23.08	−2.30	69541.7	0.5	23.0	−4.09	70627.6	1	23.08	−4.70	70504.8
80	23.08	−3.50	70020.6	1	22.9	−2.86	70891.8	2.5	23.08	−4.70	70590.7
100	23.08	−4.70	70447.6	1.5	22.8	−1.34	71107.1	5	23.08	−4.70	70646.0
c				α				β			
5	28.18	−5.39	36817.8	0.01	23.11	−4.88	70382.4	0.05	23.08	−4.70	70447.6
10	23.08	−4.70	70447.6	0.02	23.08	−4.70	70447.6	0.10	23.08	−4.70	70505.3
15	17.98	−3.57	103789.0	0.05	23.01	−4.10	70631.8	0.25	23.08	−4.70	70591.4

$\Delta = (c_s - rc)/r - K(0)$, and then the result is presented in Table 2. Note that as $c_s = \{0.2, 0.5\}$, the nonnegative solution of (t_r,t_s) which satisfies Eq. (13) does not exist. We know from Table 2 that $NPV(t_r^*,t_s^*)$ increases as any of the parameters c_s, A, c_{ho}, c_{hr}, c, α or β increases. However, $NPV(t_r^*,t_s^*)$ decreases as r or W increases.

5. Concluding remarks

In this paper, an inventory model is developed for deteriorating items with two levels of storage, permitting shortage and complete backlogging. In particular, we use the NPV of total cost as the objective function for the generalized inventory system. The analytical formulations of the problem on the general framework described have been given. The condition which guarantees the unique solution exists is obtained and the complete proof of corresponding second-order sufficient conditions for optimum is also provided. Following the previous research dealing with the two-warehouse inventory problem, due to additional cost in RW, for example, maintenance, material handling, etc. we assume that the unit inventory costs (including holding cost and deterioration cost) per unit time in RW are higher than those in OW, i.e. $c_{hr} + \beta c > c_{ho} + \alpha c$. A intuitively managerial implication of the assumption follows: it will be economical to consume the goods of RW at the earliest. However, this does not mean that the firm always takes time to search a preserving facility with a lower deterioration rate than that in OW. Hence, the condition, $D > \alpha W$, is more suitable than the assumption of deterioration rates in RW and OW described by Benkherouf [1] and Yang [9]. In addition, when the discount rate r is small, we have $1 - \mathrm{e}^{-r(t_o+t_s)} = \mathrm{e}^{-r(t_o+t_s)}[\mathrm{e}^{r(t_o+t_s)} - 1] \approx r(t_o + t_s)\mathrm{e}^{-r(t_o+t_s)}$, which implies $G(t_r) \approx rZ(t_r)$. Hence, the optimal solution based on average total cost will be a good approximation to the one based on NPV. Our theoretical results could be some complements to previous research. For example, the conditions to find the optimal solution in Yang [9] can be derived by similar methods as our Lemmas 2 and 3. Furthermore, Proposition 1 discloses the reason why the reorder interval based on the average total cost could be longer than that derived using NPV.

The proposed model can be extended in several ways. Firstly, we can easily extend the backlogging rate of unsatisfied demand to any decreasing function $\beta(x)$, where x is the waiting time up to the next replenishment, and $0 \leqslant \beta(x) \leqslant 1$ with $\beta(0) = 1$. Secondly, we can also incorporate the quantity discount, and the learning curve phenomenon into the model.

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Appendix A. The proof of Lemma 1

It is obvious that $K(t_r)$ is a continuous function of $t_r \in [0,\infty)$ because $K(t_r)$ is a polynomial function of t_r in the interval $[0,\infty)$. Next, taking the derivative of $K(t_r)$ with respect to t_r, we have

$$\frac{\mathrm{d}K(t_r)}{\mathrm{d}t_r} = [c_{hr} + (r+\beta)c]\frac{\alpha W \mathrm{e}^{-\alpha t_r}}{D}\mathrm{e}^{\beta t_r + r t_o}\left\{\frac{D\mathrm{e}^{\alpha t_r}}{\alpha W} - \frac{\alpha - \beta}{r+\beta}[1 - \mathrm{e}^{-(r+\beta)t_r}]\right\} + [(c_{hr} + \beta c) - (c_{ho} + \alpha c)]\frac{\alpha W \mathrm{e}^{-\alpha t_r}}{D}\mathrm{e}^{r(t_o - t_r)}.$$

Let

$$H(t_r) = \frac{D\mathrm{e}^{\alpha t_r}}{\alpha W} - \frac{\alpha - \beta}{r+\beta}[1 - \mathrm{e}^{-(r+\beta)t_r}], \quad t_r \geqslant 0,$$

thus

$$\frac{\mathrm{d}H(t_r)}{\mathrm{d}t_r} = \frac{D\mathrm{e}^{\alpha t_r}}{W} - (\alpha - \beta)\mathrm{e}^{-(r+\beta)t_r} > \frac{D}{W} - \alpha\mathrm{e}^{-(r+\beta)t_r} = \alpha\left[\frac{D}{\alpha W} - \mathrm{e}^{-(r+\beta)t_r}\right] > \alpha\left(\frac{D}{\alpha W} - 1\right).$$

If $D > \alpha W$, then we know $\mathrm{d}H(t_r)/\mathrm{d}t_r > 0$. Therefore, $H(t_r)$ is a strictly increasing function in the interval $[0,\infty)$, which implies

$$H(t_r) > H(0) = \frac{D}{\alpha W} > 0, \quad \text{for } t_r > 0.$$

Then, from the above result and Assumption 5, we know that $\mathrm{d}K(t_r)/\mathrm{d}t_r > 0$, for $t_r > 0$. Therefore, $K(t_r)$ is a strictly increasing function in the interval $[0,\infty)$. The fact that $K(0) = \frac{c_{ho} + (r+\alpha)c}{r+\alpha}[(1 + \frac{\alpha W}{D})^{\frac{r}{\alpha}+1} - 1]$ and $\lim_{t_r \to \infty} K(t_r) = \infty$ are trivial. This completes the proof. □

Appendix B. The proof of Lemma 2

If $K(0) \geqslant (c_s - rc)/r$, then $K(0) > (1 - \mathrm{e}^{-rt_s})(c_s - rc)/r$ for $t_s \in [0,\infty)$. On the other hand, from Lemma 1, we have $K(t_r)$ is a strictly increasing function of $t_r \in [0,\infty)$. Thus it can not be found a value of t_r in the interval $[0,\infty)$ such that $K(t_r) = (1 - \mathrm{e}^{-rt_s})(c_s - rc)/r$. This completes the proof. □

Appendix C. The proof of Lemma 3

(a) First, we consider $G(0) < 0$. Since $G(t_r)$ is a strictly increasing function in the interval $[0, \hat{t}_r)$, and $\lim_{t_r \to \hat{t}_r^-} G(t_r) = \infty$, by using the Intermediate Value Theorem, there exists a unique solution $t_r^* \in (0, \hat{t}_r)$ such that $G(t_r^*) = 0$, i.e., t_r^* is the unique solution which satisfies Eq. (14). Next, if $G(0) = 0$, then from the property that $G(t_r)$ is a strictly increasing in the interval $[0, \hat{t}_r)$, we see that $t_r^* = 0$ is the unique value which satisfies $G(t_r^*) = 0$. In the case, the inventory system reduces to the one-warehouse problem.

(b) From the property that $G(t_r)$ is a strictly increasing in the interval $[0, \hat{t}_r)$, if $G(0) > 0$, then we have $G(t_r) > 0$ for all $t_r \in [0, \hat{t}_r)$. Thus, we can not find a value $t_r^* \in [0, \hat{t}_r)$ such that $G(t_r^*) = 0$. This completes the proof. □

Appendix D. The proof of Theorem 1

(a) For $G(0) < 0$, since $\mathrm{NPV}(t_r, t_s) = \mathrm{TC}(t_r, t_s)/[1 - e^{-r(t_o+t_s)}]$, we know that the necessary conditions for minimum are

$$\frac{\partial \mathrm{NPV}(t_r,t_s)}{\partial t_r} = \frac{dt_o}{dt_r}\left\{\frac{-re^{-r(t_o+t_s)}}{[1-e^{-r(t_o+t_s)}]^2}\mathrm{TC}(t_r,t_s) + \frac{1}{1-e^{-r(t_o+t_s)}}\frac{1}{dt_o/dt_r}\frac{\partial \mathrm{TC}(t_r,t_s)}{\partial t_r}\right\} = 0,$$

and

$$\frac{\partial \mathrm{NPV}(t_r,t_s)}{\partial t_s} = \frac{-re^{-r(t_o+t_s)}}{[1-e^{-r(t_o+t_s)}]^2}\mathrm{TC}(t_r,t_s) + \frac{1}{1-e^{-r(t_o+t_s)}}\frac{\partial \mathrm{TC}(t_r,t_s)}{\partial t_s} = 0.$$

It implies that

$$\frac{1}{dt_o/dt_r}\frac{\partial \mathrm{TC}(t_r,t_s)}{\partial t_r}\bigg|_{(t_r,t_s)=(t_r^*,t_s^*)} = \frac{\partial \mathrm{TC}(t_r,t_s)}{\partial t_s}\bigg|_{(t_r,t_s)=(t_r^*,t_s^*)} = \frac{re^{-r(t_o^*+t_s^*)}}{1-e^{-r(t_o^*+t_s^*)}}\mathrm{TC}(t_r^*,t_s^*),$$

where t_o^* is defined as in Eq. (16). From Lemma 3(a), the solution $t_r^* \in (0, \hat{t}_r)$ which satisfies Eq. (14) not only exists but also is unique. Hence, the value t_s^* can be uniquely determined by Eq. (16). Furthermore, we can obtain

$$\frac{\partial^2 \mathrm{NPV}(t_r,t_s)}{\partial t_r^2}\bigg|_{(t_r,t_s)=(t_r^*,t_s^*)} = \frac{1}{1-e^{-r(t_o+t_s)}}\left[\left(r\frac{dt_o}{dt_r} - \frac{1}{dt_o/dt_r}\frac{d^2t_o}{dt_r^2}\right)\frac{\partial \mathrm{TC}(t_r,t_s)}{\partial t_r} + \frac{\partial^2 \mathrm{TC}(t_r,t_s)}{\partial t_r^2}\right]\bigg|_{(t_r,t_s)=(t_r^*,t_s^*)}$$

$$= \frac{1}{1-e^{-r(t_o+t_s)}}\left[\left(r - \frac{\alpha^2 We^{-\alpha t_r}}{D}\right)\frac{dt_o}{dt_r}\frac{\partial \mathrm{TC}(t_r,t_s)}{\partial t_r} + \frac{\partial^2 \mathrm{TC}(t_r,t_s)}{\partial t_r^2}\right]\bigg|_{(t_r,t_s)=(t_r^*,t_s^*)}$$

$$= \frac{De^{-rt_o}}{1-e^{-r(t_o+t_s)}}\frac{dt_o}{dt_r}\frac{dK(t_r)}{dt_r}\bigg|_{(t_r,t_s)(t_r^*,t_s^*)} = D(c_s - rc)\frac{e^{-r(t_o+t_s)}}{1-e^{-r(t_o+t_s)}}\frac{dt_o}{dt_r}\frac{dt_s}{dt_r}\bigg|_{(t_r,t_s)=(t_r^*,t_s^*)} > 0,$$

$$\frac{\partial^2 \mathrm{NPV}(t_r,t_s)}{\partial t_r^2}\bigg|_{(t_r,t_s)=(t_r^*,t_s^*)} = \frac{1}{1-e^{-r(t_o+t_s)}}\left(r\frac{\partial TC(t_r,t_s)}{\partial t_s} + \frac{\partial^2 \mathrm{TC}(t_r,t_s)}{\partial t_s^2}\right)\bigg|_{(t_r,t_s)=(t_r^*,t_s^*)}$$

$$= D(c_s - rc)\frac{e^{-r(t_o^*+t_s^*)}}{1-e^{-r(t_o^*+t_s^*)}} > 0,$$

and

$$\frac{\partial^2 \mathrm{NPV}(t_r,t_s)}{\partial t_r \partial t_s}\bigg|_{(t_r,t_s)=(t_r^*,t_s^*)} = \frac{1}{1-e^{-r(t_o+t_s)}}\left(r\frac{\partial TC(t_r,t_s)}{\partial t_s}\frac{dt_o}{dt_r} + \frac{\partial^2 \mathrm{TC}(t_r,t_s)}{\partial t_s \partial t_r}\right)\bigg|_{(t_r,t_s)=(t_r^*,t_s^*)}$$

$$= \frac{1}{1-e^{-r(t_o+t_s)}}\left\{rDe^{-r(t_o+t_s)}[c + (c_s - rc)t_s]\frac{dt_o}{dt_r} - rDe^{-r(t_o+t_s)}[c + (c_s - rc)t_s]\frac{dt_o}{dt_r}\right\}\bigg|_{(t_r,t_s)=(t_r^*,t_s^*)} = 0$$

Thus, the determinant of Hessian matrix **H** at the stationary point (t_r^*, t_s^*) is

$$\det(\mathbf{H}) = \left.\frac{\partial^2 \mathrm{NPV}(t_r,t_s)}{\partial t_r^2}\right|_{(t_r,t_s)=(t_r^*,t_s^*)} \times \left.\frac{\partial^2 \mathrm{NPV}(t_r,t_s)}{\partial t_s^2}\right|_{(t_r,t_s)=(t_r^*,t_s^*)} - \left[\left.\frac{\partial^2 \mathrm{NPV}(t_r,t_s)}{\partial t_r \partial t_s}\right|_{(t_r,t_s)=(t_r^*,t_s^*)}\right]^2$$
$$= \left.\frac{D(c_s - rc)\mathrm{e}^{-r(t_o+t_s)}}{1-\mathrm{e}^{-r(t_o+t_s)}}\frac{\mathrm{d}t_o}{\mathrm{d}t_r}\frac{\mathrm{d}t_s}{\mathrm{d}t_r}\right|_{(t_r,t_s)=(t_r^*,t_s^*)} \times \left.\frac{D(c_s - rc)\mathrm{e}^{-r(t_o+t_s)}}{1-\mathrm{e}^{-r(t_o+t_s)}}\right|_{(t_r,t_s)=(t_r^*,t_s^*)}$$
$$= \left[\frac{D(c_s - rc)}{\mathrm{e}^{r(t_o+t_s)}-1}\right]^2 \left.\frac{\mathrm{d}t_o}{\mathrm{d}t_r}\frac{\mathrm{d}t_s}{\mathrm{d}t_r}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} > 0.$$

Hence, the Hessian matrix **H** at point (t_r^*, t_s^*) is positive definite. Consequently, we can conclude that the stationary point for our optimization problem is a global minimum point.

(b) For $G(0) = 0$, form the proof of Lemma 3(a), we see that $t_r^* = 0$ is the unique solution which satisfies $G(t_r^*) = 0$. For $G(0) > 0$, by Eqs. (10), (13) and (17), Eq. (6) becomes

$$\frac{\partial \mathrm{NPV}(t_r,t_s)}{\partial t_r} = \frac{\mathrm{d}t_o}{\mathrm{d}t_r}\left\{\frac{-r\mathrm{e}^{-r(t_o+t_s)}}{[1-\mathrm{e}^{-r(t_o+t_s)}]^2}TC(t_r,t_s) + \frac{1}{1-\mathrm{e}^{-r(t_o+t_s)}}\frac{1}{\mathrm{d}t_o/\mathrm{d}t_r}\frac{\partial \mathrm{TC}(t_r,t_s)}{\partial t_r}\right\}$$
$$= \frac{\mathrm{e}^{-r(t_o+t_s)}}{[1-\mathrm{e}^{-r(t_o+t_s)}]^2}G(t_r)\frac{\mathrm{d}t_o}{\mathrm{d}t_r}.$$

Because $\mathrm{d}t_o/\mathrm{d}t_r > 0$ and $G(t_r)$ is a strictly increasing function, we have $\partial\mathrm{NPV}(t_r,t_s)/\partial t_r > 0$ for any $t_r \in (0, \hat{t}_r)$, which implies that for any fixed $t_s \in [0,\infty)$, a smaller value of t_r causes a lower value of $\mathrm{NPV}(t_r,t_s)$. As a result, the minimum value of $\mathrm{NPV}(t_r,t_s)$ occurs at the boundary point $t_r^* = 0$. For the special circumstance that $t_r^* = 0$, since the RW is not used, the model reduces to the one-warehouse inventory problem. This completes the proof. □

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