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Selecting the Population Most Close to a Control via Empirical Bayes Approach

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This article deals with the problem of selecting the population most equivalent to a control from among k independent normal populations using the parametric empirical Bayes approach. By combining useful information from the past data, an empirical Bayes selection procedure P_n^* is studied. It is proved that the regret of P_n^* converges to zero at a rate $O(\frac{\ln n}{n})$, where n is the number of past observations at hand. A simulation study is carried out to investigate the performance of P_n^* for small to moderate values of n.

Keywords Asymptotically optimal; Bayes selection procedure; Empirical Bayes; Equivalent to a control; Rate of convergence; Regret.

Mathematics Subject Classification Primary 62F07; Secondary 62C12, 62C10.

1. Introduction

Problems of selecting populations equivalent to a control arise frequently in many applications. For example, in the industrial manufacturing process, suppose there are k different methods of producing certain product. For a specific characteristic of an item, it is required that its measurement should be in certain specification limits (a control). Since the procedure of producing an item is complicated, the measurement of this characteristic is thus a random variable that involves several factors. We are mainly interested in selecting certain way of producing so that its associated mean of the specific characteristic is the best fit to the specification limits. For related applications, it is referred to Romano (1977) for instance. In a toxicological study, as described by Wellek and Michaelis (1991), such a selection problem arises in drug clinical trials and bioavailability trials. Consider k different ways of producing drug for certain symptom and we need only to develop one of them. From a medical study, a response of some characteristic should reach a certain quantity (a control) so that the symptom can be removed.

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However, if the response is more than that quantity, there will be some side effect. Under this situation we need to choose certain one from those k different ways so that the response from body using this drug is the most close to this quantity.

In the literature, Gupta and Singh (1979) and Gupta and Hsiao (1981) proposed Bayes, Γ-minimax, and minimax procedures for selecting populations close to a control. Mee et al. (1987) developed multiple testing procedures to compare the means of k normal populations with respect to a control. Giani and Strassburger (1994) studied testing and selection procedures for equivalence of k populations with respect to a control. Dunnett and Gent (1977) developed tests to establish equivalence between treatments. Lakshminarayanan et al. (1994) studied multi-stage test procedures for testing Blackwelder's hypothesis of equivalence. Chen et al. (1993) derived range tests for the dispersion of several location parameters. Chen and Chen (1999) investigated a range test for the equivalence of means under unequal variances. Liang (1997, 2006) derived empirical Bayes procedures for selecting populations close to a control.

Consider k independent normal populations $\pi_1, \ldots, \pi_k$, where π_i has an unknown mean θ_i and an unknown variance σ_i^2, $i = 1, \ldots, k$. For a fixed known control level θ_0, let $\delta_i = (\theta_i - \theta_0)^2$ denote the distance between π_i and the control θ_0. For a given constant $\delta_0 > 0$, π_i is said to be close to the control θ_0 if $\delta_i \leq \delta_0$, and otherwise if $\delta_i > \delta_0$. Also, let $\delta_{[1]} \leq \cdots \leq \delta_{[k]}$ denote the ordered values of $\delta_1, \ldots, \delta_k$. The exact pairing between the ordered and the unordered parameters is of course unknown. The population π_i with $\delta_i = \delta_{[1]}$ is called the population most close to the control θ_0 if it is close to π_0. It is our main interest to select the population most close to the control. If there is no such population, we select none.

In this article, we employ the parametric empirical Bayes approach for our problem and it is organized as follows. The framework of the selection problem is introduced in Sec. 2. A Bayes selection procedure is derived. By mimicking the behavior of the Bayes procedure, we propose an empirical Bayes selection procedure P_n^* in Sec. 3. The asymptotic optimality of P_n^* is studied in Sec. 4. It is shown that the regret of P_n^* converges to zero at a rate $O(\frac{\ln n}{n})$, where n is the number of past data available when the current selection problem is considered. A simulation study is carried out to investigate the performance of P_n^* for small to moderate values of n. The simulated results are reported in Sec. 5. A detailed proof of the asymptotic optimality of P_n^* are provided in Appendices A and B.

2. The Selection Problem and A Bayes Selection Procedure

Let $\Omega = \{\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_k) \mid -\infty < \theta_i < \infty, i = 1, \ldots, k\}$ be the parameter space. Let $\underset{\sim}{a} = (a_0, a_1, \ldots, a_k)$ be an action where $a_i = 0, 1$; $i = 0, 1, \ldots, k$, and $\sum_{i=0}^{k} a_i = 1$. For $a_i = 1$ $(i \neq 0)$, it means that π_i is selected as the population closest to and equivalent to the control θ_0. When $a_0 = 1$, it means that none is equivalent to the control and thus none is selected. We consider the following loss function:

$$L(\underset{\sim}{\theta}, \underset{\sim}{a}) = \sum_{i=0}^{k} a_i \delta_i - \min(\delta_{[1]}, \delta_0). \tag{2.1}$$

Since θ_0 is known, without loss of generality, we let $\theta_0 = 0$.

For each $i = 1, \ldots, k$, let $X_{i1}, \ldots, X_{im}$ be a sample of size m $(m \geq 2)$ from π_i and let Y_i denote its sample mean. For its simplicity, for given (θ_i, σ_i^2), let $h_i(y_i \mid \theta_i)$

denote the probability density function (pdf) of Y_i, i.e., $N(\theta_i, \frac{\sigma_i^2}{m})$. We are dealing with Bayes and empirical Bayes selection procedures regarding the parameters $\underset{\sim}{\theta}$. It suffices to consider selection procedures based on $\underset{\sim}{Y} = (Y_1, \ldots, Y_k)$, the sufficient statistics for $\underset{\sim}{\theta}$.

Consider that the parameter θ_i is a realization of a random variable Θ_i which has $N(\theta_0, \tau_i^2)$ as its prior distribution with unknown variance τ_i^2. The random variables $\Theta_1, \ldots, \Theta_k$ are assumed to be independent. Thus, Y_i has a marginal $N\big(\theta_0, \frac{\sigma_i^2}{m} + \tau_i^2\big)$ distribution and the corresponding marginal pdf is denoted by $h_i(y_i)$. Given $Y_i = y_i$, Θ_i follows a posterior $N\big((1-B_i)y_i, (1-B_i)\frac{\sigma_i^2}{m}\big)$ with $B_i = \frac{\sigma_i^2}{m} / \big(\frac{\sigma_i^2}{m} + \tau_i^2\big)$.

Let $\mathscr{Y}$ be the sample space of $\underset{\sim}{Y}$. A selection procedure $\underset{\sim}{p} = (p_0, p_1, \ldots, p_k)$ is a mapping defined on $\mathscr{Y}$ into the product space $[0, 1]^{k+1}$ such that for each $\underset{\sim}{y}$ in $\mathscr{Y}$, $\underset{\sim}{p}(\underset{\sim}{y}) = (p_0(\underset{\sim}{y}), \ldots, p_k(\underset{\sim}{y}))$, where $p_i(\underset{\sim}{y})$ is the probability of selecting π_i as the one closest to and equivalent to the control $\theta_0 = 0$; $p_0(\underset{\sim}{y})$ is the probability of selecting none, with $\sum_{i=0}^k p_i(\underset{\sim}{y}) = 1$. Under the error loss of (2.1), the Bayes risk of a selection procedure $\underset{\sim}{p}$ is accordingly,

$$
\begin{aligned}
R(\underset{\sim}{p}) &= \int_\Omega \int_{\mathscr{Y}} \left[\sum_{i=0}^k p_i(\underset{\sim}{y})\delta_i - \min(\delta_{[1]}, \delta_0) \right] h(\underset{\sim}{y} \mid \underset{\sim}{\theta}) dy \, d\Pi(\underset{\sim}{\theta}) \\
&= \int_{\mathscr{Y}} \left[p_0(\underset{\sim}{y})\delta_0 + \sum_{i=1}^k p_i(\underset{\sim}{y})\psi_i(y_i) \right] h(\underset{\sim}{y}) d\underset{\sim}{y} - C, \qquad (2.2)
\end{aligned}
$$

where $h(\underset{\sim}{y} \mid \underset{\sim}{\theta}) = \prod_{i=1}^k h_i(y_i \mid \theta_i)$, $h(\underset{\sim}{y}) = \prod_{i=1}^k h_i(y_i)$, $\Pi(\underset{\sim}{\theta}) = \prod_{i=1}^k \pi_i(\theta_i)$ where $\pi_i(\theta_i)$ denotes $N(\theta_0, \tau_i^2)$, $C = \int_\Omega \min(\delta_{[1]}, \delta_0) d\Pi(\underset{\sim}{\theta})$, and

$$
\psi_i(y_i) = E[\Theta_i^2 \mid Y_i = y_i] = (1-B_i)\frac{\sigma_i^2}{m} + (1-B_i)^2 y_i^2. \qquad (2.3)
$$

2.1. *A Bayes Selection Procedure*

For each $\underset{\sim}{y}$ in $\mathscr{Y}$, let

$$
I(\underset{\sim}{y}) = \left\{ i \mid \psi_i(y_i) = \min_{1 \le j \le k} \psi_j(y_j) \text{ and } \psi_i(y_i) \le \delta_0 \right\}.
$$

Define

$$
i_B \equiv i_B(\underset{\sim}{y}) = \begin{cases} \min\{j \mid j \in I(\underset{\sim}{y})\} & \text{if } I(\underset{\sim}{y}) \neq \phi, \\ 0 & \text{if } I(\underset{\sim}{y}) = \phi. \end{cases} \qquad (2.4)
$$

A Bayes selection procedure $P_B = (p_{B_0}, \ldots, p_{B_k})$ which minimizes the Bayes risks $R(\underset{\sim}{p})$ among all selection procedures can be obtained as follows:

For each $\underset{\sim}{y}$ in $\mathcal{Y}$ and $i = 0, 1, \ldots, k$,

$$p_{B_i}(\underset{\sim}{y}) = \begin{cases} 1 & \text{if } i = i_B, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

The minimum Bayes risk of this selection problem is:

$$R(P_B) = \int_{\mathcal{Y}} \left[\sum_{i=1}^{k} p_{B_i}(\underset{\sim}{y})\psi_i(y_i) + p_{B_0}(\underset{\sim}{y})\delta_0 \right] h(\underset{\sim}{y})d\underset{\sim}{y} - C. \tag{2.6}$$

3. The Proposed Empirical Bayes Selection Procedures

It is noted that the Bayes selection procedure P_B depends on the unknown parameters τ_i^2 and σ_i^2. It is thus impossible to implement the Bayes selection procedure P_B. In the empirical Bayes framework, it is assumed that certain past data are available when the present selection problem is considered. Let $X_{ij\ell}$, $j = 1, \ldots, m$, denote a sample of size m from π_i at stage ℓ, $\ell = 1, 2, \ldots$. It is assumed that conditioning on $(\theta_{i\ell}, \sigma_i^2)$, $X_{ij\ell}$, $j = 1, \ldots, m$, are *iid* $N(\theta_{i\ell}, \sigma_i^2)$ and $\theta_{i\ell}$ is a realization of a random variable $\Theta_{i\ell}$, which has $N(\theta_0, \tau_i^2)$ as its prior. For each $i = 1, \ldots, k$, we assume that $((X_{i1l}, \ldots, X_{iml}), \Theta_{i\ell}), l = 1, \ldots, n$, are independent. It is also assumed that $\Theta_{i\ell}$, $i = 1, \ldots, k$, $\ell = 1, 2, \ldots$ are mutually independent. For ease of notation, we consider the current stage as stage $n+1$ and denote $X_{ij\ n+1}$ by X_{ij}, $j = 1, \ldots, m$, $i = 1, \ldots, k$. Thus, $X_{ij\ell}$, $i = 1, \ldots, k$, $j = 1, \ldots, m$, $\ell = 1, \ldots, n$ are the past data. Denote $\theta_i = \theta_{i,n+1}$ as a realization of the current random variable $\Theta_{i,n+1}$, $i = 1, \ldots, k$ and let $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_k)$.

For each π_i, $i = 1, \ldots, k$, and $\ell = 1, \ldots, n$, denote $X_{i.\ell} = \frac{1}{m}\sum_{j=1}^{m} X_{ij\ell}$, $W_{i\ell} = \frac{1}{m-1}\sum_{j=1}^{m}(X_{ij\ell} - X_{i.\ell})^2$. Note that $X_{i.\ell}$ and $W_{i\ell}$ are mutually independent, $X_{i.\ell}$ has a marginal $N\big(\theta_0, \frac{\sigma_i^2}{m} + \tau_i^2\big)$ distribution, and $\frac{(m-1)W_{i\ell}}{\sigma_i^2}$ follows a $\chi^2_{(m-1)}$ distribution. Define $S_i(n) = \frac{1}{n}\sum_{\ell=1}^{n}(X_{i.\ell} - \theta_0)^2$, and $W_i(n) = \frac{1}{n}\sum_{\ell=1}^{n} W_{i\ell}$. Thus, $S_i(n)$ and $W_i(n)$ are independent, $\frac{nS_i(n)}{\tau_i^2 + \sigma_i^2/m} \sim \chi^2_{(n)}$, $\frac{(m-1)nW_i(n)}{\sigma_i^2} \sim \chi^2_{(n(m-1))}$. Thus, $E\ S_i(n) = \frac{\sigma_i^2}{m} + \tau_i^2$ and $E\ W_i(n) = \sigma_i^2$. So, we use $S_i(n)$ to estimate $\frac{\sigma_i^2}{m} + \tau_i^2$ and $W_i(n)$ to estimate σ_i^2. We may use $\frac{W_i(n)/m}{S_i(n)}$ to estimate $B_i = \frac{\sigma_i^2}{m}/\big(\frac{\sigma_i^2}{m} + \tau_i^2\big)$. Since $0 < B_i < 1$, we define $B_{in} = \min\big(\frac{W_i(n)/m}{S_i(n)}, 1\big)$ and use B_{in} as an estimator of B_i. Now, mimicking the form (2.3), we propose an estimator $\psi_{in}(y_i)$ for $\psi_i(y_i)$, where

$$\psi_{in}(y_i) = (1 - B_{in})\frac{W_i(n)}{m} + (1 - B_{in})^2 y_i^2. \tag{3.1}$$

Now, for each $\underset{\sim}{y}$ in $\mathcal{Y}$, define

$$I_n(\underset{\sim}{y}) = \Big\{ i \mid \psi_{in}(y_i) = \min_{1 \le j \le k} \psi_{jn}(y_j), \psi_{in}(y_i) \le \delta_0 \Big\}. \tag{3.2}$$

and

$$i_n^* \equiv i_n^*(\underset{\sim}{y}) = \begin{cases} \min I_n(\underset{\sim}{y}) & \text{if } I_n(\underset{\sim}{y}) \neq \phi, \\ 0 & \text{if } I_n(\underset{\sim}{y}) = \phi. \end{cases} \tag{3.3}$$

We propose an empirical Bayes selection procedure $P_n^* = (p_{n0}^*, p_{n1}^*, \ldots, p_{nk}^*)$ based on i_n^* as follows:

For each $\underset{\sim}{y}$ in $\mathcal{Y}$,

$$p_{ni}^*(\underset{\sim}{y}) = \begin{cases} 1 & \text{if } i = i_n^*, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

We note that P_n^* is an empirical Bayes selection procedure based on the past data $\{X_{ij\ell},\ i = 1, \ldots, k;\ j = 1, \ldots, m,\ \ell = 1, \ldots, n\}$ only through $\underset{\sim}{W}(n) = (W_1(n), \ldots, W_k(n))$ and $\underset{\sim}{S}(n) = (S_1(n), \ldots, S_k(n))$. Conditioning on $\underset{\sim}{W}(n)$ and $\underset{\sim}{S}(n)$, the conditional Bayes risk of P_n^* is

$$R(P_n^* \mid \underset{\sim}{W}(n), \underset{\sim}{S}(n)) = \int_{\mathcal{Y}} \left[\sum_{i=1}^{k} p_{ni}^*(\underset{\sim}{y})\psi_i(y_i) + p_{n0}^*(\underset{\sim}{y})\delta_0 \right] h(\underset{\sim}{y}) d\underset{\sim}{y} - C.$$

The (unconditional) Bayes risk of P_n^* is thus,

$$\begin{aligned} R(P_n^*) &= E_n R(P_n^* \mid \underset{\sim}{W}(n), \underset{\sim}{S}(n)) \\ &= \int_{\mathcal{Y}} \left[\sum_{i=1}^{k} E_n[p_{ni}^*(\underset{\sim}{y})]\psi_i(y_i) + E_n[p_{n0}^*(\underset{\sim}{y})\delta_0] \right] h(\underset{\sim}{y}) d\underset{\sim}{y} - C, \end{aligned} \tag{3.5}$$

where the expectation E_n is taken with respect to the probability measure generated by $(\underset{\sim}{W}(n), \underset{\sim}{S}(n))$.

4. Asymptotic Optimality

In this section, we study the asymptotic optimality of the empirical Bayes selection procedure P_n^*. For an empirical Bayes selection procedure P_n, let $R(P_n \mid \underset{\sim}{W}(n), \underset{\sim}{S}(n))$ and $R(P_n)$ denote its corresponding conditional and unconditional Bayes risks, respectively. Since $R(P_B)$ is the minimum Bayes risk, $R(P_n \mid \underset{\sim}{W}(n), \underset{\sim}{S}(n)) - R(P_B) \geq 0$ for all $(\underset{\sim}{W}(n), \underset{\sim}{S}(n))$ and n, therefore, $R(P_n) - R(P_B) \geq 0$ for all n. The non negative difference $R(P_n) - R(P_B)$, the regret of the selection procedure P_n, is usually used as a measure of performance of the selection procedure P_n. An empirical Bayes selection procedure P_n is said to be asymptotically optimal of order $O(\varepsilon_n)$ if $R(P_n) - R(P_B) = O(\varepsilon_n)$, where $\{\varepsilon_n\}$ is a sequence of positive numbers decreasing to zero.

From (2.5)–(2.6) and (3.3)–(3.5), the regret of P_n^* can be expressed as

$$\begin{aligned} &R(P_n^*) - R(P_B) \\ &= \int_{\mathcal{Y}} \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} P\{i_n^*(\underset{\sim}{y}) = i, i_B(\underset{\sim}{y}) = j\}[\psi_i(y_i) - \psi_j(y_j)] h(\underset{\sim}{y}) d\underset{\sim}{y} \end{aligned}$$

$$+\int_{\mathscr{Y}}\sum_{i=1}^{k}P\{i_n^*(\underset{\sim}{y})=i,i_B(\underset{\sim}{y})=0\}[\psi_i(y_i)-\delta_0]h(\underset{\sim}{y})d\underset{\sim}{y}$$

$$+\int_{\mathscr{Y}}\sum_{j=1}^{k}P\{i_n^*(\underset{\sim}{y})=0,i_B(\underset{\sim}{y})=j\}[\delta_0-\psi_j(y_j)]h(\underset{\sim}{y})d\underset{\sim}{y}. \tag{4.1}$$

Note that if $(1-B_i)\frac{\sigma_i^2}{m}\geq\delta_0$, then $\psi_i(y_i)\geq\delta_0$ for all y_i. In the following analysis, it is assumed that $(1-B_i)\frac{\sigma_i^2}{m}<\delta_0$ for each $i=1,\ldots,k$. Let $a_i=\sqrt{\delta_0-(1-B_i)\frac{\sigma_i^2}{m}/(1-B_i)}$. Thus, $\psi_i(y_i)<\delta_0$ if, and only if, $|y_i|<a_i$ and $\psi_i(a_i)=\delta_0$, and $\psi_i(y_i)>\delta_0$ if, and only if, $|y_i|>a_i$. Let $A_i=(-a_i,a_i)$ and $A_i^c=(-\infty,-a_i]\cup[a_i,\infty)$. Therefore, $i_B(\underset{\sim}{y})=0$ if, and only if, $|y_i|>a_i$ for all $i=1,\ldots,k$. Hence, by the definition of i_n^* and i_B, we have

$$\int_{\mathscr{Y}}P\{i_n^*(\underset{\sim}{y})=i,i_B(\underset{\sim}{y})=0\}[\psi_i(y_i)-\delta_0]h(\underset{\sim}{y})d\underset{\sim}{y}$$
$$\leq\int_{A_i^c}P\{\psi_{in}(y_i)<\delta_0\}[\psi_i(y_i)-\delta_0]h_i(y_i)dy_i. \tag{4.2}$$

and

$$\int_{\mathscr{Y}}P\{i_n^*(\underset{\sim}{y})=0,i_B(\underset{\sim}{y})=j\}[\delta_0-\psi_j(y_j)]h(\underset{\sim}{y})d\underset{\sim}{y}$$
$$\leq\int_{A_j}P\{\psi_{jn}(y_j)\geq\delta_0\}[\delta_0-\psi_j(y_j)]h_j(y_j)dy_j. \tag{4.3}$$

Define $A_{ij}=\{(y_i,y_j)\,|\,|y_j|\leq a_j,\psi_i(y_i)-\psi_j(y_j)>0\}$. For $(y_i,y_j)\in A_{ij}$, let $t(y_i,y_j)\equiv(\psi_i(y_i)-\psi_j(y_j))/6>0$. Thus, for each $\underset{\sim}{y}$ such that $i_B(\underset{\sim}{y})=j$ and $\psi_i(y_i)>\psi_j(y_j)$, the related (y_i,y_j) must be in A_{ij}. Hence,

$$\begin{aligned}P\{i_n^*(\underset{\sim}{y})=i,i_B(\underset{\sim}{y})=j\}&\leq P\{\psi_{in}(y_i)-\psi_{jn}(y_j)\leq 0\}\\&=P\{[\psi_{in}(y_i)-\psi_{jn}(y_j)]-[\psi_i(y_i)-\psi_j(y_j)]<-6t(y_i,y_j)\}\\&\leq P\{[\psi_{in}(y_i)-\psi_i(y_i)]<-3t(y_i,y_j)\}\\&\quad+P\{[\psi_{jn}(y_j)-\psi_j(y_j)]>3t(y_i,y_j)\}.\end{aligned}$$

Therefore,

$$\iint_{\mathscr{Y}}P\{i_n^*(\underset{\sim}{y})=i,i_B(\underset{\sim}{y})=j\}[\psi_i(y_i)-\psi_j(y_j)]h(\underset{\sim}{y})d\underset{\sim}{y}$$
$$\leq\iint_{A_{ij}}P\{[\psi_{in}(y_i)-\psi_i(y_i)]<-3t(y_i,y_j)\}6t(y_i,y_j)h_i(y_i)h_j(y_j)dy_idy_j$$
$$+\iint_{A_{ij}}P\{[\psi_{jn}(y_j)-\psi_j(y_j)]>3t(y_i,y_j)\}6t(y_i,y_j)h_i(y_i)h_j(y_j)dy_idy_j \tag{4.4}$$

Table 1
For $\delta_0 = 1$ under C_{31}

n	f_n	$\overline{D}_n$	$SE(\overline{D}_n)$
20	0.9933	3.781E-005	8.972E-004
	0.9959	1.460E-005	3.827E-004
40	0.9945	1.924E-005	4.746E-004
	0.9975	6.391E-006	2.171E-004
60	0.9947	1.190E-005	2.921E-004
	0.9976	5.463E-006	1.852E-004
80	0.9954	1.262E-005	3.046E-004
	0.9968	3.894E-006	1.641E-004
100	0.9968	7.961E-006	2.694E-004
	0.9982	1.921E-006	8.621E-005
200	0.9968	3.374E-006	9.399E-005
	0.9981	1.311E-006	5.263E-005
400	0.9980	1.476E-006	4.594E-005
	0.9990	7.785E-007	4.097E-005
600	0.9988	8.348E-007	3.067E-005
	0.9992	3.735E-007	1.996E-005
800	0.9990	1.962E-006	8.307E-005
	0.9992	2.411E-007	1.102E-005
1000	0.9990	5.809E-007	2.374E-005
	0.9989	7.796E-007	4.333E-005
1500	0.9991	5.367E-007	2.924E-005
	0.9994	1.052E-007	5.412E-006
2000	0.9996	2.810E-008	1.820E-006
	0.9992	3.858E-007	2.762E-005

Substituting the inequalities of (4.2)–(4.4) into (4.1), we obtain

$$
\begin{aligned}
&R(P_n^*) - R(P_B) \\
&\quad \le \sum\sum_{i\neq j} \iint_{A_{ij}} P\{[\psi_{in}(y_i) - \psi_i(y_i)] < -3t(y_i, y_j)\} 6t(y_i, y_j) h_i(y_i) h_j(y_j)\, dy_i\, dy_j \\
&\qquad + \sum\sum_{i\neq j} \iint_{A_{ij}} P\{[\psi_{jn}(y_j) - \psi_j(y_j)] > 3t(y_i, y_j)\} 6t(y_i, y_j) h_i(y_i) h_j(y_j)\, dy_i\, dy_j \\
&\qquad + \sum_{i=1}^{k} \int_{A_i^c} P\{\psi_{in}(y_i) < \delta_0\}[\psi_i(y_i) - \delta_0] h_i(y_i)\, dy_i \\
&\qquad + \sum_{j=1}^{k} \int_{A_j} P\{\psi_{jn}(y_j) \ge \delta_0\}[\delta_0 - \psi_j(y_j)] h_j(y_j)\, dy_j \\
&\quad = I + II + III + IV.
\end{aligned} \tag{4.5}
$$

Thus, to study the asymptotic optimality of P_n^*, it suffices to investigate the asymptotic behaviors of the four terms I, II, III, and IV. It is noted that the analysis of the four terms are similar, though the analysis of I and II is somewhat more complicated than that of III and IV. We thus provide only detailed analysis and discussion for the term of I in Appendix A. It follows then that

$$I = O\left(\frac{\ln n}{n}\right). \tag{4.6}$$

Following a similar analysis and discussion as that of I, we can also obtain the following results:

$$II = O\left(\frac{\ln n}{n}\right). \tag{4.7}$$

$$III = O\left(\frac{1}{n}\right). \tag{4.8}$$

$$IV = O\left(\frac{1}{n}\right). \tag{4.9}$$

We summarize the preceding results as follows.

Table 2
For $m = 30$ under C_{32}

n	f_n	$\overline{D}_n$	$SE(\overline{D}_n)$	U(C_{32}/C_{31})
20	0.9925	5.417E-005	9.930E-004	1.0008
	0.9926	1.075E-004	2.703E-003	
40	0.9936	3.868E-005	1.059E-003	1.0009
	0.9952	3.797E-005	9.063E-004	
60	0.9960	1.671E-005	3.893E-004	0.9987
	0.9957	4.191E-005	1.166E-003	
80	0.9966	1.338E-005	3.513E-004	0.9988
	0.9964	1.543E-005	3.787E-004	
100	0.9965	1.720E-005	3.861E-004	1.0003
	0.9972	1.236E-005	3.714E-004	
200	0.9968	1.038E-005	2.684E-004	1.0000
	0.9969	8.529E-006	2.577E-004	
400	0.9981	3.115E-006	1.047E-004	0.9999
	0.9985	3.840E-006	2.017E-004	
600	0.9988	1.514E-006	5.643E-005	1.0000
	0.9992	5.023E-007	2.449E-005	
800	0.9988	1.724E-006	5.214E-005	1.0002
	0.9987	2.844E-006	1.327E-004	
1000	0.9990	1.359E-006	5.001E-005	1.0000
	0.9989	7.924E-007	3.055E-005	
1500	0.9988	1.688E-006	6.934E-005	1.0003
	0.9989	3.712E-006	1.725E-004	
2000	0.9991	4.354E-007	2.169E-005	1.0005
	0.9989	1.185E-006	6.649E-005	

Theorem 4.1. *Suppose that* $(1 - B_i)\frac{\sigma_i^2}{m} < \delta_0$ *for each* $i = 1, \ldots, k$. *Let* P_n^* *be the empirical Bayes selection procedure constructed in Sec. 3. Then,* P_n^* *is asymptotically optimal and* $R(P_n^*) - R(P_B) = O\left(\frac{\ln n}{n}\right)$.

5. Simulation Study

In this study, we consider two cases, i.e., $k = 3$ and $k = 5$, taking $\theta_0 = 0$.

For $k = 3$, we consider two situations of its parameters:

$$C_{31} : \sigma_i^2 = 1, \tau_i^2 = i, i = 1, 2, 3$$
$$C_{32} : \sigma_i^2 = \tau_i^2 = i, i = 1, 2, 3.$$

And for $k = 5$, we also consider two situations:

$$C_{51} : \sigma_i^2 = 1, \tau_i^2 = i, i = 1, 2, 3, 4, 5$$
$$C_{52} : \sigma_i^2 = \tau_i^2 = i, i = 1, 2, 3, 4, 5.$$

Table 3
For $\delta_0 = 1$ under C_{51}

n	f_n	$\overline{D}_n$	$SE(\overline{D}_n)$
20	0.9894	2.483E-005	4.257E-004
	0.9939	2.094E-005	5.646E-004
40	0.9936	9.591E-006	2.666E-004
	0.9953	1.287E-005	4.316E-004
60	0.9946	7.470E-006	1.756E-004
	0.9967	3.119E-006	8.419E-005
80	0.9941	8.197E-006	1.656E-004
	0.9958	2.112E-006	4.773E-005
100	0.9945	4.131E-006	8.170E-005
	0.9972	1.617E-006	4.387E-005
200	0.9964	1.875E-006	4.726E-005
	0.9981	4.917E-007	1.449E-005
400	0.9966	2.531E-006	1.052E-004
	0.9982	4.529E-007	1.403E-005
600	0.9973	1.287E-006	4.177E-005
	0.9993	8.666E-008	3.530E-006
800	0.9979	6.933E-007	1.876E-005
	0.9980	6.030E-007	1.927E-005
1000	0.9982	4.342E-007	1.203E-005
	0.9995	1.173E-007	7.349E-006
1500	0.9991	1.675E-007	6.961E-006
	0.9990	3.583E-007	2.322E-005
2000	0.9984	3.806E-007	1.863E-005
	0.9997	5.335E-008	3.989E-006

Table 4
For $m = 30$ under C_{52}

n	f_n	$\overline{D}_n$	$SE(\overline{D}_n)$	$U(C_{52}/C_{51})$
20	0.9895	3.984E-005	6.990E-004	0.9998
	0.9900	3.240E-005	7.349E-004	
40	0.9922	1.061E-005	2.389E-004	1.0014
	0.9950	1.130E-005	2.800E-004	
60	0.9941	1.357E-005	3.678E-004	1.0005
	0.9956	1.429E-005	4.811E-004	
80	0.9927	1.102E-005	2.594E-004	1.0014
	0.9947	1.215E-005	5.328E-004	
100	0.9947	3.924E-006	8.562E-005	0.9997
	0.9954	7.469E-006	1.926E-004	
200	0.9968	5.200E-006	3.577E-004	0.9995
	0.9970	2.580E-006	7.573E-005	
400	0.9979	1.672E-006	6.554E-005	0.9986
	0.9980	1.145E-006	6.224E-005	
600	0.9976	1.507E-006	6.630E-005	0.9996
	0.9989	3.289E-007	1.092E-005	
800	0.9980	1.332E-006	4.983E-005	0.9998
	0.9984	2.967E-007	8.792E-006	
1000	0.9975	4.630E-007	1.242E-005	1.0007
	0.9986	1.217E-006	5.262E-005	
1500	0.9990	6.524E-007	3.588E-005	1.0001
	0.9986	3.282E-007	1.066E-005	
2000	0.9986	2.993E-007	1.073E-005	0.9997
	0.9987	3.253E-007	1.271E-005	

All simulations are repeated 10,000 times. For stage n, we denote f_n as the frequency of correct selection, $\overline{D}_n$ its corresponding average loss and $SE(\overline{D}_n)$ the standard deviation of $\overline{D}_n$, where the correct selection means the event that the Bayes rule P_B selects the same population which the empirical Bayes rule P_n^* does.

In Table 1, we take $\delta_0 = 1$ under the case of C_{31}. The upper and lower entries are respectively associated with $m = 30$ and $m = 50$, the sample size of each population for each stage. In Table 2, we consider $m = 30$, $\delta_0 = 1$ (upper entry) and $\delta_0 = 2$ (lower entry) under C_{32}.

To see the role of σ_i^2 in the frequency of correct selection in C_{31}, we define $U(C_{32}/C_{31}) = f_n(C_{31})/f_n(C_{32})$, where $f_n(C_{3i})$ denotes the frequency of correct selection under C_{3i} when $\delta_0 = 1$, $m = 30$ and $\sigma_i^2 = 1$, for $i = 1, 2$. The value of U is tabulated in the last column of Table 2.

In Table 3, we consider $\delta_0 = 1$, $m = 30$ (upper entry) and $m = 50$ (lower entry) under C_{51}. Values of $U(C_{52}/C_{51})$ are also tabulated for $\delta_0 = 1$ and $m = 30$. For Table 4, we take $m = 30$ and consider $\delta_0 = 2$ (upper entry) and $\delta_0 = 3$ (lower entry) under C_{52}.

To give some ideas how the expected loss behaves relating to stage number n, we define $r_{20}(n) = \frac{\overline{D}_{20} - \overline{D}_n}{\overline{D}_{20}}$, the percentage of the decrease of $\overline{D}_n$ with respect to

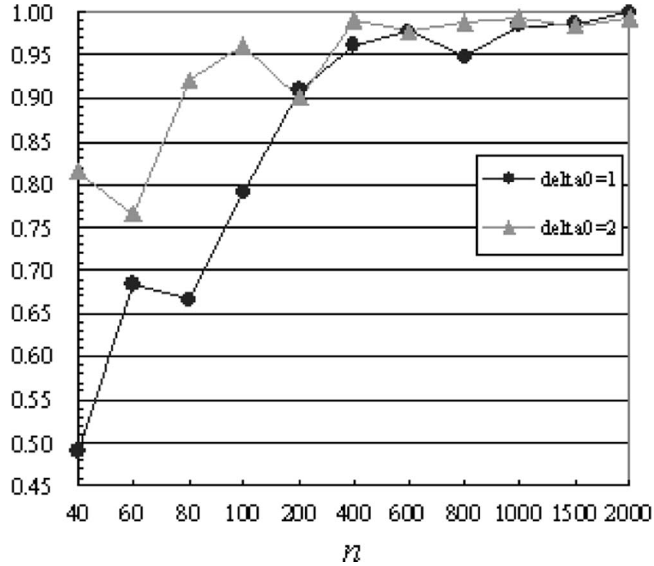

Figure 1. Plot of $r_{20}(n)$ under C_{31}.

$\overline{D}_{20}$with $n = 20$. Values of $r_{20}(n)$ are plotted in Figs. 1 and 2 for different values of δ_0 and different configurations. For some given δ_0 and configuration, to compare $r_{20}(n)$ for $m = 30$ to that of $m = 50$, we consider the ratio $R(n) = \frac{S_{30}(n)}{S_{50}(n)}$, $S_{30}(n)$ and $S_{50}(n)$ are, respectively, $r_{20}(n)$ for $m = 30$ and $m = 50$. We plot $R(n)$ respectively, for two different values of δ_0 under different configuration in Figs. 3 and 4. To see the variations of expected loss with respect to sample size m for given $n = 20$

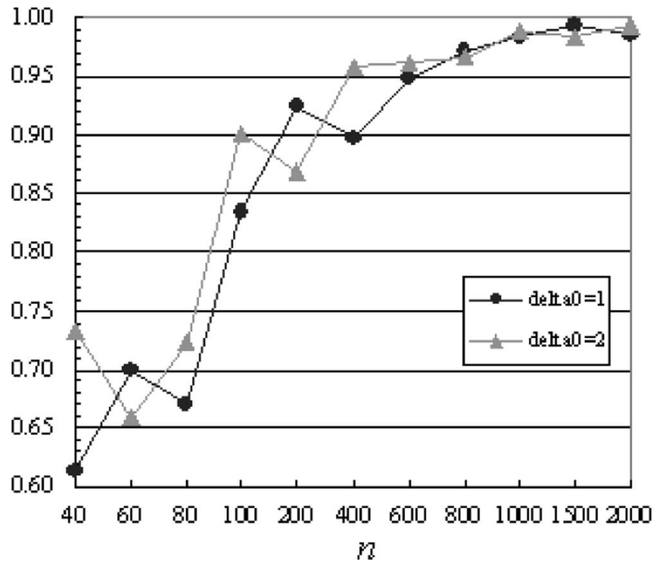

Figure 2. Plot of $r_{20}(n)$ under C_{51}.

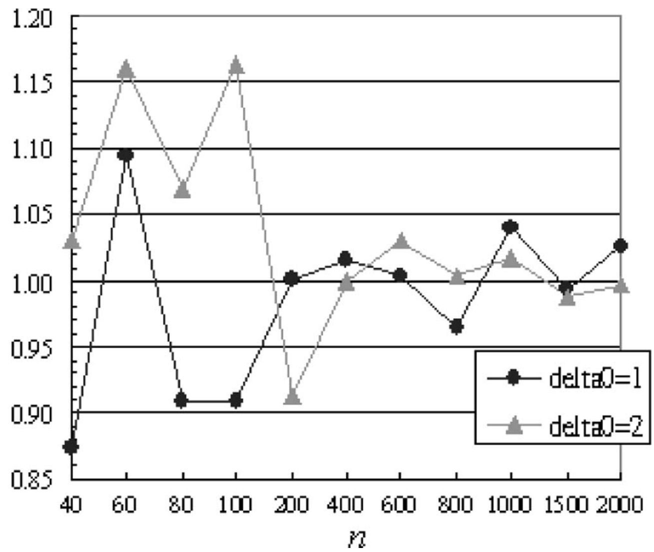

Figure 3. Plot of $R(n)$ under C_{31}.

and $n = 40$, respectively, we define $t_5(m) = (D_5 - D_m)/D_5$, where D_m denotes the expected loss $\overline{D}_n$ with sample size m under some given n. $t(m)$ are plotted in Figs. 5 and 6 for C_{31} and C_{51}, respectively.

It is easy to see that the expected loss becomes stable when $m \geq 35$ for both C_{31} and C_{51} and the frequency of correct selection seems stable, though it is slowly increasing in n, even for small n.

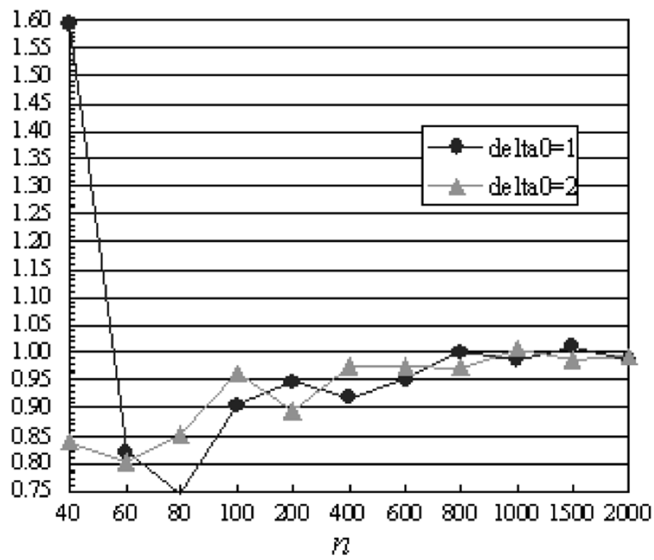

Figure 4. Plot of $R(n)$ under C_{51}.

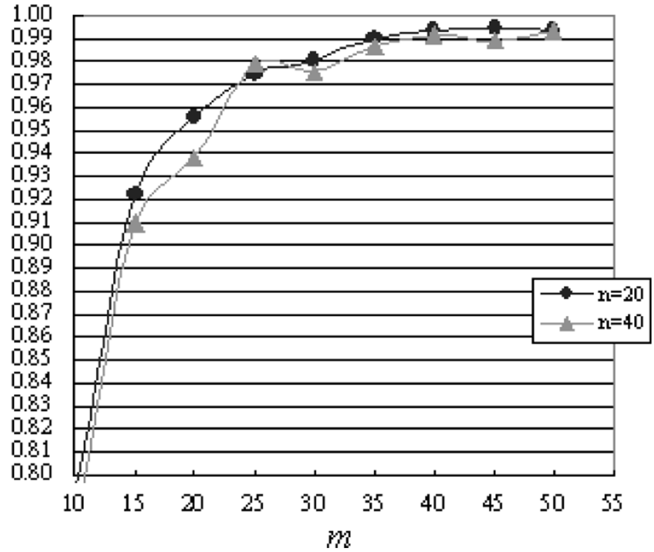

Figure 5. Plot of $t_5(m)$ under C_{31}.

To see the effect of σ_i^2 for the frequency of correct selection, we can see that values of $\mathrm{U}(C_{32}/C_{31})$ are stable and close to 1 for $\delta_0 = 1$ and $m = 30$ for both small and large values of n. So is the case of C_{52} against C_{51}.

To summarize, the expected loss depends heavily on n and also on δ_0, rather than m. It decreases steadily as n increases. However, the rate of decrease closely relates to m and δ_0.

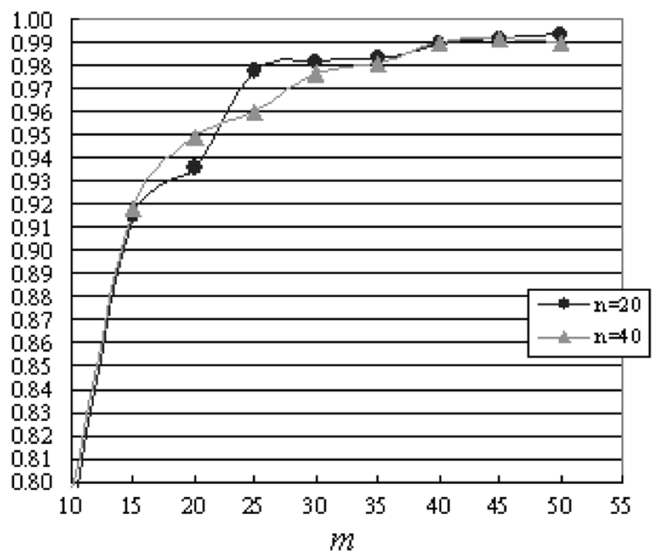

Figure 6. Plot of $t_5(m)$ under C_{51}.

Appendix A

Proof of $I = O(\frac{\ln n}{n})$. Note that $I = \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} I_{ij}$, where

$$I_{ij} = \iint_{A_{ij}} P\{[\psi_{in}(y_i) - \psi_i(y_i)] < -3t(y_i, y_j)\} 6t(y_i, y_j) h_i(y_i) h_j(y_j) dy_i \, dy_j. \tag{A.1}$$

By Lemma B.3 (see Appendix B), for each (y_i, y_j) in A_{ij}, and noting that $g_1(x) \equiv x + \ln(1-x)$ for $0 < x < 1$ and $g_2(x) \equiv x - \ln(1+x)$ for $x > 0$,

$$\begin{aligned}
&P\{\psi_{in}(y_i) - \psi_i(y_i) < -3t(y_i, y_j)\} \\
&\quad \leq \exp\left\{\frac{n(m-1)}{2} g_1\left(\frac{mt(y_i, y_j)}{\sigma_i^2}\right)\right\} I(\sigma_i^2 - mt(y_i, y_j)) \\
&\qquad + \exp\left\{\frac{-n(m-1)}{2} g_2\left(\frac{mt(y_i, y_j)}{2B_i\sigma_i^2}\right)\right\} I((1-B_i)\sigma_i^2 - mt(y_i, y_j)) \\
&\qquad + \exp\left\{\frac{n}{2} g_1\left(\frac{mt(y_i, y_j)}{2B_i\sigma_i^2 + 2mt(y_i, y_j)}\right)\right\} I((1-B_i)\sigma_i^2 - mt(y_i, y_j)) \\
&\qquad + \exp\left\{\frac{-n(m-1)}{2} g_2\left(\frac{t(y_i, y_j)}{4B_i y_i^2}\right)\right\} I(2(1-B_i)y_i^2 - t(y_i, y_j)) \\
&\qquad + \exp\left\{\frac{n}{2} g_1\left(\frac{t(y_i, y_j)}{4B_i y_i^2 + 2t(y_i, y_j)}\right)\right\} I(2(1-B_i)y_i^2 - t(y_i, y_j)).
\end{aligned} \tag{A.2}$$

Let

$$\begin{aligned}
A_{ij1} &= \{(y_i, y_j) \in A_{ij} \mid \sigma_i^2 - mt(y_i, y_j) > 0,\ \ y_i > 0,\ \ y_j > 0\}, \\
A_{ij2} &= \{(y_i, y_j) \in A_{ij} \mid (1-B_i)\sigma_i^2 - mt(y_i, y_j) > 0,\ \ y_i > 0,\ \ y_j > 0\}, \\
A_{ij3} &= \{(y_i, y_j) \in A_{ij} \mid 2(1-B_i)y_i^2 - t(y_i, y_j) > 0,\ \ y_i > 0,\ \ y_j > 0\}.
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\alpha_1(y_i, y_j, n) &= 24\exp\left\{\frac{n(m-1)}{2} g_1\left(\frac{mt(y_i, y_j)}{\sigma_i^2}\right)\right\} t(y_i, y_j) h_i(y_i) h_j(y_j), \\
\alpha_2(y_i, y_j, n) &= 24\exp\left\{\frac{-n(m-1)}{2} g_2\left(\frac{mt(y_i, y_j)}{2B_i\sigma_i^2}\right)\right\} t(y_i, y_j) h_i(y_i) h_j(y_j), \\
\alpha_3(y_i, y_j, n) &= 24\exp\left\{\frac{n}{2} g_1\left(\frac{mt(y_i, y_j)}{2B_i\sigma_i^2 + 2mt(y_i, y_j)}\right)\right\} t(y_i, y_j) h_i(y_i) h_j(y_j), \\
\alpha_4(y_i, y_j, n) &= 24\exp\left\{\frac{-n(m-1)}{2} g_2\left(\frac{t(y_i, y_j)}{4B_i y_i^2}\right)\right\} t(y_i, y_j) h_i(y_i) h_j(y_j), \\
\alpha_5(y_i, y_j, n) &= 24\exp\left\{\frac{n}{2} g_1\left(\frac{t(y_i, y_j)}{4B_i y_i^2 + 2t(y_i, y_j)}\right)\right\} t(y_i, y_j) h_i(y_i) h_j(y_j).
\end{aligned} \tag{A.4}$$

Substitute the inequality of (A.2) into (A.1). Note that $\psi_i(y_i)$ and $h_j(y_j)$ are symmetric about $\theta_0 = 0$. By this symmetry property, we obtain:

$$I_{ij} \leq \iint_{A_{ij1}} \alpha_1(y_i, y_j, n) dy_i \, dy_j + \iint_{A_{ij2}} \alpha_2(y_i, y_j, n) dy_i \, dy_j + \iint_{A_{ij2}} \alpha_3(y_i, y_j, n) dy_i \, dy_j$$

$$+ \iint_{A_{ij3}} \alpha_4(y_i, y_j, n) dy_i\, dy_j + \iint_{A_{ij3}} \alpha_5(y_i, y_j, n) dy_i\, dy_j$$
$$= J_1 + J_2 + J_3 + J_4 + J_5. \tag{A.5}$$

Let $S_{ij} = \{y_j \,|\, 0 \le y_j \le a_j,\ \psi_i(0) - \psi_j(y_j) > 0\}$ and

$$s_{ij} = \begin{cases} \max_{u \in S_{ij}} u & \text{if } S_{ij} \ne \phi, \\ 0 & \text{if } S_{ij} = \phi. \end{cases}$$

Note that $\psi_i(y_i)$ is increasing in y_i for $y_i \ge 0$.

Case 1. If $s_{ij} > 0$, then for each $y_i \ge 0$, $\psi_i(y_i) - \psi_j(\frac{s_{ij}}{2}) \ge \psi_i(0) - \psi_j(\frac{s_{ij}}{2}) \equiv c_2 > 0$.

Case 2. If $s_{ij} = 0$, then $\psi_i(0) - \psi_j(0) = (1 - B_i)\frac{\sigma_i^2}{m} - (1 - B_j)\frac{\sigma_j^2}{m} \equiv c_3 \le 0$.
In the following, we study the asymptotic behaviors of the five terms J_i, $i = 1, 2, 3, 4, 5$.

Case 1. If $s_{ij} > 0$.

Let $D_1 = \{(y_i, y_j) \in A_{ij1} \,|\, y_j \le \frac{s_{ij}}{2}\}$ and $C_1 = \{(y_i, y_j) \in A_{ij1} \,|\, y_j > \frac{s_{ij}}{2}\}$. Thus,

$$J_1 = \iint_{D_1} \alpha_1(y_i, y_j, n) dy_i\, dy_j + \iint_{C_1} \alpha_1(y_i, y_j, n) dy_i\, dy_j = J_{11} + J_{12}. \tag{A.6}$$

On D_1, $t(y_i, y_j) \ge t(0, \frac{s_{ij}}{2}) = \frac{c_2}{6} > 0$. Since $g_1(x)$ is decreasing in x for $0 < x < 1$, thus,

$$\begin{aligned} J_{11} &\le \iint_{D_1} 24 \exp\left(\frac{n(m-1)}{2} g_1\left(\frac{mc_2}{6\sigma_i^2}\right)\right) t(y_i, y_j) h_i(y_i) h_j(y_j) dy_i\, dy_j \\ &= \exp\left(\frac{n(m-1)}{2} g_1\left(\frac{mc_2}{6\sigma_i^2}\right)\right) \iint_{D_1} 24 t(y_i, y_j) h_i(y_i) h_j(y_j) dy_i\, dy_j \\ &\le 4 \exp\left(\frac{n(m-1)}{2} g_1\left(\frac{mc_2}{6\sigma_i^2}\right)\right) \tau^{*^2}, \end{aligned} \tag{A.7}$$

where the last inequality is obtained due to Lemma B.6 and where $\tau^{*2} = \max_i \tau_i^2$ and $\tau_i^2 = E\Theta_i^2$. Note that $g_1(\frac{mc_2}{6\sigma_i^2}) \le \frac{-1}{2}(\frac{mc_2}{6\sigma_i^2})^2 < 0$.

Again, note that $h_i(y_i) \le \ell_0$ for all y_i, for some positive value ℓ_0, and $i = 1, \ldots, k$. On C_1, $\left|\frac{\partial t(y_i, y_j)}{\partial y_j}\right| = \frac{2(1-B_j)^2 y_j}{6} \ge \frac{(1-B_j)^2 s_{ij}}{6} > 0$. By using the preceding inequality, changing the variable by letting $x = \frac{mt(y_i, y_j)}{\sigma_i^2}$ and by Lemma B.4, we obtain

$$\begin{aligned} J_{12} &= 24 \iint_{C_1} \exp\left\{\frac{n(m-1)}{2} g_1\left(\frac{mt(y_i, y_j)}{\sigma_i^2}\right)\right\} \frac{mt(y_i, y_j)}{\sigma_i^2} \left|\frac{\partial}{\partial y_j} \frac{mt(y_i, y_j)}{\sigma_i^2}\right| \frac{\sigma_i^4}{m^2} \\ &\quad \times \frac{3 h_j(y_j)}{(1 - B_j)^2 y_j} h_i(y_i) dy_i\, dy_j \\ &\le \frac{144 \sigma_i^4 \ell_0}{m^2 (1 - B_j)^2 s_{ij}} \int_{y_i} \int_{x=0}^{1} \exp\left(\frac{n(m-1)}{2} g_1(x)\right) x d\, x h_i(y_i) dy_i \\ &\le \frac{288 \sigma_i^4 \ell_0}{m^2 (1 - B_j)^2 s_{ij}} \times \frac{1}{n(m-1)}. \end{aligned} \tag{A.8}$$

Combining (A.6)–(A.8) yields that

$$J_1 = O\left(\frac{1}{n}\right). \tag{A.9}$$

Let $D_2 = \left\{(y_i, y_j) \in A_{ij2} \mid y_j < \frac{s_{ij}}{2}\right\}$ and $C_2 = \left\{(y_i, y_j) \in A_{ij2} \mid y_j \geq \frac{s_{ij}}{2}\right\}$. Thus,

$$J_2 = \iint_{D_2} \alpha_2(y_i, y_j, n)dy_i\, dy_j + \iint_{C_2} \alpha_2(y_i, y_j, n)dy_i\, dy_j = J_{21} + J_{22}. \tag{A.10}$$

Note that $g_2(x)$ is increasing in x for $x > 0$. Thus, on D_2, $g_2\left(\frac{mt(y_i,y_j)}{2B_i\sigma_i^2}\right) \geq g_2\left(\frac{mt(0,\frac{s_{ij}}{2})}{2B_i\sigma_i^2}\right) = g_2\left(\frac{mc_2}{12B_i\sigma_i^2}\right) > 0$. By a discussion similar to that of J_{11}, we can obtain:

$$J_{21} \leq \exp\left(\frac{-n(m-1)}{2} g_2\left(\frac{mc_2}{12B_i\sigma_i^2}\right)\right)\tau^{*^2}. \tag{A.11}$$

For the term J_{22}, by a discussion similar to that of J_{12}, by changing variable by letting $x = \frac{mt(y_i,y_j)}{2B_i\sigma_i^2}$ and Lemma B.5, we have

$$\begin{aligned} J_{22} &= 24 \iint_{C_2} \exp\left\{\frac{-n(m-1)}{2} g_2\left(\frac{mt(y_i, y_j)}{2B_i\sigma_i^2}\right)\right\} \frac{mt(y_i, y_j)}{2B_i\sigma_i^2} \left|\frac{\partial}{\partial y_j} \frac{mt(y_i, y_j)}{2B_i\sigma_i^2}\right| \\ &\quad \times \frac{12B_i^2\sigma_i^4 h_j(y_j)}{m^2(1-B_j)^2 y_j} h_i(y_i)dy_j\, dy_i \\ &\leq \frac{288B_i^2\sigma_i^4\ell_0}{m^2(1-B_j)^2 s_{ij}} \int_{y_i}\int_{x=0}^{\infty} \exp\left(\frac{-n(m-1)}{2} g_2(x)\right) x dx h_i(y_i)dy_i \\ &\leq \frac{288B_i^2\sigma_i^4\ell_0}{m^2(1-B_j)^2 s_{ij}}\left[\frac{2}{n(m-1)} + \frac{16}{n^2(m-1)^2}\right] = O\left(\frac{1}{n}\right). \end{aligned} \tag{A.12}$$

Combining (A.10)–(A.12) yields that

$$J_2 = O\left(\frac{1}{n}\right). \tag{A.13}$$

$$J_3 = \iint_{D_2} \alpha_3(y_i, y_j, n)dy_i\, dy_j + \iint_{C_2} \alpha_3(y_i, y_j, n)dy_i\, dy_j = J_{31} + J_{32}. \tag{A.14}$$

Note that $\frac{mt(y_i,y_j)}{2B_i\sigma_i^2+2mt(y_i,y_j)}$ is increasing in y_i and decreasing in y_j for (y_i, y_j) in D_2. Thus, on D_2, $\frac{mt(y_i,y_j)}{2B_i\sigma_i^2+2mt(y_i,y_j)} \geq \frac{mt\left(0,\frac{s_{ij}}{2}\right)}{2B_i\sigma_i^2+2mt\left(0,\frac{s_{ij}}{2}\right)} \equiv c_4 > 0$. Therefore,

$$J_{31} \leq \exp\left(\frac{n}{2} g_1(c_4)\right)\tau^{*^2}. \tag{A.15}$$

Let y_{i0} be the point such that $(1-B_i)\sigma_i^2 - mt(y_{i0}, s_{ij}) = 0$. By the definition of C_2, if (y_i, y_j) is in C_2, then $0 < y_i < y_{i0}$ and $y_j \geq \frac{s_{ij}}{2}$. Thus for (y_i, y_j) in C_2,

$\frac{mt(y_i,y_j)}{2B_i\sigma_i^2+2mt(y_i,y_j)} \geq \frac{mt(y_i,y_j)}{2B_i\sigma_i^2+2mt(y_{i0},\frac{s_{ij}}{2})}$. Therefore,

$$J_{32} \leq \iint_{C_2} 24\exp\left\{\frac{n}{2}g_1\left(\frac{mt(y_i, y_j)}{2B_i\sigma_i^2 + 2mt(y_{i0}, \frac{s_{ij}}{2})}\right)\right\}t(y_i, y_j)h_i(y_i)h_j(y_j)dy_i\, dy_j \equiv J_{32}^*. \tag{A.16}$$

Now, the form of J_{32}^* is similar to that of J_{12}. Thus, following a discussion similar to that of J_{12}, we can obtain:

$$J_{32} \leq J_{32}^* \leq \frac{144[2B_i\sigma_i^2 + 2mt(y_{i0}, \frac{s_{ij}}{2})]^2\ell_0}{m^2(1-B_j)^2 s_{ij}} \times \frac{1}{n}. \tag{A.17}$$

Combining (A.14)–(A.17), it leads to

$$J_3 = O\left(\frac{1}{n}\right). \tag{A.18}$$

Define $D_3 = \{(y_i, y_j) \in A_{ij3} \mid y_j < \frac{s_{ij}}{2}\}$ and $C_3 = \{(y_i, y_j) \in A_{ij3} \mid y_j \geq \frac{s_{ij}}{2}\}$. Thus,

$$J_4 = \iint_{D_3} \alpha_4(y_i, y_j, n)dy_i\, dy_j + \iint_{C_3} \alpha_4(y_i, y_j, n)dy_i\, dy_j = J_{41} + J_{42}. \tag{A.19}$$

On D_3, $\frac{t(y_i,y_j)}{4B_iy_i^2} \geq \frac{t(y_i,\frac{s_{ij}}{2})}{4B_iy_i^2} = \frac{(1-B_i)^2}{24B_i} + \frac{\psi_i(0)-\psi_j(\frac{s_{ij}}{2})}{24B_iy_i^2} \geq \frac{(1-B_i)^2}{24B_i}$ since $\psi_i(0) - \psi_j(\frac{s_{ij}}{2}) > 0$. By increasing property of $g_2(x)$ for $x > 0$, we can obtain:

$$J_{41} \leq \exp\left\{\frac{-n(m-1)}{2}g_2\left(\frac{(1-B_i)^2}{24B_i}\right)\right\}\tau^{*^2}. \tag{A.20}$$

For J_{42}, following a discussion similar to that of J_{22}, we obtain:

$$\begin{aligned} J_{42} &= 24\iint_{C_3} \exp\left\{\frac{-n(m-1)}{2}g_2\left(\frac{t(y_i, y_j)}{4B_iy_i^2}\right)\right\}\frac{t(y_i, y_j)}{4B_iy_i^2}\left|\frac{\partial}{\partial y_j}\frac{t(y_i, y_j)}{4B_iy_i^2}\right| \\ &\quad \times \frac{48B_i^2y_i^4}{(1-B_j)^2y_j}h_i(y_i)h_j(y_j)dy_i\, dy_j \\ &\leq \frac{2304B_i^2}{(1-B_j)^2s_{ij}}\int_0^\infty \exp\left(\frac{-n(m-1)}{2}g_2(x)\right)x\, dx \int_{-\infty}^\infty y_i^4h_i(y_i)dy_i \\ &\leq \frac{2304B_i^2}{(1-B_j)^2s_{ij}}\left[\frac{2}{n(m-1)} + \frac{16}{n^2(m-1)^2}\right]M_4, \end{aligned} \tag{A.21}$$

where $M_4 = \max_{1\leq i\leq k}\{EY_i^4\}$.

Combining (A.19)–(A.21) yields that

$$J_4 = O\left(\frac{1}{n}\right). \tag{A.22}$$

On A_{ij3}, $0 \le y_j \le s_{ij}$. Thus, we have: $\frac{t(y_i, y_j)}{4B_i y_i^2 + 2t(y_i, y_j)} \ge \frac{t(y_i, s_{ij})}{4B_i y_i^2 + 2t(y_i, s_{ij})} = \frac{y_i^2(1-B_i)^2 + \psi_i(0) - \psi_j(s_{ij})}{y_i^2[4B_i + 2(1-B_i)^2] + 2[\psi_i(0) - \psi_j(s_{ij})]} = \frac{(1-B_i)^2}{4B_i + 2(1-B_i)^2}$, since $\psi_i(0) - \psi_j(s_{ij}) = 0$. Since $g_1(x)$ is decreasing in (0, 1), therefore,

$$\begin{aligned}
J_5 &= \iint_{A_{ij3}} 24 \exp\left(\frac{n}{2} g_1\left(\frac{t(y_i, y_j)}{4B_i y_i^2 + 2t(y_i, y_j)}\right)\right) t(y_i, y_j) h_i(y_i) h_j(y_j) dy_i\, dy_j \\
&\le \iint_{A_{ij3}} 24 \exp\left(\frac{n}{2} g_1\left(\frac{(1-B_i)^2}{4B_i + 2(1-B_i)^2}\right)\right) t(y_i, y_j) h_i(y_i) h_j(y_j) dy_i\, dy_j \\
&\le 24 \exp\left\{\frac{n}{2} g_1\left(\frac{(1-B_i)^2}{4B_i + 2(1-B_i)^2}\right)\right\} \tau^{*^2}. \qquad \text{(A.23)}
\end{aligned}$$

Combining (A.5), (A.9), (A.13), (A.18), (A.22), and (A.23), we conclude that under Case 1 that $s_{ij} > 0$,

$$I_{ij} = O\left(\frac{1}{n}\right). \qquad \text{(A.24)}$$

Case 2. $s_{ij} = 0$.

When $s_{ij} = 0$, $\psi_j(0) - \psi_i(0) \ge 0$. For (y_i, y_j) in A_{ij}, $\psi_i(y_i) = \psi_i(0) + (1-B_i)^2 y_i^2 > \psi_j(0) + (1-B_j)^2 y_j^2 = \psi_j(y_j)$. Thus, $|y_i| \ge \frac{(1-B_j)}{(1-B_i)} |y_j|$.

Define $D_1(n) = \{(y_i, y_j) \in A_{ij1} \mid y_j \ge \frac{1}{n}\}$ and $C_1(n) = \{(y_i, y_j) \in A_{ij1} \mid y_j < \frac{1}{n}\}$. Thus,

$$J_1 = \iint_{D_1(n)} \alpha_1(y_i, y_j, n) dy_i\, dy_j + \iint_{C_1(n)} \alpha_1(y_i, y_j, n) dy_i\, dy_j = J_{11}(n) + J_{12}(n). \qquad \text{(A.25)}$$

On $D_1(n)$, $\frac{\partial t(y_i, y_j)}{\partial y_i} = \frac{2}{6}(1-B_i)^2 y_i \ge \frac{1}{3}(1-B_i)(1-B_j) y_j$. Thus,

$$\begin{aligned}
J_{11}(n) &\le 24 \iint_{D_1(n)} \exp\left(\frac{n(m-1)}{2} g_1\left(\frac{mt(y_i, y_j)}{\sigma_i^2}\right)\right) \frac{mt(y_i, y_j)}{\sigma_i^2} \left|\frac{\partial}{\partial y_i} \frac{mt(y_i, y_j)}{\sigma_i^2}\right| \frac{\sigma_i^4}{m^2} \\
&\quad \times \frac{3h_i(y_i) h_j(y_j)}{(1-B_i)(1-B_j) y_j} dy_i\, dy_j \\
&\le \frac{72 \sigma_i^4 \ell_0^2}{m^2 (1-B_i)(1-B_j)} \int_{\frac{1}{n}}^{a_j} \int_0^1 \exp\left(\frac{n(m-1)}{2} g_1(x)\right) x\, dx \frac{1}{y_j} dy_j \\
&\le \frac{144 \sigma_i^4 \ell_0^2}{m^2 (1-B_i)(1-B_j)} \frac{1}{n(m-1)} [\ln n + \ln a_j] = O\left(\frac{\ln n}{n}\right). \qquad \text{(A.26)}
\end{aligned}$$

Also, by the definition of $C_1(n)$, we can obtain

$$J_{12}(n) \le \frac{\ell_0 \tau^{*^2}}{n}. \qquad \text{(A.27)}$$

Therefore, we have

$$J_1 = O\left(\frac{\ln n}{n}\right). \qquad \text{(A.28)}$$

Let $D_2(n) = \{(y_i, y_j) \in A_{ij2} \mid y_j \geq \frac{1}{n}\}$ and $C_2(n) = \{(y_i, y_j) \in A_{ij2} \mid y_j < \frac{1}{n}\}$. Thus,

$$J_2 = \iint_{D_2(n)} \alpha_2(y_i, y_j, n) dy_i\, dy_j + \iint_{C_2(n)} \alpha_2(y_i, y_j, n) dy_i\, dy_j = J_{21}(n) + J_{22}(n), \tag{A.29}$$

where

$$J_{22}(n) \leq \frac{\ell_0 \tau^{*^2}}{n}, \tag{A.30}$$

and by a discussion similar to that of $J_{11}(n)$, we can obtain

$$J_{21}(n) = O\left(\frac{\ln n}{n}\right). \tag{A.31}$$

Therefore, we have

$$J_2 = O\left(\frac{\ln n}{n}\right). \tag{A.32}$$

$$J_3 = \iint_{D_2(n)} \alpha_3(y_i, y_j, n) dy_i\, dy_j + \iint_{C_2(n)} \alpha_3(y_i, y_j, n) dy_i\, dy_j = J_{31}(n) + J_{32}(n), \tag{A.33}$$

where

$$J_{32}(n) \leq \frac{\ell_0 \tau^{*^2}}{n}. \tag{A.34}$$

Note that for (y_i, y_j) in $D_2(n)$, $(1 - B_i)\sigma_i^2 - mt(y_i, y_j) > 0$. Thus, $\frac{\partial}{\partial y_i} \frac{mt(y_i, y_j)}{2B_i\sigma_i^2 + 2mt(y_i, y_j)} = \frac{24mB_i\sigma_i^2(1-B_i)^2 y_i}{[12B_i\sigma_i^2 + 2m(\psi_i(y_i) - \psi_j(y_j))]^2} \geq \frac{24mB_i(1-B_i)(1-B_j)y_j}{(10B_i+2)^2\sigma_i^2}$. By the preceding inequality and by changing variable taking $x = \frac{mt(y_i, y_j)}{2B_i\sigma_i^2 + 2mt(y_i, y_j)}$, we can obtain:

$$\begin{aligned} J_{31}(n) &\leq 24 \int_{\frac{1}{n}}^{a_j} \int_0^1 \exp\left(\frac{n}{2} g_1(x)\right) x\, dx \frac{2\sigma_i^2}{m} \times \frac{\ell_0^2 (10B_i + 2)^2 \sigma_i^2}{24mB_i(1 - B_i)(1 - B_j) y_j} dy_j \\ &= \frac{2\sigma_i^2 \ell_0^2 (10B_i + 2)^2 \sigma_i^2}{m^2 B_i (1 - B_i)(1 - B_j)} \times \frac{1}{n} [\ln n + \ln a_j] \\ &= O\left(\frac{\ln n}{n}\right). \end{aligned} \tag{A.35}$$

Hence, we obtain

$$J_3 = O\left(\frac{\ln n}{n}\right). \tag{A.36}$$

Let $D_3 = \{(y_i, y_j) \in A_{ij3} \mid y_i \geq 2a_i\}$ and $C_3 = \{(y_i, y_j) \in A_{ij3} \mid y_i < 2a_i\}$. Thus,

$$J_4 = \iint_{D_3} \alpha_4(y_i, y_j, n) dy_i\, dy_j + \iint_{C_3} \alpha_4(y_i, y_j, n) dy_i\, dy_j = J_{41} + J_{42}. \tag{A.37}$$

Since $\psi_j(0) - \psi_i(0) \geq 0$, on D_3, $\frac{t(y_i,y_j)}{4B_i y_i^2}$ is increasing in y_i for $y_i > 0$. Thus, $\frac{t(y_i,y_j)}{4B_i y_i^2} \geq \frac{t(2a_i,y_j)}{16B_i a_i^2} \geq \frac{t(2a_i,a_j)}{16B_i a_i^2} > 0$. Also, $g_2(x)$ is increasing in x for $x > 0$, so, $g_2\left(\frac{t(y_i,y_j)}{4B_i y_i^2}\right) \geq g_2\left(\frac{t(2a_i,a_j)}{16B_i a_i^2}\right)$. Hence,

$$J_{41} \leq \exp\left\{\frac{-n(m-1)}{2} g_2\left(\frac{t(2a_i, a_j)}{16B_i a_i^2}\right)\right\}\tau^{*2}. \tag{A.38}$$

On C_3, $y_i < 2a_i$. Thus, $\frac{t(y_i,y_j)}{4B_i y_i^2} \geq \frac{t(y_i,y_j)}{16B_i a_i^2}$. Therefore analogous to J_2, we have

$$\begin{aligned} J_{42} &\leq 24 \iint_{C_3} \exp\left\{\frac{-n(m-1)}{2} g_2\left(\frac{t(y_i, y_j)}{16B_i a_i^2}\right)\right\} t(y_i, y_j) h_i(y_i) h_j(y_j) dy_i\, dy_j \\ &= O\left(\frac{\ln n}{n}\right). \end{aligned} \tag{A.39}$$

Combining (A.37)–(A.39) yields that

$$J_4 = O\left(\frac{\ln n}{n}\right) \tag{A.40}$$

$$J_5 = \iint_{D_3} \alpha_5(y_i, y_j, n) dy_i\, dy_j + \iint_{C_3} \alpha_5(y_i, y_j, n) dy_i\, dy_j = J_{51} + J_{52}. \tag{A.41}$$

On A_{ij3}, $\frac{t(y_i,y_j)}{4B_i y_i^2 + 2t(y_i,y_j)}$ is increasing in y_i for $y_i > 0$. Thus, on D_3, $\frac{t(y_i,y_j)}{4B_i y_i^2 + 2t(y_i,y_j)} \geq \frac{t(2a_i,y_j)}{16B_i a_i^2 + 2t(2a_i,y_j)} \geq \frac{t(2a_i,a_j)}{16B_i a_i^2 + 2t(2a_i,a_j)}$. Therefore,

$$J_{51} \leq \exp\left\{\frac{n}{2} g_1\left(\frac{t(2a_i, a_j)}{16B_i a_i^2 + 2t(2a_i, a_j)}\right)\right\}\tau^{*2}. \tag{A.42}$$

On C_3, $\frac{t(y_i,y_j)}{4B_i y_i^2 + 2t(y_i,y_j)} \geq \frac{t(y_i,y_j)}{16B_i a_i^2 + 2t(2a_i,0)}$. Again, analogous to J_1, we can obtain

$$\begin{aligned} J_{52} &\leq 24 \iint_{C_3} \exp\left\{\frac{n}{2} g_1\left(\frac{t(y_i, y_j)}{16B_i a_i^2 + 2t(2a_i, 0)}\right)\right\} t(y_i, y_j) h_i(y_i) h_j(y_j) dy_i\, dy_j \\ &= O\left(\frac{\ln n}{n}\right). \end{aligned} \tag{A.43}$$

Combining the preceding results leads to

$$J_5 = O\left(\frac{\ln n}{n}\right). \tag{A.44}$$

Now combining (A.5), (A.28), (A.32), (A.36), (A.40), and (A.44), we conclude that under Case 2 that $s_{ij} = 0$,

$$I_{ij} = O\left(\frac{\ln n}{n}\right). \tag{A.45}$$

Finally, since there are $\frac{k(k-1)}{2}$ terms in I, each of them has order $O(\frac{\ln n}{n})$, therefore we conclude that $I = O(\frac{\ln n}{n})$.

Appendix B

Lemma A.1 is cited from Liang (1997).

Lemma B.1. *Let S be a $\chi^2(n)$ random variable. Then, the following inequalities hold.*

(a) $P\{\frac{S}{n} - 1 \leq -c\} \leq \exp\{\frac{n}{2}[c + \ln(1-c)]\}$ *if* $0 \leq c < 1$, $P\{\frac{S}{n} - 1 \leq -c\} = 0$ *if* $c \geq 1$.
(b) $P\{\frac{S}{n} - 1 \geq c\} \leq \exp\{\frac{-n}{2}[c - \ln(1+c)]\}$ *for* $c > 0$.

Corollary B.1.

(a) $\frac{n(m-1)W_i(n)}{\sigma_i^2} \sim \chi^2(n(m-1))$. *Thus,*

$$P\left\{\frac{W_i(n)}{\sigma_i^2} - 1 \leq -c\right\} \begin{cases} \leq \exp\left\{\dfrac{n(m-1)}{2}[c + \ln(1-c)]\right\} & \text{if } 0 \leq c < 1, \\ = 0 & \text{if } c \geq 1. \end{cases}$$

$$P\left\{\frac{W_i(n)}{\sigma_i^2} - 1 \geq c\right\} \leq \exp\left\{\frac{-n(m-1)}{2}[c - \ln(1+c)]\right\} \quad \text{for } c > 0.$$

(b) $\frac{nS_i(n)}{\tau_i^2 + \sigma_i^2/m} \sim \chi^2(n)$. *Thus,*

$$P\left\{\frac{S_i(n)}{\tau_i^2 + \sigma_i^2/m} - 1 \leq -c\right\} \begin{cases} \leq \exp\left\{\dfrac{n}{2}[c + \ln(1-c)]\right\} & \text{if } 0 \leq c < 1, \\ = 0 & \text{if } c \geq 1. \end{cases}$$

$$P\left\{\frac{S_i(n)}{\tau_i^2 + \sigma_i^2/m} - 1 \geq c\right\} \leq \exp\left\{\frac{-n}{2}[c - \ln(1+c)]\right\} \quad \text{for } c > 0.$$

Lemma B.2.

(a) *For $b > 0$ and $B_i + b < 1$,*

$$\begin{aligned} &P\{B_{in} - B_i > b\} \\ &\quad \leq P\left\{\frac{W_i(n)}{\sigma_i^2} - 1 > \frac{b}{2B_i}\right\} + P\left\{\frac{S_i(n)}{\tau_i^2 + \sigma_i^2/m} - 1 < \frac{-b}{2(B_i + b)}\right\} \\ &\quad \leq \exp\left\{\frac{-n(m-1)}{2}\left[\frac{b}{2B_i} - \ln\left(1 + \frac{b}{2B_i}\right)\right]\right\} \\ &\qquad + \exp\left\{\frac{n}{2}\left[\frac{b}{2(B_i+b)} + \ln\left(1 - \frac{b}{2(B_i+b)}\right)\right]\right\}, \end{aligned}$$

and

$$P\{B_{in} - B_i > b\} = 0 \quad \text{if } B_i + b \geq 1.$$

(b) *For* $b > 0$ *such that* $-b + B_i > 0$,

$$
\begin{aligned}
&P\{B_{in} - B_i < -b\} \\
&\quad \le P\left\{\frac{W_i(n)}{\sigma_i^2} - 1 < \frac{-b}{2B_i}\right\} + P\left\{\frac{S_i(n)}{\tau_i^2 + \sigma_i^2/m} - 1 > \frac{b}{2(B_i - b)}\right\} \\
&\quad \le \exp\left\{\frac{n(m-1)}{2}\left[\frac{b}{2B_i} + \ln\left(1 - \frac{b}{2B_i}\right)\right]\right\} \\
&\qquad + \exp\left\{\frac{-n}{2}\left[\frac{b}{2(B_i - b)} - \ln\left(1 + \frac{b}{2(B_i - b)}\right)\right]\right\},
\end{aligned}
$$

and

$$P\{B_{in} - B_i < -b\} = 0 \quad \text{if } B_i - b \le 0.$$

Lemma B.3. *For* $c > 0$, *and* $\psi_i(y_i) - 3c > 0$, *then,*

$$
\begin{aligned}
&P\{\psi_{in}(y_i) - \psi_i(y_i) < -3c\} \\
&\quad \le P\{W_i(n) - \sigma_i^2 < -mc\} + P\left\{B_{in} - B_i > \frac{mc}{\sigma_i^2}\right\} + P\left\{B_{in} - B_i > \frac{c}{2y_i^2}\right\} \\
&\quad \le \exp\left\{\frac{n(m-1)}{2}\left[\frac{mc}{\sigma_i^2} + \ln\left(1 - \frac{mc}{\sigma_i^2}\right)\right]\right\} I(\sigma_i^2 - mc) \\
&\qquad + \exp\left\{\frac{-n(m-1)}{2}\left[\frac{mc}{2B_i\sigma_i^2} - \ln\left(1 + \frac{mc}{2B_i\sigma_i^2}\right)\right]\right\} I\left(1 - B_i - \frac{mc}{\sigma_i^2}\right) \\
&\qquad + \exp\left\{\frac{n}{2}\left[\frac{mc}{2B_i\sigma_i^2 + 2mc} + \ln\left(1 - \frac{mc}{2B_i\sigma_i^2 + 2mc}\right)\right]\right\} I\left(1 - B_i - \frac{mc}{\sigma_i^2}\right) \\
&\qquad + \exp\left\{\frac{-n(m-1)}{2}\left[\frac{c}{4B_i y_i^2} - \ln\left(1 + \frac{c}{4B_i y_i^2}\right)\right]\right\} I\left(1 - B_i - \frac{c}{2y_i^2}\right) \\
&\qquad + \exp\left\{\frac{n}{2}\left[\frac{c}{4B_i y_i^2 + 2c} + \ln\left(1 - \frac{c}{4B_i y_i^2 + 2c}\right]\right\} I\left(1 - B_i - \frac{c}{2y_i^2}\right),
\end{aligned}
$$

where $I(x) = 1$ *if* $x > 0$, *and* 0 *otherwise, and* $P\{W_{in}(y_i) - \psi_i(y_i) < -3c\} = 0$ *if* $\psi_i(y_i) - 3c \le 0$.

For $0 < x < 1$, define $g_1(x) = x + \ln(1 - x)$. For $x > 0$, define $g_2(x) = x - \ln(1 + x)$.

Lemma B.4.

(a) $g_1(x)$ *is decreasing in* $(0, 1)$ *and* $g_1(x) \le \frac{-x^2}{2}$ *in* $(0, 1)$.
(b) *For* $0 < t < 1$, *and* $c > 0$,

$$\int_0^t x \exp(cn(x + \ln(1 - x)))dx = \int_0^t x \exp(cng_1(x))dx \le \frac{1}{nc}.$$

Lemma B.5.

(a) *$g_2(x)$ is increasing in x for $x > 0$, $g_2(x) \geq \frac{x^2}{4}$ for $0 < x \leq 1$ and $g_2(x) \geq \frac{x}{4} + c_1$ for $x \geq 1$, where $c_1 = \frac{3}{4} - \ln 2 > 0$.*

(b) $0 < t < 1$, *and* $c > 0$,

$$\int_0^t x\exp(-cn(x - \ln(1+x)))dx = \int_0^t x\exp(-cng_2(x))dx \leq \frac{2}{nc}.$$

(c) *For* $t > 1$,

$$\int_1^t x\exp(-cng_2(x))dx \leq \frac{16}{n^2c^2}.$$

Lemma B.6.

$$\iint_{A_{ij}} [\psi_i(y_i) - \psi_j(y_j)]h_i(y_i)h_j(y_j)dy_i dy_j \leq \int \psi_i(y_i)h_i(y_i)dy_i = E[\Theta_i^2] = \tau_i^2 \leq \tau^{*^2},$$

where $\tau^{*^2} = \max(\tau_1^2, \ldots, \tau_k^2)$.

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