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An economic order quantity model for deteriorating items with partially permissible delay in payments linked to order quantity

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Abstract

To attract more sales suppliers frequently offer a permissible delay in payments if the retailer orders more than or equal to a predetermined quantity W. In this paper, we generalize [Goyal, S.K., 1985. EOQ under conditions of permissible delay in payments. Journal of the Operational Research Society 36, 335–338] economic order quantity (EOQ) model with permissible delay in payment to reflect the following real-world situations: (1) the retailer's selling price per unit is significantly higher than unit purchase price, (2) the interest rate charged by a bank is not necessarily higher than the retailer's investment return rate, (3) many items such as fruits and vegetables deteriorate continuously, and (4) the supplier may offer a partial permissible delay in payments even if the order quantity is less than W. We then establish the proper mathematical model, and derive several theoretical results to determine the optimal solution under various situations and use two approaches to solve this complex inventory problem. Finally, a numerical example is given to illustrate the theoretical results.

Keywords: Inventory; EOQ; Trade credits; Partially permissible delay in payment; Finance

1. Introduction

The classical inventory economic order quantity (or EOQ) model is based on the assumption that the supplier is paid for the items immediately after the items are received. However, in practice, the supplier may provide the retailer many incentives such as a cash discount to motivate faster payment and stimulate sales, or a permissible delay in payments to attract new customers and increase sales. Goyal (1985) developed an EOQ model under the conditions of permissible delay in payments. Aggarwal and Jaggi (1995) extended Goyal's (1985) model to consider the deteriorating items. Jamal et al. (1997) further generalized Aggarwal and Jaggi's (1995) model to allow for shortages. Teng (2002) then amended Goyal's (1985) model by considering the difference between unit price and unit cost, and found that it makes economic sense for a well-established retailer to order less quantity and take the benefits of payment delay more frequently. Chang et al. (2003) then established an EOQ model for deteriorating items under supplier trade credits linked to order quantity. Concurrently, Chung and Liao (2004) studied a similar lot-sizing problem under supplier's trade credits depending on the retailer's order quantity. Recently, Huang (2007) established an EOQ model in which the supplier offers a partially permissible delay in

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Table 1
Major characteristics of inventory models on selected researches

Author(s) and published year	Allowing for deterioration	Assuming unrealistic $p = c$	Assuming unrealistic $I_k \geqslant I_e$	Payment linked to order quantity	Allowing for partial payments
Aggarwal and Jaggi (1995)	Yes	Yes	Yes	No	No
Jamal et al. (1997)	Yes	Yes	Yes	No	No
Chang et al. (2003)	Yes	No	No	Yes	No
Chung and Liao (2004)	Yes	Yes	Yes	Yes	No
Goyal (1985)	No	Yes	Yes	No	No
Huang (2007)	No	Yes	Yes	Yes	Yes
Teng (2002)	No	No	No	No	No
Present paper	Yes	No	No	Yes	Yes

payments when the order quantity is smaller than the predetermined quantity W. The major assumptions used in the above research articles are summarized in Table 1.

In this paper, we develop a generalized mathematical model in which we complement the shortcoming of all the previous models. The primary difference of this paper as compared to previous studies is that we first introduce a generalized inventory model by relaxing the traditional EOQ model in the following four ways: (1) the retailer's selling price per unit is higher than its purchase unit cost, (2) the interest rate charged by a bank is not necessarily higher than the retailer's investment return rate, (3) many selling items deteriorate continuously such as fresh fruits and vegetables, and (4) the supplier may offer a partial permissible delay in payments even if the order quantity is less than W. We then establish several theoretical results to characterize the optimal solutions and determine the optimal solution under various situations and use two approaches to solve this complex inventory problem. Finally, a numerical example is given to illustrate the theoretical results.

2. Mathematical formulation

For simplicity, we use the same notation and assumptions as in Huang's (2007) model, except the selling price per unit p and the deterioration rate θ.

D the annual demand
A the ordering cost per order
W the quantity at which the fully delay payment permitted per order
c the purchasing cost per unit
h the unit holding cost per year excluding interest charge
p the selling price per unit
I_e the interest earned per dollar per year
I_k the interest charged per dollar in stocks per year
M the period of permissible delay in settling accounts
α the fraction of the delay payments permitted by the supplier per order, $0 \leqslant \alpha \leqslant 1$
θ the deterioration rate, $0 \leqslant \theta < 1$
T the replenishment cycle time in years
Q the order quantity
TRC(T) the annual total relevant cost, which is a function of T
T^* the optimal replenishment cycle time of TRC(T)
Q^* the optimal order quantity

The inventory level decreases owing to demand as well as deterioration. Thus, the change of inventory level can be represented by the following differential equation:

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} + \theta I(t) = -D, \quad 0 < t < T, \tag{1}$$

with the boundary condition $I(T) = 0$. The solution of Eq. (1) is

$$I(t) = \frac{D}{\theta}[\mathrm{e}^{\theta(T-t)} - 1], \quad 0 \leqslant t \leqslant T. \tag{2}$$

Hence, the order quantity for each cycle is

$$Q = I(0) = \frac{D}{\theta}(\mathrm{e}^{\theta T} - 1). \tag{3}$$

From Eq. (3), we can obtain the time interval that W units are depleted to zero due to both demand and deterioration as

$$T_W = \frac{1}{\theta}\ln\left(\frac{\theta}{D}W + 1\right). \tag{4}$$

If $Q \geqslant W$ (i.e., $T \geqslant T_W$), then fully delayed payment is permitted. Otherwise, the partially delayed payment is permitted. Hence, if $Q < W$ (i.e., $T < T_W$), then the retailer must take a loan (with the interest charged of I_k) to pay the supplier the partial payment of $(1-\alpha)cQ$ when the order is filled at time 0. From the constant sales revenue pD, the retailer will be able to pay off the loan $(1-\alpha)cQ$ at time $(1-\alpha)(c/p)(e^{\theta T}-1)/\theta$.

Note that (1) if $T \geqslant T_W$, then the fully delayed payment is permitted, and (2) if the payoff time of the partial payment at $(1-\alpha)(c/p)(e^{\theta T}-1)/\theta$ is shorter or equal to the permissible delay M, then $T \leqslant T_0 \equiv \frac{1}{\theta}\ln\left(\frac{\theta pM}{(1-\alpha)c}+1\right)$, and vice versa. Note that $T_0 > M$. Consequently, based on the values of M, T_W, and T_0, we have three possible cases: (1) $T_0 > M \geqslant T_W$, (2) $T_0 \geqslant T_W > M$, and (3) $T_W > T_0 > M$.

Case 1 $T_0 > M \geqslant T_W$

(a) Annual ordering cost $= \frac{A}{T}$.

(b) Annual stock holding cost excluding interest charge

$$= \frac{h}{T}\int_0^T \frac{D}{\theta}[e^{\theta(T-t)}-1]\,dt = \frac{hD}{\theta^2 T}(e^{\theta T}-\theta T-1).$$

(c) Annual deteriorating cost $= \frac{c}{T}(Q-DT) = \frac{cD}{\theta T}(e^{\theta T}-\theta T-1)$.

(d) There are three sub-cases in terms of annual opportunity cost of the capital.

(i) $M \leqslant T$.

When the credit period M is shorter than or equal to the replenishment cycle time T, the retailer starts paying the interest for the items in stock after time M with rate I_k. Hence, the annual interest payable is

$$\frac{cI_k}{T}\int_M^T \frac{D}{\theta}[e^{\theta(T-t)}-1]\,dt = \frac{cI_kD}{\theta^2 T}[e^{\theta(T-M)}-\theta(T-M)-1].$$

However, during time 0 through M, the retailer sells the goods and continues to accumulate sales revenue and earns the interest with rate I_e. Therefore, the annual interest earned starts from time 0 to M and is $\frac{pI_eDM^2}{2T}$. Consequently, the annual opportunity cost of capital in this sub-case is

$$\frac{cI_kD}{\theta^2 T}[e^{\theta(T-M)}-\theta(T-M)-1] - \frac{pI_eDM^2}{2T}.$$

(ii) $T_W \leqslant T \leqslant M$.

If $T_W \leqslant T \leqslant M$, there is no interest paid for financing inventory in stock, and the annual interest earned starts from time 0 to M and is $\frac{pI_eDT}{2} + pI_eD(M-T) = pI_eD\left(M-\frac{T}{2}\right)$. Therefore, in this sub-case, we can obtain that the annual opportunity cost of capital $= -pI_eD\left(M-\frac{T}{2}\right)$.

(iii) $0 < T < T_W$.

If $T < T_W$, then the retailer must borrow the partial payment $(1-\alpha)cQ$ at time 0 to pay the supplier, and then pays off the loan from sales revenue at time $(1-\alpha)(c/p)(e^{\theta T}-1)/\theta$. Consequently, the interest charged on the partial payment is from time 0 to $(1-\alpha)(c/p)(e^{\theta T}-1)/\theta$. Hence, the annual interest payable is

$$\frac{cI_k(1-\alpha)^2(c/p)(e^{\theta T}-1)Q}{2\theta T} = \frac{cI_k(1-\alpha)^2(c/p)D}{2\theta^2 T}(e^{\theta T}-1)^2. \tag{5}$$

Similarly, the interest earned starts from time $(1-\alpha)(c/p)(e^{\theta T}-1)/\theta$ to M, and thus the annual interest earned is

$$\frac{pI_eD}{2T}[T-(1-\alpha)(c/p)(e^{\theta T}-1)/\theta]^2 + \frac{pI_eD(M-T)}{T}[T-(1-\alpha)(c/p)(e^{\theta T}-1)/\theta]. \tag{6}$$

As a result, in this sub-case, the annual opportunity cost of capital is

$$\frac{cI_k(1-\alpha)^2(c/p)D}{2\theta^2 T}(e^{\theta T}-1)^2 - \frac{pI_eD}{2T}[T-(1-\alpha)(c/p)(e^{\theta T}-1)/\theta]^2 - \frac{pI_eD(M-T)}{T}[T-(1-\alpha)(c/p)(e^{\theta T}-1)/\theta]. \tag{7}$$

Therefore, the annual total relevant cost for the retailer in Case 1 can be expressed as

$$\mathrm{TRC}(T) = \begin{cases} \mathrm{TRC}_1(T), & M \leqslant T, \\ \mathrm{TRC}_2(T), & T_W \leqslant T \leqslant M, \\ \mathrm{TRC}_3(T), & 0 < T < T_W, \end{cases} \tag{8}$$

where

$$\mathrm{TRC}_1(T) = \frac{A}{T} + \frac{(c\theta + h)D}{\theta^2 T}(\mathrm{e}^{\theta T} - \theta T - 1) + \frac{cI_k D}{\theta^2 T}[\mathrm{e}^{\theta(T-M)} - \theta(T - M) - 1] - \frac{pI_e DM^2}{2T}, \tag{9}$$

$$\mathrm{TRC}_2(T) = \frac{A}{T} + \frac{(c\theta + h)D}{\theta^2 T}(\mathrm{e}^{\theta T} - \theta T - 1) - pI_e D\left(M - \frac{T}{2}\right), \tag{10}$$

and

$$\mathrm{TRC}_3(T) = \frac{A}{T} + \frac{(c\theta + h)D}{\theta^2 T}(\mathrm{e}^{\theta T} - \theta T - 1) + \frac{cI_k(c/p)(1-\alpha)^2 D}{2\theta^2 T}(\mathrm{e}^{\theta T} - 1)^2 - \frac{pI_e D}{2T}[T - (1-\alpha)(c/p)(\mathrm{e}^{\theta T} - 1)/\theta]^2$$
$$- \frac{pI_e D(M - T)}{T}[T - (1-\alpha)(c/p)(\mathrm{e}^{\theta T} - 1)/\theta]. \tag{11}$$

Case 2 $T_0 \geqslant T_W > M$.

Similar to the approach used in Case 1, the annual total relevant cost for the retailer in this case can be expressed as

$$\mathrm{TRC}(T) = \begin{cases} \mathrm{TRC}_1(T), & T_W \leqslant T, \\ \mathrm{TRC}_4(T), & M \leqslant T < T_W, \\ \mathrm{TRC}_3(T), & T \leqslant M, \end{cases} \tag{12}$$

where

$$\mathrm{TRC}_4(T) = \frac{A}{T} + \frac{(c\theta + h)D}{\theta^2 T}(\mathrm{e}^{\theta T} - \theta T - 1) + \frac{cI_k(c/p)(1-\alpha)^2 D}{2\theta^2 T}(\mathrm{e}^{\theta T} - 1)^2 + \frac{cI_k D}{\theta^2 T}[\mathrm{e}^{\theta(T-M)} - \theta(T - M) - 1]$$
$$- \frac{pI_e D}{2T}[M - (1-\alpha)(c/p)(\mathrm{e}^{\theta T} - 1)/\theta]^2. \tag{13}$$

Case 3 $T_W > T_0 > M$.

If $T_W > T \geqslant T_0$, then $M < (1-\alpha)(c/p)(\mathrm{e}^{\theta T} - 1)/\theta$. In this sub-case, the retailer must take a loan to pay the supplier the partial payment of $(1-\alpha)cQ$ at time 0, and then take another loan to pay the rest of αcQ at time M. The first loan will be paid from the revenue received until $t = (1-\alpha)(c/p)(\mathrm{e}^{\theta T} - 1)/\theta$ ($>M$). Hence, the retailer gets the second loan at time M but starts paying off from the sales revenue after $t = (1-\alpha)(c/p)(\mathrm{e}^{\theta T} - 1)/\theta$ ($>M$). As a result, there is no interest earned, and the annual interest payable is

$$\frac{cI_k(1-\alpha)^2(c/p)(\mathrm{e}^{\theta T} - 1)Q}{2\theta T} + \frac{cI_k \alpha Q}{T}[(1-\alpha)(c/p)(\mathrm{e}^{\theta T} - 1)/\theta - M] + \frac{cI_k \alpha^2 (c/p)(\mathrm{e}^{\theta T} - 1)Q}{2\theta T}. \tag{14}$$

For the other sub-case, we can also obtain the corresponding annual opportunity cost of capital. Therefore, we have the annual total relevant cost for the retailer in Case 3 is

$$\mathrm{TRC}(T) = \begin{cases} \mathrm{TRC}_1(T), & T_W \leqslant T \\ \mathrm{TRC}_5(T) & T_0 \leqslant T < T_W, \\ \mathrm{TRC}_4(T), & M \leqslant T \leqslant T_0, \\ \mathrm{TRC}_3(T), & T \leqslant M, \end{cases} \tag{15}$$

where

$$\mathrm{TRC}_5(T) = \frac{A}{T} + \frac{(c\theta + h)D}{\theta^2 T}(\mathrm{e}^{\theta T} - \theta T - 1) + \frac{cI_k(c/p)(1 - 2\alpha + 2\alpha^2)D}{2\theta^2 T}(\mathrm{e}^{\theta T} - 1)^2 + \frac{cI_k \alpha D(\mathrm{e}^{\theta T} - 1)}{\theta T}[(1-\alpha)(c/p)$$
$$\times (\mathrm{e}^{\theta T} - 1)/\theta - M]. \tag{16}$$

3. Theoretical results

Now, we shall determine the optimal replenishment cycle time that minimizes the annual total relevant cost.

Case 1 $T_0 > M \geqslant T_W$.

The first-order condition for $\mathrm{TRC}_1(T)$ in (9) to be minimized is $\mathrm{dTRC}_1(T)/\mathrm{d}T = 0$, which implies that

$$\frac{(c\theta+h)D}{\theta^2}(\theta T\mathrm{e}^{\theta T}-\mathrm{e}^{\theta T}+1)+\frac{cI_kD}{\theta^2}[\theta T\mathrm{e}^{\theta(T-M)}-\mathrm{e}^{\theta(T-M)}+1]+\frac{pI_eDM^2}{2}-\frac{cI_kDM}{\theta}-A=0. \tag{17}$$

To show that there exists a value of T in the interval $[M,\infty)$ at which minimizes $\mathrm{TRC}_1(T)$, we let

$$\Delta_1 \equiv \frac{(c\theta+h)D}{\theta^2}(\theta M\mathrm{e}^{\theta M}-\mathrm{e}^{\theta M}+1)+\frac{pI_eDM^2}{2}-A. \tag{18}$$

Then we have the following lemma.

Lemma 1

(a) *If* $\Delta_1 \leqslant 0$, *then the annual total relevant cost* $TRC_1(T)$ *has the unique minimum value at the point* $T = T_1$, *where* $T_1 \in [M,\infty)$ *and satisfies Eq.* (17).
(b) *If* $\Delta_1 > 0$, *then the annual total relevant cost* $TRC_1(T)$ *has a minimum value at the boundary point* $T = M$.

Proof. See the Appendix A.

Similarly, the first-order necessary condition for $\mathrm{TRC}_2(T)$ in (10) to be minimum is $\mathrm{dTRC}_2(T)/\mathrm{d}T = 0$, which leads to

$$\frac{(c\theta+h)D}{\theta^2}(\theta T\mathrm{e}^{\theta T}-\mathrm{e}^{\theta T}+1)+\frac{pI_eDT^2}{2}-A=0. \tag{19}$$

To prove that there exists a value of T in the interval $[T_W, M]$ at which minimizes $\mathrm{TRC}_2(T)$, we let

$$\Delta_2 \equiv \frac{(c\theta+h)D}{\theta^2}(\theta T_W\mathrm{e}^{\theta T_W}-\mathrm{e}^{\theta T_W}+1)+\frac{pI_eDT_W^2}{2}-A. \tag{20}$$

It is obvious that $\Delta_2 \leqslant \Delta_1$ if $M \geqslant T_W$. Then we have following lemma. □

Lemma 2

(a) *If* $\Delta_2 \leqslant 0 \leqslant \Delta_1$, *then the annual total relevant cost* $TRC_2(T)$ *has the unique minimum value at the point* $T = T_2$, *where* $T_2 \in [T_W, M]$ *and satisfies* (19).
(b) *If* $\Delta_2 > 0$, *then the annual total relevant cost* $TRC_2\ (T)$ *has a minimum value at the lower boundary point* $T = T_W$.
(c) *If* $\Delta_1 < 0$, *then the annual total relevant cost* $TRC_2\ (T)$ *has a minimum value at the upper boundary point* $T = M$.

Proof. The proof is similar to that in Lemma 1. Hence we omit it.

Likewise, the first-order necessary condition for $\mathrm{TRC}_3(T)$ in (11) to be minimum is $\mathrm{dTRC}_3(T)/\mathrm{d}T = 0$, which leads to

$$\begin{aligned}&\frac{(c\theta+h)D}{\theta^2}(\theta T\mathrm{e}^{\theta T}-\mathrm{e}^{\theta T}+1)+\frac{cI_k(c/p)(1-\alpha)^2D}{2\theta^2}(\mathrm{e}^{\theta T}-1)(2\theta T\mathrm{e}^{\theta T}-\mathrm{e}^{\theta T}+1)-\frac{pI_eD}{2\theta}[T-(1-\alpha)(c/p)(\mathrm{e}^{\theta T}-1)/\theta]\\&\quad\times[\theta T-(1-\alpha)(c/p)(2\theta T\mathrm{e}^{\theta T}-\mathrm{e}^{\theta T}+1)]+\frac{pI_eD}{\theta}\{\theta T^2[1-(1-\alpha)(c/p)\mathrm{e}^{\theta T}]\\&\quad+(1-\alpha)(c/p)M(\theta T\mathrm{e}^{\theta T}-\mathrm{e}^{\theta T}+1)\}-A=0.\end{aligned} \tag{21}$$

Again, to show that there exists a value of T which satisfies (21) and minimizes $\mathrm{TRC}_3(T)$, we let

$$\begin{aligned}\Delta_3 \equiv{}& \frac{(c\theta+h)D}{\theta^2}(\theta T_W\mathrm{e}^{\theta T_W}-\mathrm{e}^{\theta T_W}+1)+\frac{cI_k(c/p)(1-\alpha)^2D}{2\theta^2}(\mathrm{e}^{\theta T_W}-1)(2\theta T_W\mathrm{e}^{\theta T_W}-\mathrm{e}^{\theta T_W}+1)-\frac{pI_eD}{2\theta}[T_W-(1-\alpha)\\&\times(c/p)(\mathrm{e}^{\theta T_W}-1)/\theta][\theta T_W-(1-\alpha)(c/p)(2\theta T_W\mathrm{e}^{\theta T_W}-\mathrm{e}^{\theta T_W}+1)]+\frac{pI_eD}{\theta}\{\theta T_W^2[1-(1-\alpha)(c/p)\mathrm{e}^{\theta T_W}]\\&+(1-\alpha)(c/p)M(\theta T_W\mathrm{e}^{\theta T_W}-\mathrm{e}^{\theta T_W}+1)\}-A.\end{aligned} \tag{22}$$

Then we have following lemma. □

Lemma 3

(a) *If* $\Delta_3 \geqslant 0$, *then the annual total relevant cost* $TRC_3(T)$ *has the unique minimum value at the point* $T = T_3$, *where* $T_3 \in (0, T_W)$ *and satisfies* (21).
(b) *If* $\Delta_3 < 0$, *then the value of* $T \in (0, T_W)$ *which minimizes* $TRC_3(T)$ *does not exist.*

Proof. See the Appendix B.

From (20) and (22), it is obvious that $\Delta_3 \geqslant \Delta_2$ for $1 \geqslant \alpha \geqslant 0$. Moreover, since $M \geqslant T_W$, we know that $\Delta_1 \geqslant \Delta_2$. Consequently, combining Lemmas 1–3 and the fact that $\mathrm{TRC}_1(M) = \mathrm{TRC}_2(M)$, we can obtain the following theoretical result to determine the optimal cycle time T^* for Case 1. □

Theorem 1. *For* $T_0 > M \geqslant T_W$, *the optimal replenishment cycle time* T^* *that minimizes the annual total relevant cost is given as follows*:

Situations	$\mathrm{TRC}(T^*)$	T^*
$\Delta_1 \leqslant 0$ and $\Delta_3 < 0$	$\mathrm{TRC}_1(T_1)$	T_1
$\Delta_1 \leqslant 0$ and $\Delta_3 \geqslant 0$	$\min\{\mathrm{TRC}_1(T_1), \mathrm{TRC}_3(T_3)\}$	T_1 or T_3
$\Delta_1 > 0$, $\Delta_2 < 0$ and $\Delta_3 \geqslant 0$	$\min\{\mathrm{TRC}_2(T_2), \mathrm{TRC}_3(T_3)\}$	T_2 or T_3
$\Delta_2 \geqslant 0$	$\min\{\mathrm{TRC}_2(T_W), \mathrm{TRC}_3(T_3)\}$	T_W or T_3
$\Delta_1 > 0$ and $\Delta_3 < 0$	$\mathrm{TRC}_2(T_2)$	T_2

Case 2 $T_0 \geqslant T_W > M$

Similar to the approach used in Case 1, the first-order condition for $\mathrm{TRC}_1(T)$ (9) is the same as (17). Similarly, to show that there exists a unique value of T in the interval $[T_W, \infty)$ at which $\mathrm{TRC}_1(T)$ is minimized, we let

$$\Delta_4 \equiv \frac{(c\theta + h)D}{\theta}(T_W \mathrm{e}^{\theta T_W} - W/D) + \frac{cI_k D}{\theta^2}[\theta T_W \mathrm{e}^{\theta(T_W - M)} - \mathrm{e}^{\theta(T_W - M)} + 1] - \frac{cI_k DM}{\theta} + \frac{pI_e DM^2}{2} - A. \tag{23}$$

Consequently, we have following lemma.

Lemma 4

(a) *If* $\Delta_4 \leqslant 0$, *then the annual total relevant cost* $TRC_1(T)$ *has the unique minimum value at the point* $T = T_1$, *where* $T_1 \in [T_W, \infty)$ *and satisfies* (17).
(b) *If* $\Delta_4 > 0$, *then the annual total relevant cost* $TRC_1(T)$ *has a minimum value at the boundary point* $T = T_W$.

Proof. The proof is similar to that in Lemma 1. For simplicity, we omit it.

Likewise, the first-order condition for $\mathrm{TRC}_4(T)$ in (13) is $\mathrm{dTRC}_4(T)/\mathrm{d}T = 0$, which implies that

$$\begin{aligned} &\frac{(c\theta + h)D}{\theta^2}(\theta T \mathrm{e}^{\theta T} - \mathrm{e}^{\theta T} + 1) + \frac{cI_k D}{\theta^2}[\theta T \mathrm{e}^{\theta(T-M)} - \mathrm{e}^{\theta(T-M)} + 1] - \frac{cI_k DM}{\theta} + \frac{cI_k (c/p)(1-\alpha)^2 D}{2\theta^2}(\mathrm{e}^{\theta T} - 1)(2\theta T \mathrm{e}^{\theta T} - \mathrm{e}^{\theta T} + 1) \\ &\quad + \frac{pI_e D}{2\theta}[M - (1-\alpha)(c/p)(\mathrm{e}^{\theta T} - 1)/\theta][\theta M + (1-\alpha)(c/p)(2\theta T \mathrm{e}^{\theta T} - \mathrm{e}^{\theta T} + 1)] - A = 0. \end{aligned} \tag{24}$$

To prove that there exists a unique value of T in the interval $[M, T_W)$ at which $\mathrm{TRC}_4(T)$ is minimized, we let

$$\begin{aligned} \Delta_5 \equiv &\frac{(c\theta + h)D}{\theta^2}(\theta M \mathrm{e}^{\theta M} - \mathrm{e}^{\theta M} + 1) + \frac{cI_k (c/p)(1-\alpha)^2 D}{2\theta^2}(\mathrm{e}^{\theta M} - 1)(2\theta M \mathrm{e}^{\theta M} - \mathrm{e}^{\theta M} + 1) + \frac{pI_e D}{2\theta}[M - (1-\alpha)(c/p) \\ &\times (\mathrm{e}^{\theta M} - 1)/\theta][\theta M + (1-\alpha)(c/p)(2\theta M \mathrm{e}^{\theta M} - \mathrm{e}^{\theta M} + 1)] - A, \end{aligned} \tag{25}$$

and

$$\Delta_6 \equiv \frac{(c\theta+h)D}{\theta^2}(\theta T_W e^{\theta T_W} - e^{\theta T_W} + 1) + \frac{cI_k(c/p)(1-\alpha)^2 D}{2\theta^2}(e^{\theta T_W} - 1)(2\theta T_W e^{\theta T_W} - e^{\theta T_W} + 1)$$
$$+ \frac{pI_e D}{2\theta}[M - (1-\alpha)(c/p)(e^{\theta T_W} - 1)/\theta][\theta M + (1-\alpha)(c/p)(2\theta T_W e^{\theta T_W} - e^{\theta T_W} + 1)] - A. \tag{26}$$

Then we have the following lemma. □

Lemma 5

(a) *If $\Delta_5 \leqslant 0 \leqslant \Delta_6$, then the annual total relevant cost $TRC_4(T)$ has the unique minimum value at the point $T = T_4$, where $T_4 \in [M, T_W)$ and satisfies (24).*
(b) *If $\Delta_5 > 0$, then the annual total relevant cost $TRC_4(T)$ has a minimum value at the lower boundary point $T = M$.*
(c) *If $\Delta_6 < 0$, then the value of $T \in [M, T_W)$ which minimizes $TRC_4(T)$ does not exist.*

Proof. The proof of either (a) or (b) is similar to that in Lemma 1. However, the proof of (c) is similar to that in Lemma 3(b).
Since the first-order condition for $TRC_3(T)$ in (11) to be minimized is $dTRC_3(T)/dT = 0$ which is the same as in (21), we have the following lemma: □

Lemma 6

(a) *If $\Delta_5 \geqslant 0$, then the annual total relevant cost $TRC_3(T)$ has the unique minimum value at the point $T = T_3$, where $T_3 \in (0, M]$ and satisfies (21).*
(b) *If $\Delta_5 < 0$, then the annual total relevant cost $TRC_3(T)$ has a minimum value at the boundary point $T = M$.*

Proof. We omit the proof because it is similar to that in Lemma 1.

From (23) and (26), it is obvious that $\Delta_6 \geqslant \Delta_4$ for $1 \geqslant \alpha \geqslant 0$. Moreover, since $M < T_W$, we know that $\Delta_6 \geqslant \Delta_5$. Consequently, combining Lemmas 4–6 and the facts that $TRC_4(M) = TRC_3(M)$, we can obtain a theoretical result to determine the optimal cycle time T^* for Case 2. That is, we have the following result. □

Theorem 2. *For $T_0 \geqslant T_W > M$, the optimal replenishment cycle time T^* that minimizes the annual total relevant cost is given as follows*:

Conditions	TRC(T^*)	T^*
$\Delta_6 < 0$	min{$TRC_1(T_1)$,$TRC_3(M)$}	T_1 or M
$\Delta_4 < 0$, $\Delta_5 < 0$ and $\Delta_6 \geqslant 0$	min{$TRC_1(T_1)$, $TRC_4(T_4)$}	T_1 or T_4
$\Delta_4 < 0$ and $\Delta_5 \geqslant 0$	min{$TRC_1(T_1)$,$TRC_3(T_3)$}	T_1 or T_3
$\Delta_4 \geqslant 0$ and $\Delta_5 < 0$	min{$TRC_1(T_W)$,$TRC_4(T_4)$}	T_W or T_4
$\Delta_4 \geqslant 0$ and $\Delta_5 \geqslant 0$	min{$TRC_1(T_W)$,$TRC_3(T_3)$}	T_W or T_3

Case 3 $T_W > T_0 > M$

From Lemma 4, we know that if $\Delta_4 \leqslant 0$, then the annual total relevant cost $TRC_1(T)$ in (9) has the unique minimum value at the point $T = T_1$, where $T_1 \in [T_W, \infty)$ and satisfies (17). Otherwise, $TRC_1(T)$ has a minimum value at the boundary point $T = T_W$.

Next, the first-order condition for $TRC_5(T)$ in (16) to be minimized is $dTRC_5(T)/dT = 0$, which implies that

$$\frac{(c\theta+h)D}{\theta^2}(\theta T e^{\theta T} - e^{\theta T} + 1) + \frac{cI_k(c/p)(1-2\alpha+2\alpha^2)D}{2\theta^2}(e^{\theta T} - 1)(2\theta T e^{\theta T} - e^{\theta T} + 1) + \frac{cI_k\alpha D}{\theta}\{(\theta T e^{\theta T} - e^{\theta T} + 1)$$
$$\times [(1-\alpha)(c/p)(e^{\theta T} - 1)/\theta - M] + (e^{\theta T} - 1)(1-\alpha)(c/p)T e^{\theta T}\} - A$$
$$= 0. \tag{27}$$

To prove that there exists a value of T in the interval $[T_0, T_W)$ at which minimizes $TRC_5(T)$, we let

$$\Delta_7 \equiv \frac{(c\theta+h)D}{\theta^2}(\theta T_0 e^{\theta T_0} - e^{\theta T_0} + 1) + \frac{cI_k(c/p)(1-2\alpha+2\alpha^2)D}{2\theta^2}(e^{\theta T_0} - 1)(2\theta T_0 e^{\theta T_0} - e^{\theta T_0} + 1) + \frac{cI_k\alpha D}{\theta}$$
$$\times \{(\theta T_0 e^{\theta T_0} - e^{\theta T_0} + 1)[(1-\alpha)(c/p)(e^{\theta T_0} - 1)/\theta - M] + (e^{\theta T_0} - 1)(1-\alpha)(c/p)T_0 e^{\theta T_0}\} - A, \tag{28}$$

and

$$\Delta_8 \equiv \frac{(c\theta+h)D}{\theta^2}(\theta T_W e^{\theta T_W} - e^{\theta T_W} + 1) + \frac{cI_k(c/p)(1-2\alpha+2\alpha^2)D}{2\theta^2}(e^{\theta T_W} - 1)(2\theta T_W e^{\theta T_W} - e^{\theta T_W} + 1) + \frac{cI_k\alpha D}{\theta} \times \{(\theta T_W e^{\theta T_W} - e^{\theta T_W} + 1)[(1-\alpha)(c/p)(e^{\theta T_W} - 1)/\theta - M] + (e^{\theta T_W} - 1)(1-\alpha)(c/p)T_W e^{\theta T_W}\} - A. \quad (29)$$

Consequently, we have the following lemma:

Lemma 7

(a) *If $\Delta_7 \leqslant 0 \leqslant \Delta_8$, then the annual total relevant cost $TRC_5(T)$ has the unique minimum value at the point $T = T_5$, where $T_5 \in [T_0, T_W)$ and satisfies* (27).
(b) *If $\Delta_7 > 0$, then the annual total relevant cost TRC_5 (T) has a minimum value at the lower boundary point $T = T_0$.*
(c) *If $\Delta_8 < 0$, then the value of $T \in [T_0, T_W)$ minimizes $TRC_5(T)$ does not exist.*

Proof. The proof of either (a) or (b) is similar to that in Lemma 1. As to (c), it is similar to that in Lemma 3(b).

Likewise, the first-order condition for $\mathrm{TRC}_4(T)$ in (13) to be minimized is $\mathrm{dTRC}_4(T)/\mathrm{d}T = 0$, which is the same as in (24). To prove that there exists a unique value of T in the interval $[M,T_0]$ at which minimizes $\mathrm{TRC}_4(T)$, we let □

$$\Delta_9 \equiv \frac{(c\theta+h)D}{\theta^2}(\theta T_0 e^{\theta T_0} - e^{\theta T_0} + 1) + \frac{cI_k(c/p)(1-2\alpha+2\alpha^2)D}{2\theta^2}(e^{\theta T_0} - 1)(2\theta T_0 e^{\theta T_0} - e^{\theta T_0} + 1) + \frac{cI_k\alpha D}{\theta} \times \{(\theta T_0 e^{\theta T_0} - e^{\theta T_0} + 1)[(1-\alpha)(c/p)(e^{\theta T_0} - 1)/\theta - M] + (e^{\theta T_0} - 1)(1-\alpha)(c/p)T_0 e^{\theta T_0}\} - A. \quad \square \quad (30)$$

Then we have the following lemma:

Lemma 8

(a) *If $\Delta_5 \leqslant 0 \leqslant \Delta_9$, then the annual total relevant cost $TRC_4(T)$ has the unique minimum value at the point $T = T_4$, where $T_4 \in [M,T_0]$ and satisfies* (24).
(b) *If $\Delta_5 > 0$, then the annual total relevant cost TRC_4 (T) has a minimum value at the lower boundary point $T = M$.*
(c) *If $\Delta_9 < 0$, then the annual total relevant cost TRC_4 (T) has a minimum value at the boundary point $T = T_0$.*

Proof. The proof is similar to that in Lemma 1.

From Lemma 6, we know that if $\Delta_5 \geqslant 0$, then the annual total relevant cost $\mathrm{TRC}_3(T)$ has the unique minimum value at the point $T = T_3$, where $T_3 \in (0,M]$ and satisfies (21). Otherwise, $\Delta_5 < 0$, then the annual total relevant cost $\mathrm{TRC}_3(T)$ has a minimum value at the boundary point $T = M$.

From (28) and (30), it is obvious that $\Delta_9 \geqslant \Delta_7$. Since $T_W > T_0 > M$, we know that $\Delta_9 \geqslant \Delta_5$ and $\Delta_8 \geqslant \Delta_7 \geqslant \Delta_5$. Consequently, combining Lemmas 4, 6, 7, and 8 and the fact that $\mathrm{TRC}_4(M) = \mathrm{TRC}_3(M)$, we can obtain the following theoretical result to determine the optimal cycle time T^* for Case 3. □

Theorem 3. *For $T_W > T_0 > M$, the optimal replenishment cycle time T^* that minimizes the annual total relevant cost is given as follows*:

Situation	Conditions	$\mathrm{TRC}(T^*)$	T^*
$\Delta_4 < 0$	$\Delta_8 < 0$ and $\Delta_9 < 0$	$\min\{\mathrm{TRC}_1(T_1), \mathrm{TRC}_4(T_0)\}$	T_1 or T_0
	$\Delta_8 < 0$ and $\Delta_9 \geqslant 0$	$\min\{\mathrm{TRC}_1(T_1), \mathrm{TRC}_4(T_4)\}$	T_1 or T_4
	$\Delta_8 \geqslant 0$ and $\Delta_9 < 0$	$\min\{\mathrm{TRC}_1(T_1), \mathrm{TRC}_4(T_0), \mathrm{TRC}_5(T_5)\}$	T_1 or T_0 or T_5
	$\Delta_7 < 0$, $\Delta_8 \geqslant 0$ and $\Delta_9 \geqslant 0$	$\min\{\mathrm{TRC}_1(T_1), \mathrm{TRC}_4(T_4), \mathrm{TRC}_5(T_5)\}$	T_1 or T_4 or T_5
	$\Delta_5 < 0$ and $\Delta_7 \geqslant 0$	$\min\{\mathrm{TRC}_1(T_1), \mathrm{TRC}_4(T_4), \mathrm{TRC}_5(T_0)\}$	T_1 or T_4 or T_0
	$\Delta_5 \geqslant 0$	$\min\{\mathrm{TRC}_1(T_1), \mathrm{TRC}_3(T_3), \mathrm{TRC}_5(T_0)\}$	T_1 or T_3 or T_0
$\Delta_4 \geqslant 0$	$\Delta_8 < 0$ and $\Delta_9 < 0$	$\min\{\mathrm{TRC}_1(T_W), \mathrm{TRC}_4(T_0)\}$	T_W or T_0
	$\Delta_8 < 0$ and $\Delta_9 \geqslant 0$	$\min\{\mathrm{TRC}_1(T_W), \mathrm{TRC}_4(T_4)\}$	T_W or T_4
	$\Delta_8 \geqslant 0$ and $\Delta_9 < 0$	$\min\{\mathrm{TRC}_1(T_W), \mathrm{TRC}_4(T_0), \mathrm{TRC}_5(T_5)\}$	T_W or T_0 or T_5
	$\Delta_7 < 0$, $\Delta_8 \geqslant 0$ and $\Delta_9 \geqslant 0$	$\min\{\mathrm{TRC}_1(T_W), \mathrm{TRC}_4(T_4), \mathrm{TRC}_5(T_5)\}$	T_W or T_4 or T_5
	$\Delta_5 < 0$ and $\Delta_7 \geqslant 0$	$\min\{\mathrm{TRC}_1(T_W), \mathrm{TRC}_4(T_4), \mathrm{TRC}_5(T_0)\}$	T_W or T_4 or T_0
	$\Delta_5 \geqslant 0$	$\min\{\mathrm{TRC}_1(T_W), \mathrm{TRC}_3(T_3), \mathrm{TRC}_5(T_0)\}$	T_W or T_3 or T_0

4. Some special cases

In this section, we discuss several special cases and make descriptions of these cases.

(i) When $\theta \to 0$ and $p = c$, the model can be reduced to EOQ model under partially payments delay and the result is similar to Huang (2007).
(ii) When $\alpha = 0$, the model is similar to Chang et al. (2003).
(iii) When $\alpha = 0$ and $p = c$, the model can be reduced to Chung and Liao (2004).
(iv) When $\alpha = 1$, $p = c$ and $W = 0$, the model is similar to Aggarwal and Jaggi (1995).
(v) When $\theta \to 0$, $\alpha = 1$ and $W = 0$ the model can be reduced to EOQ model under payments delay and the result is the same as that in Teng (2002).
(vi) When $\theta \to 0$, $\alpha = 1$, $p = c$ and $W = 0$, the model is the same as Goyal (1985).

Therefore, our model is in general framework that includes numerous previous models such as Goyal (1985), Aggarwal and Jaggi (1995), Teng (2002), Chang et al. (2003), Chung and Liao (2004) and Huang (2007), as special cases.

5. Solution procedures

In this section, we develop two solution approaches to solve the problem. The first approach is to use any standard non-linear programming software to solve the following 10 sub-cases in which the objective function is non-linear but the constraints are linear. Consequently, the non-linear problem in each sub-case should have a unique global minimum. Based on $T_0 > M$, and the values of M, T_W, and T_0, we have three possible sets of sub-problems out of 10 sub-cases: (1) $T_0 > M \geqslant T_W$, (2) $T_0 \geqslant T_W > M$, and (3) $T_W > T_0 > M$ which are shown below.

Case 1 $T_0 > M \geqslant T_W$

S-1 Min $$\mathrm{TRC}_1(T) = \frac{A}{T} + \frac{(c\theta + h)D}{\theta^2 T}(\mathrm{e}^{\theta T} - \theta T - 1) + \frac{cI_k D}{\theta^2 T}[\mathrm{e}^{\theta(T-M)} - \theta(T-M) - 1] - \frac{pI_e DM^2}{2T}$$

s.t. $T \geqslant M$.

S-2 Min $$\mathrm{TRC}_2(T) = \frac{A}{T} + \frac{(c\theta + h)D}{\theta^2 T}(\mathrm{e}^{\theta T} - \theta T - 1) - pI_e D\left(M - \frac{T}{2}\right)$$

s.t. $T_W \leqslant T \leqslant M$.

S-3 Min $$\begin{aligned}\mathrm{TRC}_3(T) = {}& \frac{A}{T} + \frac{(c\theta + h)D}{\theta^2 T}(\mathrm{e}^{\theta T} - \theta T - 1) + \frac{cI_k(c/p)(1-\alpha)^2 D}{2\theta^2 T}(\mathrm{e}^{\theta T} - 1)^2 \\ & - \frac{pI_e D}{2T}[T - (1-\alpha)(c/p)(\mathrm{e}^{\theta T} - 1)/\theta]^2 \\ & - \frac{pI_e D(M-T)}{T}[T - (1-\alpha)(c/p)(\mathrm{e}^{\theta T} - 1)/\theta]\end{aligned}$$

s.t. $0 \leqslant T \leqslant T_W$.

Case 2 $T_0 \geqslant T_W > M$

S-4 Min $$\mathrm{TRC}_1(T) = \frac{A}{T} + \frac{(c\theta + h)D}{\theta^2 T}(\mathrm{e}^{\theta T} - \theta T - 1) + \frac{cI_k D}{\theta^2 T}[\mathrm{e}^{\theta(T-M)} - \theta(T-M) - 1] - \frac{pI_e DM^2}{2T}$$

s.t. $T \geqslant T_W$.

S-5 Min $$\begin{aligned}\mathrm{TRC}_4(T) = {}& \frac{A}{T} + \frac{(c\theta + h)D}{\theta^2 T}(\mathrm{e}^{\theta T} - \theta T - 1) \\ & + \frac{cI_k(c/p)(1-\alpha)^2 D}{2\theta^2 T}(\mathrm{e}^{\theta T} - 1)^2 + \frac{cI_k D}{\theta^2 T}[\mathrm{e}^{\theta(T-M)} - \theta(T-M) - 1] \\ & - \frac{pI_e D}{2T}[M - (1-\alpha)(c/p)(\mathrm{e}^{\theta T} - 1)/\theta]^2\end{aligned}$$

s.t. $M \leqslant T \leqslant T_W$.

$$\text{S-6}\quad \text{Min}\quad \mathrm{TRC}_3(T)=\frac{A}{T}+\frac{(c\theta+h)D}{\theta^2T}(\mathrm{e}^{\theta T}-\theta T-1)+\frac{cI_k(c/p)(1-\alpha)^2D}{2\theta^2T}(\mathrm{e}^{\theta T}-1)^2$$
$$-\frac{pI_eD}{2T}[T-(1-\alpha)(c/p)(\mathrm{e}^{\theta T}-1)/\theta]^2$$
$$-\frac{pI_eD(M-T)}{T}[T-(1-\alpha)(c/p)(\mathrm{e}^{\theta T}-1)/\theta]$$
$$\text{s.t.}\quad 0\leqslant T\leqslant M.$$

Case 3 $T_W > T_0 > M$

$$\text{S-7}\quad \text{Min}\quad \mathrm{TRC}_1(T)=\frac{A}{T}+\frac{(c\theta+h)D}{\theta^2T}(\mathrm{e}^{\theta T}-\theta T-1)+\frac{cI_kD}{\theta^2T}[\mathrm{e}^{\theta(T-M)}-\theta(T-M)-1]-\frac{pI_eDM^2}{2T}$$
$$\text{s.t.}\quad T\geqslant T_W.$$

$$\text{S-8}\quad \text{Min}\quad \mathrm{TRC}_5(T)=\frac{A}{T}+\frac{(c\theta+h)D}{\theta^2T}(\mathrm{e}^{\theta T}-\theta T-1)$$
$$+\frac{cI_k(c/p)(1-2\alpha+2\alpha^2)D}{2\theta^2T}(\mathrm{e}^{\theta T}-1)^2+\frac{cI_k\alpha D(\mathrm{e}^{\theta T}-1)}{\theta T}[(1-\alpha)(c/p)(\mathrm{e}^{\theta T}-1)/\theta-M]$$
$$\text{s.t.}\quad T_0\leqslant T\leqslant T_W.$$

$$\text{S-9}\quad \text{Min}\quad \mathrm{TRC}_4(T)=\frac{A}{T}+\frac{(c\theta+h)D}{\theta^2T}(\mathrm{e}^{\theta T}-\theta T-1)$$
$$+\frac{cI_k(c/p)(1-\alpha)^2D}{2\theta^2T}(\mathrm{e}^{\theta T}-1)^2+\frac{cI_kD}{\theta^2T}[\mathrm{e}^{\theta(T-M)}-\theta(T-M)-1]$$
$$-\frac{pI_eD}{2T}[M-(1-\alpha)(c/p)(\mathrm{e}^{\theta T}-1)/\theta]^2$$
$$\text{s.t.}\quad M\leqslant T\leqslant T_0.$$

$$\text{S-10}\quad \text{Min}\quad \mathrm{TRC}_3(T)=\frac{A}{T}+\frac{(c\theta+h)D}{\theta^2T}(\mathrm{e}^{\theta T}-\theta T-1)+\frac{cI_k(c/p)(1-\alpha)^2D}{2\theta^2T}(\mathrm{e}^{\theta T}-1)^2$$
$$-\frac{pI_eD}{2T}[T-(1-\alpha)(c/p)(\mathrm{e}^{\theta T}-1)/\theta]^2$$
$$-\frac{pI_eD(M-T)}{T}[T-(1-\alpha)(c/p)(\mathrm{e}^{\theta T}-1)/\theta]$$
$$\text{s.t.}\quad 0\leqslant T\leqslant M.$$

The decision rule to determine the optimal solution is as follows. If $T_0 > M \geqslant T_W$, then we solve sub-problems S-1, S-2 and S-3, and then compare their values to find the optimal minimum solution. If $T_0 \geqslant T_W > M$, then we solve sub-problems S-4, S-5 and S-6, and then find the optimal minimum solution among them. Finally, if $T_W > T_0 > M$, then we solve sub-problems S-7, S-8, S-9 and S-10, and then the optimal minimum solution can be found by comparing those four values.

The second approach is to develop the following algorithm to solve this complex inventory problem by using the characteristics of Theorems 1–3 above.

Algorithm

Step 1 Compare the values of T_0, M and T_W. If $T_0 > M \geqslant T_W$, then go to Step 2. If $T_0 \geqslant T_W > M$, then go to Step 3. Otherwise, if $T_W > T_0 > M$, then go to Step 4.

Step 2 Calculate Δ_1, Δ_2 and Δ_3 which are shown as in Eqs. (18), (20) and (22), respectively.

(1) If $\Delta_1 < 0$ and $\Delta_3 < 0$, then $\mathrm{TRC}(T^*) = \mathrm{TRC}_1(T_1)$ and $T^* = T_1$. Go to Step 5.
(2) If $\Delta_1 < 0$ and $\Delta_3 \geqslant 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_1(T_1), \mathrm{TRC}_3(T_3)\}$ and $T^* = T_1$ or T_3. Go to Step 5.
(3) If $\Delta_1 \geqslant 0$, $\Delta_2 < 0$ and $\Delta_3 \geqslant 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_2(T_2), \mathrm{TRC}_3(T_3)\}$ and $T^* = T_2$ or T_3. Go to Step 5.
(4) If $\Delta_2 \geqslant 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_2(T_W), \mathrm{TRC}_3(T_3)\}$ and $T^* = T_W$ or T_3. Go to Step 5.
(5) If $\Delta_1 \geqslant 0$ and $\Delta_3 < 0$, then $\mathrm{TRC}(T^*) = \mathrm{TRC}_2(T_2)$ and $T^* = T_2$. Go to Step 5.

Step 3. Calculate Δ_4, Δ_5 and Δ_6 which are shown as in Eqs. (23), (25) and (26), respectively.

(1) If $\Delta_6 < 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_1(T_1), \mathrm{TRC}_3(M)\}$ and $T^* = T_1$ or M. Go to Step 5.
(2) If $\Delta_4 < 0$, $\Delta_5 < 0$ and $\Delta_6 \geqslant 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_1(T_1),\mathrm{TRC}_4(T_4)\}$ and $T^* = T_1$ or T_4. Go to Step 5.
(3) If $\Delta_4 < 0$ and $\Delta_5 \geqslant 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_1(T_1),\mathrm{TRC}_3(T_3)\}$ and $T^* = T_1$ or T_3. Go to Step 5.
(4) If $\Delta_4 \geqslant 0$ and $\Delta_5 < 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_1(T_W),\mathrm{TRC}_4(T_4)\}$ and $T^* = T_W$ or T_4. Go to Step 5.
(5) If $\Delta_4 \geqslant 0$ and $\Delta_5 \geqslant 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_1(T_W),\mathrm{TRC}_3(T_3)\}$ and $T^* = T_W$ or T_3. Go to Step 5.

Step 4 Calculate Δ_4, Δ_5, Δ_7, Δ_8 and Δ_9 which are shown as in Eqs. (23), (25), (28), (29) and (30), respectively. If $\Delta_4 < 0$, then go to Step 4-1. Otherwise, go to Step 4-2.

Step 4-1

(1) If$\Delta_8 < 0$ and $\Delta_9 < 0$, then $\mathrm{TRC}(T^*) = \min\ \{\mathrm{TRC}_1(T_1),\mathrm{TRC}_4(T_0)\}$ and $T^* = T_1$ or T_0. Go to Step 5.
(2) If $\Delta_8 < 0$ and $\Delta_9 \geqslant 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_1(T_1),\mathrm{TRC}_4(T_4)\}$ and $T^* = T_1$ or T_4. Go to Step 5.
(3) If $\Delta_8 \geqslant 0$ and $\Delta_9 < 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_1(T_1),\mathrm{TRC}_4(T_0),\mathrm{TRC}_5(T_5)\}$ and $T^* = T_1$ or T_0 or T_5. Go to Step 5.
(4) If, $\Delta_7 < 0$, $\Delta_8 \geqslant 0$ and $\Delta_9 \geqslant 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_1(T_1),\mathrm{TRC}_4(T_4),\mathrm{TRC}_5(T_5)\}$ and $T^* = T_1$ or T_4 or T_5. Go to Step 5.
(5) If $\Delta_5 < 0$ and $\Delta_7 \geqslant 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_1(T_1),\mathrm{TRC}_4(T_4),\mathrm{TRC}_5(T_0)\}$ and $T^* = T_1$ or T_4 or T_0. Go to Step 5.
(6) If $\Delta_5 \geqslant 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_1(T_1), \mathrm{TRC}_3(T_3),\mathrm{TRC}_5(T_0)\}$ and $T^* = T_1$ or T_3 or T_0. Go to Step 5.

Step 4-2

(1) If $\Delta_8 < 0$ and $\Delta_9 < 0$, then $\mathrm{TRC}(T^*) = \min\ \{\mathrm{TRC}_1(T_W),\mathrm{TRC}_4(T_0)\}$ and $T^* = T_W$ or T_0. Go to Step 5.
(2) If $\Delta_8 < 0$ and $\Delta_9 \geqslant 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_1(T_W),\mathrm{TRC}_4(T_4)\}$ and $T^* = T_W$ or T_4. Go to Step 5.
(3) If $\Delta_8 \geqslant 0$ and $\Delta_9 < 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_1(T_W),\mathrm{TRC}_4(T_0),\mathrm{TRC}_5(T_5)\}$ and $T^* = T_W$ or T_0 or T_5. Go to Step 5.
(4) If $\Delta_7 < 0$, $\Delta_8 \geqslant 0$ and $\Delta_9 \geqslant 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_1(T_W),\mathrm{TRC}_4(T_4),\mathrm{TRC}_5(T_5)\}$ and $T^* = T_W$ or T_4 or T_5. Go to Step 5.
(5) If $\Delta_5 < 0$ and $\Delta_7 \geqslant 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_1(T_W),\mathrm{TRC}_4(T_4),\mathrm{TRC}_5(T_0)\}$ and $T^* = T_W$ or T_4 or T_0. Go to Step 5.
(6) If $\Delta_5 \geqslant 0$, then $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_1(T_W), \mathrm{TRC}_3(T_3),\mathrm{TRC}_5(T_0)\}$ and $T^* = T_W$ or T_3 or T_0. Go to Step 5.

Step 5 Stop.

6. Numerical examples

To illustrate the results, we use the same numerical example as shown in Huang (2007). However, we assume that the selling price per unit p is \$50 and deteriorating rate $\theta = 0.05$.

Example. Given$A = \$50$/order, $D = 1000$ unit/year, $h = \$5$/unit/year, $I_k = \$\ 0.1/\$/\text{year}$, $I_e = \$0.07/\$/\text{year}$, and $M = 0.12$ year, we obtain the optimal cycle time and the optimal order quantity for different parameters of $\alpha(0.2, 0.5, 0.8)$, $W(50, 150, 250)$ and $c(10, 20, 30)$ as shown in Table 2.

Based on the computational results as shown in Table 2, we can obtain the following managerial insights:

(1) If the retailer's optimal order quantity is less than W (i.e., the partially delayed payment is permitted) and the fraction of the delay payments permitted α is increasing, then the optimal replenishment cycle time T^* and order quantity Q^* will be increasing while the optimal annual total relevant cost $\mathrm{TRC}(T^*)$ will be decreasing.
(2) When the quantity at which the fully delayed payment is permitted per order W increases, the retailer should take the partially delayed payment (i.e., the optimal order quantity $Q^* < W$) instead of the fully delayed payment (i.e., $Q^* \geqslant W$).
(3) In general, when the purchasing cost per unit c increases, then the optimal replenishment cycle time T^* and order quantity Q^* will be decreasing while the optimal annual total relevant cost $\mathrm{TRC}(T^*)$ will be increasing. However, for the case with $\alpha = 0.2$ and $W = 150$ in the numerical example, the optimal replenishment cycle and order quantity are fixed and are not affected by the increase in the unit purchase price. The reason is that in this situation, the retailer

Table 2
Optimal solutions under different parametric values

α	W	c	T^*	Q^*	$\mathrm{TRC}(T^*)$
0.2	50	10	$T_2 = 0.1053$	105.574	529.193
		20	$T_2 = 0.1025$	102.750	555.206
		30	$T_2 = 0.0999$	100.142	580.542
	150	10	$T_W = 0.1494$	150.000	581.840
		20	$T_W = 0.1494$	150.000	621.195
		30	$T_W = 0.1494$	150.000	661.550
	250	10	$T_3 = 0.1051$	105.327	598.600
		20	$T_3 = 0.1016$	101.886	697.827
		30	$T_3 = 0.0982$	98.392	799.836
0.5	50	10	$T_2 = 0.1053$	105.574	529.193
		20	$T_2 = 0.1025$	102.750	555.206
		30	$T_2 = 0.0999$	100.142	580.542
	150	10	$T_3 = 0.1052$	105.473	572.097
		20	$T_W = 0.1494$	150.000	621.195
		30	$T_W = 0.1494$	150.000	661.550
	250	10	$T_3 = 0.1052$	105.473	572.097
		20	$T_3 = 0.1016$	101.886	697.827
		30	$T_3 = 0.0992$	99.435	713.608
0.8	50	10	$T_2 = 0.1053$	105.574	529.193
		20	$T_2 = 0.1025$	102.750	555.206
		30	$T_2 = 0.0999$	100.142	580.542
	150	10	$T_3 = 0.1053$	105.555	546.164
		20	$T_3 = 0.1024$	102.689	589.386
		30	$T_3 = 0.0998$	100.020	632.151
	250	10	$T_3 = 0.1053$	105.555	546.164
		20	$T_3 = 0.1024$	102.689	589.386
		30	$T_3 = 0.0998$	100.020	632.151

trades off the benefits of full delay in payment against the partial delay in payment and always enjoys the full delay in payment.

7. Conclusion

In this paper, we developed an EOQ model under the conditions of permissible delay in payment by considering the following situations simultaneously: (1) the retailer's selling price per unit is higher than the unit purchase price, (2) the interest rate charged by a bank is not necessarily higher than the retailer's investment return rate, (3) many selling items deteriorate continuously such as fresh fruits and vegetables, and (4) the supplier may offer a partial permissible delay in payments even if the order quantity is less than W. Furthermore, we established several theoretical results which are given as Theorems 1–3 to determine the optimal solution under various conditions. Finally, a numerical example is given to illustrate the theoretical results and obtain some managerial insights. Our model is in general framework that includes numerous previous models such as Goyal (1985), Aggarwal and Jaggi (1995), Teng (2002), Chang et al. (2003), Chung and Liao (2004), Huang (2007) and Teng et al. (2007) as special cases. We believe that our work will provide a basic foundation for the further study of this kind of important inventory models with trade credits.

The proposed model can be extended in several ways. For instance, we may extend the model for deteriorating items with a two-parameter Weibull distribution. Also, we could consider the demand as a function of selling price as well as quality such as in Teng et al. (2005, 2006). Finally, we could generalize the model to allow for shortages, quantity discounts, discount and inflation rates, and others.

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Appendix A. Proof of Lemma 1

To prove Lemma 1, we set

$$F_1(T) = \frac{(c\theta+h)D}{\theta^2}(\theta T e^{\theta T} - e^{\theta T} + 1) + \frac{cI_k D}{\theta^2}[\theta T e^{\theta(T-M)} - e^{\theta(T-M)} + 1] + \frac{pI_e DM^2}{2} - \frac{cI_k DM}{\theta} - A, \quad T \in [M, \infty). \tag{A1}$$

Taking the derivative of $F_1(T)$ with respect to $T \in (M,\infty)$, we get

$$\frac{dF_1(T)}{dT} = [(c\theta+h)e^{\theta T} + cI_k e^{\theta(T-M)}]DT > 0, \tag{A2}$$

Thus $F_1(T)$ is a strictly increasing function of T in the interval $[M,\infty)$. Moreover, from (A1), we know that

$$F_1(M) = \Delta_1, \quad \text{and} \quad \lim_{T\to\infty} F_1(T) = +\infty. \tag{A3}$$

Therefore, if $F_1(M) = \Delta_1 \leqslant 0$, then by applying the Intermediate Value Theorem, there exists a unique $T_1 \in [M,\infty)$ such that $F_1(T_1) = 0$. Furthermore, taking the second -order derivative of $\mathrm{TRC}_1(T)$ with respect to T at the point T_1 we have

$$\left.\frac{d^2\mathrm{TRC}_1(T)}{dT^2}\right|_{T=T_1} = \frac{D}{T_1}[(c\theta+h)e^{\theta T_1} + cI_k e^{\theta(T_1-M)}] > 0. \tag{A4}$$

Thus, $T_1 \in [M,\infty)$ is the unique minimum solution to $\mathrm{TRC}_1(T)$.

On the other hand, if $F_1(M) = \Delta_1 > 0$ then we have $F_1(T) > 0$ for all $T \in [M,\infty)$. Consequently, we know that $\frac{d\mathrm{TRC}_1(T)}{dT} = \frac{F_1(T)}{T^2} > 0$ for all $T \in (M,\infty)$. Thus, $\mathrm{TRC}_1(T)$ is a strictly increasing function of T in the interval $[M,\infty)$. Therefore, $\mathrm{TRC}_1(T)$ has a minimum value at the boundary point $T = M$. This completes the proof. □

Appendix B. Proof of Lemma 3

To prove Lemma 3, we set

$$\begin{aligned} F_3(T) = {} & \frac{(c\theta+h)D}{\theta^2}(\theta T e^{\theta T} - e^{\theta T} + 1) + \frac{cI_k(c/p)(1-\alpha)^2 D}{2\theta^2}(e^{\theta T} - 1)(2\theta T e^{\theta T} - e^{\theta T} + 1) - \frac{pI_e D}{2\theta}[T - (1-\alpha)(c/p) \\ & \times (e^{\theta T} - 1)/\theta][\theta T - (1-\alpha)(c/p)(2\theta T e^{\theta T} - e^{\theta T} + 1)] + \frac{pI_e D}{\theta}\{\theta T^2[1 - (1-\alpha)(c/p)e^{\theta T}] + (1-\alpha) \\ & \times (c/p)M(\theta T e^{\theta T} - e^{\theta T} + 1)\} - A. \end{aligned} \tag{B1}$$

Taking the first derivative of $F_3(T)$ with respect to $T \in (0, T_W)$, we have

$$\frac{dF_3(T)}{dT} = \{(c\theta+h)e^{\theta T} + (cI_k - cI_e)(c/p)(1-\alpha)^2 e^{\theta T}(2e^{\theta T} - 1) + pI_e[1 + (1-\alpha)(c/p)e^{\theta T}\theta M]\}DT. \tag{B2}$$

Since

$$\begin{aligned} & (c\theta+h)e^{\theta T} + (cI_k - cI_e)(c/p)(1-\alpha)^2 e^{\theta T}(2e^{\theta T} - 1) + pI_e[1 + (1-\alpha)(c/p)e^{\theta T}\theta M] \\ & \quad \geqslant (c\theta+h) + (cI_k - cI_e)(c/p)(1-\alpha)^2 + pI_e[1 + (1-\alpha)(c/p)\theta M] \\ & \quad = (c\theta+h) + cI_k(c/p)(1-\alpha)^2 + pI_e - cI_e(c/p)(1-\alpha)^2 + pI_e(1-\alpha)(c/p)\theta M] \\ & \quad > (c\theta+h) + cI_k(c/p)(1-\alpha)^2 + pI_e - cI_e + pI_e(1-\alpha)(c/p)\theta M] > 0, \end{aligned}$$

we obtain $\frac{dF_3(T)}{dT} > 0$, which implies $F_3(T)$ is a strictly increasing function of T in the interval $(0, T_W)$. Moreover, from (B1), we know that

$$\lim_{T\to 0} F_3(T) = -A < 0, \quad \text{and} \quad \lim_{T\to T_W-} F_3(T) = \Delta_3. \tag{B3}$$

Therefore, if $\lim_{T\to T_W-} F_3(T) = \Delta_3 \geqslant 0$, then by applying the Intermediate Value Theorem, there exists a unique $T_3 \in (0, T_W)$ such that $F_3(T) = 0$. Furthermore, taking the second derivative of $\mathrm{TRC}_3(T)$ with respect to T at the point T_3, we have

$$\left.\frac{d^2\mathrm{TRC}_3(T)}{dT^2}\right|_{T=T_3} = \frac{D}{T_3}\{(c\theta+h)e^{\theta T_3} + (cI_k - cI_e)(c/p)(1-\alpha)^2 e^{\theta T_3}(2e^{\theta T_3} - 1) + pI_e[1 + (1-\alpha)(c/p)e^{\theta T_3}\theta M]\} > 0. \tag{B4}$$

Thus, $T_3 \in (0, T_W)$ is the unique minimum solution to $\mathrm{TRC}_3(T)$.

However, if $\lim_{T \to T_W -} F_3(T) = \Delta_3 < 0$, then $F_3(T) < 0$ for all $T \in (0, T_W)$. Consequently, we have $\frac{\mathrm{dTRC}_3(T)}{\mathrm{d}T} = \frac{F_3(T)}{T^2} < 0$ for all $T \in (0, T_W)$. Thus $\mathrm{TRC}_3(T)$ is a strictly decreasing function of T in the open interval $(0, T_W)$. Therefore, we cannot find a value of T in the open interval $(0, T_W)$ that minimizes $\mathrm{TRC}_3(T)$. This completes the proof. $\square$

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