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The optimal pricing and ordering policy for an integrated inventory model when trade credit linked to order quantity

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ABSTRACT

In traditional inventory models, it is implicitly assumed that the buyer must pay for the purchased items as soon as they have been received. However, in many practical situations, the vendor is willing to provide the buyer with a permissible delay period when the buyer's order quantity exceeds a given threshold. Therefore, to incorporate the concept of vendor–buyer integration and order-size-dependent trade credit, we present a stylized model to determine the optimal strategy for an integrated vendor–buyer inventory system under the condition of trade credit linked to the order quantity, where the demand rate is considered to be a decreasing function of the retail price. By analyzing the total channel profit function, we developed some useful results to characterize the optimal solution and provide an iterative algorithm to find the retail price, buyer's order quantity, and the numbers of shipment per production run from the vendor to the buyer. Numerical examples and sensitivity analysis are given to illustrate the theoretical results, and some managerial insights are also obtained.

1. Introduction

The traditional economic order quantity (EOQ) model assumes that the buyer must pay for the purchased items when these items are received. This is not always true in the actual business world. In fact, offering trade credit to buyers is a common strategy for vendors. Offering a credit period to the buyer will promote the vendor's sale and reduce the on-hand stock level. Simultaneously gaining capital, materials and service without a primary payment, the buyer can take advantage of a credit period to reduce cost. In the literature, several authors have examined the credit terms and proposed various analytical models to obtain more insights into trade credit and inventory policy. Goyal [1] was the first to establish an EOQ model with a constant demand rate under the condition of permissible delay in payments. Aggarwal and Jaggi [2] extended Goyal's [1] model to include deteriorating items. Kim et al. [3] examined the effects of a credit period upon ordering policies from the view of the vendor. Jamal et al. [4] further generalized this issue with an allowable shortage. Teng [5] modified Goyal's [1] model by considering differences between the sales price and purchase cost, and found that the economic replenishment interval and order quantity decrease under the permissible delay in payments in certain cases. Many research articles dealing with the trade credit problem can be found in Arcelus et al. [6], Arcelus and Srinivasan [7,8], Biskup et al. [9], Salameh et al. [10] and Teng et al. [11,12], etc. All previous models implicitly assumed that credit terms are independent of the order quantity.

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In reality, vendors may offer favorable credit terms to encourage buyers to order large quantities. Khouja and Mehrez [13] were the first proponents to discuss the vendor credit policies on the EOQ model where credit terms are linked to the order quantity. Later, Chang et al. [14] established an EOQ model for deteriorating items, in which the vendor provides a permissible delay to the buyer if the order quantity is greater than or equal to a predetermined quantity. Chang [15] extended Chang et al.'s [14] model by considering the effect of the inflation rate and deterioration rate in an EOQ model when vendor credits linked to the order quantity. Other research articles dealing with the order-size-dependent trade credit problem can be found in Shinn and Hwang [16], Chung et al. [17], and Chung and Liao [18,19], etc.

Previous studies about trade credit [1–19] focused on determining optimal policy for the buyer or the vendor only. However, these one-sided optimal inventory models neglected the complicated interaction and cooperation opportunity between the buyer and the vendor. In practice, many companies learn that actions taken by one member of the chain can influence the success of all others in the same value chain. Recognizing this principle, the vendor and buyer may consider how to relieve the conflict relationship and attempt to become partners to create a win–win strategy. Goyal [20] first developed a single-vendor single-buyer integrated inventory model. Subsequently, Banerjee [21] extended Goyal's [20] model and assumed that the vendor followed a lot-for-lot shipment policy with respect to a buyer. Goyal [22] relaxed the lot-for-lot policy and illustrated that the inventory cost can be reduced significantly if the vendor's economic production quantity is an integer multiple of the buyer's purchase quantity. Many researchers (e.g., Lu [23], Goyal [24], Viswanathan [25], Hill [26,27], Kelle et al. [28], Yang and Wee [29], and Wee and Chung [30]) continued to propose more batching and shipping policies for an integrated inventory model. These researches on the integrated vendor–buyer inventory problem considered the production–distribution schedule in terms of the number and size of batches transferred between both parties. Lately, some researches discussed the impact of delay payment strategy on the integrated inventory models. Abad and Jaggi [31] provided a seller–buyer integrated inventory model under trade credit and followed a lot-for-lot shipment policy. Jaber and Osman [32] proposed a supplier–retailer supply chain model in which the permissible delay in payments is considered as a decision variable. Yang and Wee [33] developed a vendor–buyer integrated inventory model for deteriorating items with permissible delay in payment. Ho et al. [34] investigated the production and ordering policy under a two-part trade credit in an integrated supplier–buyer inventory model. Given above, the order-size-dependent trade credit policy has not been examined in an integrated inventory model.

On the other hand, in the classical inventory models the replenishment rate or production rate is often assumed to be constant. However, it has been observed that the production rate is flexible in many practical situations. Silver [35] discussed the effects of slowing down production rate in saving potential costs under controllable production rates. Schweitzer and Seidmann [36] provided the concept of flexibility in production rate and discussed processing rate optimization for a flexible manufacturing system. Bhunia and Maiti [37] presented two inventory models in which the production rate depends upon the on-hand inventory for the first model and upon the demand for the second one. Manna and Chaudhuri [38] discussed an EOQ model with deteriorating items in which the production rate is proportional to the time dependent demand rate.

In this article, extending the previous studies contributed by Khouja and Mehrez [13], Abad and Jaggi [31], and Teng et al. [11], we develop an integrated vendor–buyer inventory model taking into account the following factors: (i) the demand rate is retail price dependent; (ii) the production rate is finite and proportional to the demand rate; (iii) the credit terms are linked to the order quantity. To optimize the joint total profit per unit time, the retail price and two basic issues will be determined in this study. These issues are how large should the replenishment order be, and how often should the vendor ship to the buyer during a production run. An interactive procedure is developed to help determine the optimal solution. Finally, numerical examples and sensitivity analysis are presented to illustrate the proposed model.

2. Notation and assumptions

The following notation is adopted throughout this paper.

R	production rate
S_V	vendor's setup cost per setup
S_B	buyer's ordering cost per order
c	production cost per unit
v	the unit price charged by the vendor to the buyer
p	the unit retail price to customers, where $p > v > c$
$D(p)$	the market demand rate for the item is a decreasing function of the retail price p and is given by $D(p) = ap^{-\delta}$, where $a > 0$ is a scaling factor, and $\delta > 1$ is a price-elasticity coefficient. For notational simplicity, $D(p)$ and D will be used interchangeably
r_V	vendor's holding cost rate, excluding interest charges
r_B	buyer's holding cost rate, excluding interest charges
I_{Vp}	vendor's opportunity cost per dollar per unit time
I_{Bp}	buyer's opportunity cost per dollar per unit time
I_{Be}	buyer's interest earned per dollar per unit time
ρ	the capacity utilization, $\rho = D/R$, where $\rho < 1$ and given
M	credit period offered to the buyer per order
Q	buyer's order quantity per order

Q_d the minimum order quantity at which the delay in payments is permitted
T replenishment cycle length, where $T = Q/D$
T_d the time length that Q_d units are depleted to zero
n number of shipments from the vendor to the buyer per production run, a positive integer
TVP vendor's total profit per unit time
TBP buyer's total profit per unit time

In addition, the following assumptions are made in deriving the model:

1. There is single-vendor and single-buyer for a single product in this model.
2. Shortages are not permitted.
3. The vendor sets a threshold Q_d for offering delay payment. If the buyer's order exceeds or equal to Q_d, the buyer will obtain a credit period M. Otherwise, the buyer must pay for the items immediately upon receiving them.
4. During the credit period, the buyer sells the items and uses the sales revenue to earn interest at a rate of I_{Be}. At the end of the permissible delay period, the buyer pays the purchasing cost to the vendor and incurs an opportunity cost at a rate of I_{Bp} for the items in stock.
5. The buyer orders Q units for each order and incurs an ordering cost S_B. The vendor manufactures, at rate R, in batches of size nQ and incurs a batch setup cost S_V. Each batch is dispatched to the buyer in n equal sized shipments.

3. Model formulation

In this section, we formulate an integrated inventory model with a retail price sensitive demand, where the delay in payments is only permitted if the order quantity is greater than or equal to a predetermined quantity. The inventory holding cost in our model contains two components: unit holding cost and interest charge. The unit holding cost relates to the actual ownership of the goods and includes storage and maintenance expenses, which is accounted on a per-unit-of-inventory basis. The interest charge is considered on the money value of the inventory on hand.

3.1. Vendor's total profit per unit time

For the vendor, the total profit per unit time is composed of sales profit, setup cost, holding cost, and opportunity cost. These components are evaluated as following:

(1) *Sales profit*: The total sales profit per unit time is given by $(v - c)D$.
(2) *Setup cost*: The vendor manufactures nQ in one production run. The cycle length is $nQ/D = nT$. Therefore, the setup cost per unit time is $S_V/(nT)$.
(3) *Holding cost*: The vendor's inventory per cycle can be calculated by subtracting the buyer's accumulated inventory level from the vendor's accumulated inventory level. Hence, the vendor's average inventory per unit time is given by

$$\left\{nQ\left[\frac{Q}{R}+(n-1)\frac{Q}{D}\right]-\frac{n^2Q^2}{2R}-\frac{Q^2}{D}[1+2+\cdots+(n-1)]\right\}\Big/\frac{nQ}{D}=\frac{Q}{2R}[(n-1)(R-D)+D]$$
$$=\frac{DT}{2}[(n-1)(1-\rho)+\rho],\quad \text{where}\quad \rho=D/R.$$

So the holding cost per unit time is $c(r_V + I_{Vp})DT[(n-1)(1-\rho)+\rho]/2$. Note that a similar derivation in the vendor's average inventory using a manufacturing lot size of nQ units can be found in Joglekar [39].
(4) *Opportunity cost*: If the buyer orders up to Q_d or more, then a credit period M will be offered. Under this situation, the vendor endures a capital opportunity cost within the time gap between delivery and payment received of the product. That is, when $T \geqslant T_d$, the delay in payments is permitted; the opportunity cost per unit time for offering trade credit is $vI_{Vp}DM$. Conversely, when $T < T_d$, no opportunity cost will occur since a delay in payments is not permitted.

Consequently, the total profit per unit time for the vendor is the sales profit minus the total relevant costs, which can be expressed as following:

$$TVP(n)=\begin{cases}TVP_1(n), & T<T_d\\ TVP_2(n), & T\geqslant T_d\end{cases}, \tag{1}$$

where

$$TVP_1(n)=(v-c)D-\frac{S_V}{nT}-\frac{c(r_V+I_{Vp})DT}{2}[(n-1)(1-\rho)+\rho], \tag{2}$$
$$TVP_2(n)=(v-c)D-\frac{S_V}{nT}-\frac{c(r_V+I_{Vp})DT}{2}[(n-1)(1-\rho)+\rho]-vI_{Vp}DM. \tag{3}$$

3.2. Buyer's total profit per unit time

For the buyer, the total profit per unit time is composed of sales profit, ordering cost, holding cost, opportunity cost, and interest earned. These components are evaluated as following:

(1) *Sales profit*: The total sales profit per unit time is given by $(p - v)D$.
(2) *Ordering cost*: The ordering cost per unit time is S_B/T.
(3) *Holding cost*: With the unit purchasing cost v, the holding cost rate r_B and the average inventory over the cycle $Q/2$, the buyer's holding cost (excluding interest charges) per unit time is expressed as $vr_BQ/2 = vr_BDT/2$.
(4) *Opportunity cost*: Based on the values of T, M and T_d, there are four cases to be considered: (i) $0 < T < T_d$, (ii) $T_d \leqslant T \leqslant M$, (iii) $T_d \leqslant M \leqslant T$ and (iv) $M \leqslant T_d \leqslant T$.

Case 1. $0 < T < T_d$

When buyer's order quantity Q less than Q_d (i.e., $T < T_d$), buyer must pay the purchasing cost as soon as the items are received. The opportunity cost per unit time for these items is $vI_{Bp}DT/2$.

Case 2. $T_d \leqslant T \leqslant M$

When buyer's order quantity Q greater than or equal to Q_d (i.e., $T \geqslant T_d$), then a credit period M will be offered. If the permissible payment time expires on or after the inventory is completely depleted (i.e. $M \geqslant T$), the buyer pays no opportunity cost for the purchased items.

Case 3. $T_d \leqslant M \leqslant T$, shown in Fig. 1.

When buyer's order quantity greater than or equal to Q_d (i.e., $T \geqslant T_d$) and permissible payment time expires on or before the inventory is depleted completely (i.e. $M \leqslant T$), the buyer still has some inventory on hand when paying the total purchasing amount to the vendor. Hence, for the items still in stock, buyer endures a capital opportunity cost at a rate of I_{Bp}; the opportunity cost per unit time for these items is $\frac{vI_{Bp}}{T}\int_M^T D(T-t)dt = \frac{vI_{Bp}D(T-M)^2}{2T}$.

Case 4. $M \leqslant T_d \leqslant T$

Case 4 is similar to Case 3 because T is also greater than or equal to M.

(5) *Interest earned*: Same as the opportunity cost, there are four cases to be considered in terms of the interest.

Case 1. $0 < T < T_d$

In this case, buyer pays the purchasing cost when the items are received, and hence, no interest is earned.

Case 2. $T_d \leqslant T \leqslant M$, shown in Fig. 2.

When order quantity greater than or equal to Q_d, the vendor offers a credit period M without interest charged to the buyer. During the credit period, buyer sells the products and uses the sales revenue to earn interest at a rate of I_{Be}. Thus, the interest earned per unit time is

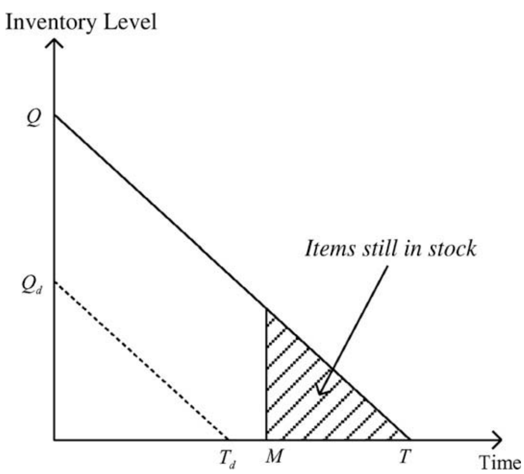

Fig. 1. Buyer's inventory model when $T_d \leqslant M \leqslant T$.

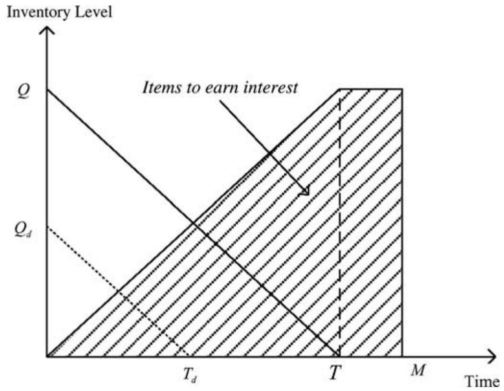

Fig. 2. Buyer's inventory model of interest earned when $T_d \leqslant T \leqslant M$.

$$\frac{1}{T}\left[pI_{Be}\int_0^T Dt\,dt + pI_{Be}DT(M-T)\right] = DpI_{Be}\left(M-\frac{T}{2}\right).$$

Case 3. $T_d \leqslant M \leqslant T$, shown in Fig. 3.

In this case, the buyer can sell the items and earn interest with rate I_{Be} until the end of the credit period M. Thus, the interest earned per unit time is $\frac{pI_{Be}}{T}\int_0^M Dt\,dt = \frac{DpI_{Be}M^2}{2T}$.

Case 4. $M \leqslant T_d \leqslant T$

In this case, M is less than or equal to T. Thus, Case 4 is similar to Case 3.

The total profit per unit time for the buyer is the sales profit plus the interest earned, minus the total relevant costs. From above arguments, we have

$$TBP(p,T) = \begin{cases} TBP_1(p,T), & 0 < T < T_d, \\ TBP_2(p,T), & T_d \leqslant T \leqslant M, \\ TBP_3(p,T), & T_d \leqslant M \leqslant T, \\ TBP_4(p,T), & M \leqslant T_d \leqslant T, \end{cases} \tag{4}$$

where

$$TBP_1(p,T) = D\left[p - v - \frac{v(r_B + I_{Bp})T}{2}\right] - \frac{S_B}{T}, \tag{5}$$

$$TBP_2(p,T) = D\left[p - v - \frac{vr_BT}{2} + pI_{Be}\left(M - \frac{T}{2}\right)\right] - \frac{S_B}{T}, \tag{6}$$

$$TBP_3(p,T) = TBP_4(p,T) = D\left[p - v - \frac{vr_BT}{2} + \frac{pI_{Be}M^2}{2T} - \frac{vI_{Bp}(T-M)^2}{2T}\right] - \frac{S_B}{T}. \tag{7}$$

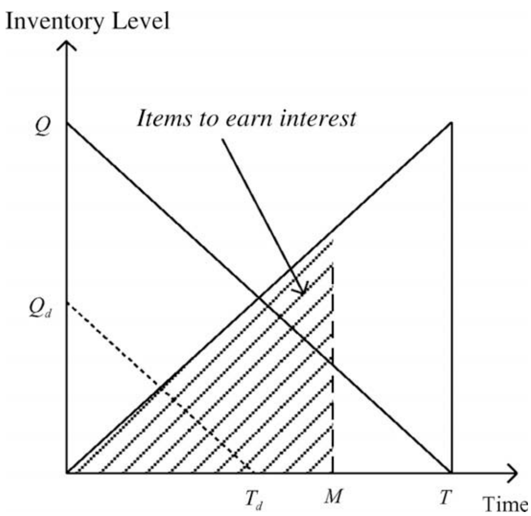

Fig. 3. Buyer's inventory model of interest earned when $T_d \leqslant M \leqslant T$.

3.3. The joint total profit per unit time

Once the vendor and the buyer have established a long-term strategic partnership and contracted to the relationship, they will jointly determine the best policy for the entire supply chain system. Under this circumstance, the joint total profit per unit time for the integrated system is

$$\Pi(n,p,T)=\begin{cases}\Pi_1(n,p,T)=TVP_1(n)+TBP_1(p,T), & 0<T<T_d,\\ \Pi_2(n,p,T)=TVP_2(n)+TBP_2(p,T), & T_d\leqslant T\leqslant M,\\ \Pi_3(n,p,T)=TVP_2(n)+TBP_3(p,T), & T_d\leqslant M\leqslant T,\\ \Pi_4(n,p,T)=TVP_2(n)+TBP_3(p,T), & M\leqslant T_d\leqslant T.\end{cases} \tag{8}$$

For notational convenience, let $\overline{S}=\frac{S_V}{n}+S_B$ and $\varphi=c(r_V+I_{Vp})[(n-1)(1-\rho)+\rho]$, then

$$\Pi_1(n,p,T)=-\frac{\overline{S}}{T}+D\left\{p-c-\frac{T}{2}[\nu(r_B+I_{Bp})+\varphi]\right\}, \tag{9}$$

$$\Pi_2(n,p,T)=-\frac{\overline{S}}{T}+D\left[p-c+(pI_{Be}-\nu I_{Vp})M-\frac{T}{2}(\nu r_B+pI_{Be}+\varphi)\right], \tag{10}$$

$$\Pi_3(n,p,T)=\Pi_4(n,p,T)=-\frac{\overline{S}}{T}+D\left\{p-c+\nu(I_{Bp}-I_{Vp})M-\frac{(\nu I_{Bp}-pI_{Be})M^2}{2T}-\frac{T}{2}[\nu(r_B+I_{Bp})+\varphi]\right\}. \tag{11}$$

4. Solution procedure

To find the optimal solution, say (n^*,p^*,T^*), for $\Pi(n,p,T)$, the following procedures are taken. First, for fixed p and T, we note that $\Pi(n,p,T)$ is a concave function of n (by the fact $\partial^2\Pi(n,p,T)/\partial n^2=-2S_V/(n^3T)<0$). Therefore, the search for the optimal shipment number, n^*, is reduced to find a local optimal solution.

Case 1. $0<T<T_d$

For fixed n and p, the first and second order partial derivatives of $\Pi_1(n,p,T)$ in (9) with respect to (w.r.t.) T are as following:

$$\frac{\partial\Pi_1(n,p,T)}{\partial T}=\frac{\overline{S}}{T^2}-\frac{D}{2}[\nu(r_B+I_{Bp})+\varphi], \tag{12}$$

$$\frac{\partial^2\Pi_1(n,p,T)}{\partial T^2}=-\frac{2\overline{S}}{T^3}<0. \tag{13}$$

Hence, for fixed n and p, $\Pi_1(n,p,T)$ is concave in T. Thus, a unique value T exists (denoted by $T_1(n,p)$) which maximizes $\Pi_1(n,p,T)$. $T_1(n,p)$ can be obtained by solving the first order necessary condition (FONC), i.e., $\partial\Pi_1(n,p,T)/\partial T=0$, and is given by

$$T_1(n,p)=\sqrt{\frac{2\overline{S}}{D[\nu(r_B+I_{Bp})+\varphi]}}, \\ =\sqrt{\frac{2(S_V/n+S_B)}{ap^{-\delta}\{\nu(r_B+I_{Bp})+c(r_V+I_{Vp})[(n-1)(1-\rho)+\rho]\}}}. \tag{14}$$

To ensure $T_1(n,p)<T_d$, substituting (14) into this inequality, we have

$$2\overline{S}<D[\nu(r_B+I_{Bp})+\varphi]T_d^2 \text{ if and only if } T_1(n,p)<T_d. \tag{15}$$

Next, motivated by (15), we let $f_1(p)=D[\nu(r_B+I_{Bp})+\varphi]T_d^2=[\nu(r_B+I_{Bp})+\varphi]\frac{Q_d^2}{ap^{-\delta}}$. It can be proved that $f_1(p)$ is an increasing function of p. Furthermore, by the fact that $\lim_{p\to 0^+}f_1(p)=0$ and $\lim_{p\to\infty}f_1(p)=\infty$, we can find a unique value $\hat{p}_1$ such that $f_1(\hat{p}_1)=2\overline{S}$, that is

$$2\overline{S}=[\nu(r_B+I_{Bp})+\varphi]\frac{Q_d^2}{a\hat{p}_1^{-\delta}}. \tag{16}$$

Thus, we have

$$2\overline{S}<D[\nu(r_B+I_{Bp})+\varphi]T_d^2 \text{ if and only if } p>\hat{p}_1. \tag{17}$$

Therefore, the following result can be obtained.

Lemma 1. *For any given n and p,*

(i) *If $2\overline{S}<D[\nu(r_B+I_{Bp})+\varphi]T_d^2$, or equivalently $p>\hat{p}_1$, then the optimal replenishment cycle length is $T_1(n,p)$.*
(ii) *If $2\overline{S}\geqslant D[\nu(r_B+I_{Bp})+\varphi]T_d^2$, or equivalently $p\leqslant\hat{p}_1$, then the optimal replenishment cycle length is T_d^-.*

Proof

(i) It immediately follows from (15) and (17).
(ii) If $2\overline{S} \geqslant D[v(r_B + I_{Bp}) + \varphi]T_d^2$, from (12), we obtain

$$\frac{\partial \Pi_1(n,p,T)}{\partial T} \geqslant \frac{D[v(r_B + I_{Bp}) + \varphi]T_d^2}{2T^2} - \frac{D}{2}[v(r_B + I_{Bp}) + \varphi], = \frac{D[v(r_B + I_{Bp}) + \varphi]}{2}\left[\frac{T_d^2}{T^2} - 1\right] > 0,$$

because $0 < T < T_d$. Thus, $\Pi_1(n,p,T)$ is a strictly increasing function in $T \in (0,T_d)$. Therefore, $\Pi_1(n,p,T)$ has a maximum value at point $T = T_d^-$. □

Alternative 1-1. The buyer's optimal replenishment cycle length is $T_1(n,p)$

Substituting (14) into (9), we can get

$$\Pi_1(n,p) \equiv \Pi_1(n,p,T_1(n,p)) = ap^{-\delta}(p-c) - \sqrt{2ap^{-\delta}\left(\frac{S_V}{n} + S_B\right)\{v(r_B + I_{Bp}) + c(r_V + I_{Vp})[(n-1)(1-\rho) + \rho]\}}. \tag{18}$$

The optimal value of p (denoted as p_{11}) which maximizes $\Pi_1(n,p)$ in (18) should satisfy $p_{11} > \hat{p}_1$, the FONC, and the second order sufficient condition (SOSC) for concavity, i.e.

$$\frac{\partial \Pi_1(n,p)}{\partial p} = ap^{-\delta} - ap^{-\delta-1}\delta(p-c) + \frac{\delta\sqrt{ap^{-\delta}\overline{S}[v(r_B + I_{Bp}) + \varphi]}}{\sqrt{2}p} = 0, \tag{19}$$

and

$$\frac{\partial^2 \Pi_1(n,p)}{\partial p^2} = a\delta(\delta+1)p^{-\delta-2}(p-c) - 2a\delta p^{-\delta-1} - \frac{\delta(2+\delta)\sqrt{ap^{-\delta}\overline{S}[v(r_B + I_{Bp}) + \varphi]}}{2\sqrt{2}p^2} < 0. \tag{20}$$

Alternative 1-2. The buyer's optimal replenishment cycle length is T_d^-

Substituting $T = T_d^- = Q_d/(ap^{-\delta})$ into (9), we can obtain

$$\Pi_1(n,p) \equiv \Pi_1(n,p,Q_d/(ap^{-\delta})) = -\frac{ap^{-\delta}\overline{S}}{Q_d} + ap^{-\delta}\left\{p - c - \frac{Q_d}{2ap^{-\delta}}[v(r_B + I_{Bp}) + \varphi]\right\}. \tag{21}$$

The optimal value of p (denoted as p_{12}) which maximizes $\Pi_1(n,p)$ in (21) should satisfy $p_{12} \leqslant \hat{p}_1$, the FONC, and the SOSC for concavity, i.e.

$$\frac{\partial \Pi_1(n,p)}{\partial p} = ap^{-\delta-1}\delta\left(c + \frac{\overline{S}}{Q_d}\right) - ap^{-\delta}(\delta - 1) = 0, \tag{22}$$

and

$$\frac{\partial^2 \Pi_1(n,p)}{\partial p^2} = -ap^{-\delta-2}\delta(\delta+1)\left(c + \frac{\overline{S}}{Q_d}\right) + ap^{-\delta-1}(\delta-1)\delta < 0. \tag{23}$$

From above arguments, we obtain the following result.

Result 1. For any given n, if there exists a value p_{11} which satisfies (19), (20) and $p_{11} > \hat{p}_1$, then p_{11} is the optimal retail price such that $\Pi_1(n,p)$ in (18) has a maximum value. Otherwise, the maximum value of $\Pi_1(n,p,T)$ is $\Pi_1(n,p_{12})$ as shown in (21), and the optimal retail price is p_{12}, which satisfies (22) and (23) and $p_{12} \leqslant \hat{p}_1$.

The similar procedure as described in Case 1 can be applied to solve the remaining three cases.

Case 2. $T_d \leqslant T \leqslant M$

For fixed n and p, by taking the second order partial derivative of $\Pi_2(n,p,T)$ in Eq. (10) w.r.t. T, we have $\partial^2\Pi_2(n,p,T)/\partial T^2 = -2\overline{S}/T^3 < 0$. Therefore, $\Pi_2(n,p,T)$ is concave in T. Thus, there exists a unique value of T (denoted by $T_2(n,p)$) which maximizes $\Pi_2(n,p,T)$. Solving $\partial\Pi_2(n,p,T)/\partial T = 0$, we obtain

$$T_2(n,p) = \sqrt{\frac{2\overline{S}}{D(vr_B + pI_{Be} + \varphi)}} = \sqrt{\frac{2(S_V/n + S_B)}{ap^{-\delta}\{vr_B + pI_{Be} + c(r_V + I_{Vp})[(n-1)(1-\rho) + \rho]\}}}. \tag{24}$$

To ensure $T_d \leqslant T_2(n,p) \leqslant M$, substituting (24) into this inequality results in

$$D(\upsilon r_B + pI_{Be} + \varphi)T_d^2 \leqslant 2\bar{S} \leqslant D(\upsilon r_B + pI_{Be} + \varphi)M^2 \text{if and only if} T_d \leqslant T_2(n,p) \leqslant M. \tag{25}$$

Next, motivated by (25), let $f_{21}(p) = D(\upsilon r_B + pI_{Be} + \varphi)T_d^2 = (\upsilon r_B + pI_{Be} + \varphi)\frac{Q_d^2}{ap^{-\delta}}$ and $f_{22}(p) = D(\upsilon r_B + pI_{Be} + \varphi)M^2 = ap^{-\delta}(\upsilon r_B + pI_{Be} + \varphi)M^2$. It can be proved that $f_{21}(p)$ increases in p and $f_{22}(p)$ decreases in p; also, $\lim_{p\to\infty} f_{21}(p) = \lim_{p\to 0^+} f_{22}(p) = \infty$ and $\lim_{p\to 0^+} f_{21}(p) = \lim_{p\to\infty} f_{22}(p) = 0$. Hence, $\hat{p}_{21}$ and $\hat{p}_{22}$ exist such that $f_{21}(\hat{p}_{21}) = 2\bar{S}$ and $f_{22}(\hat{p}_{22}) = 2\bar{S}$, respectively. That is

$$2\bar{S} = (\upsilon r_B + \hat{p}_{21}I_{Be} + \varphi)\frac{Q_d^2}{a\hat{p}_{21}^{-\delta}}, \tag{26}$$

$$2\bar{S} = a\hat{p}_{22}^{-\delta}(\upsilon r_B + \hat{p}_{22}I_{Be} + \varphi)M^2. \tag{27}$$

Thus, we have

$$D(\upsilon r_B + pI_{Be} + \varphi)T_d^2 \leqslant 2\bar{S} \leqslant D(\upsilon r_B + pI_{Be} + \varphi)M^2 \text{ if and only if } p \leqslant Min\{\hat{p}_{21}, \hat{p}_{22}\}. \tag{28}$$

Therefore, we obtain the following result.

Lemma 2. *For any given n and p,*

1. *If $D(\upsilon r_B + pI_{Be} + \varphi)T_d^2 \leqslant 2\bar{S} \leqslant D(\upsilon r_B + pI_{Be} + \varphi)M^2$, or equivalently $p \leqslant Min\{\hat{p}_{21}, \hat{p}_{22}\}$, then the optimal replenishment cycle length is $T_2(n,p)$.*
2. *If $2\bar{S} < D(\upsilon r_B + pI_{Be} + \varphi)T_d^2$, or equivalently $p > \hat{p}_{21}$, then the optimal replenishment cycle length is T_d.*
3. *If $2\bar{S} > D(\upsilon r_B + pI_{Be} + \varphi)M^2$, or equivalently $p > \hat{p}_{22}$, then the optimal replenishment cycle length is M.*

Proof. The proof is similar to Lemma 1, we omit it here. □

Alternative 2-1. The buyer's optimal replenishment cycle length is $T_2(n,p)$

Substituting (24) into (10), we can get

$$\begin{aligned}\Pi_2(n,p) &\equiv \Pi_2(n,p,T_2(n,p)) \\ &= ap^{-\delta}[p - c + (pI_{Be} - \upsilon I_{Vp})M] - \sqrt{2ap^{-\delta}\bar{S}\{\upsilon r_B + pI_{Be} + c(r_V + I_{Vp})[(n-1)(1-\rho) + \rho]\}}.\end{aligned} \tag{29}$$

Alternative 2-2. The buyer's optimal replenishment cycle length is T_d.

Substituting $T = T_d = Q_d/(ap^{-\delta})$ into (10), we can obtain

$$\Pi_2(n,p) \equiv \Pi_2(n,p,Q_d/(ap^{-\delta})) = -\frac{ap^{-\delta}\bar{S}}{Q_d} + ap^{-\delta}\left[p - c + (pI_{Be} - \upsilon I_{Vp})M - \frac{Q_d}{2ap^{-\delta}}(\upsilon r_B + pI_{Bp} + \varphi)\right]. \tag{30}$$

Alternative 2-3. The buyer's optimal replenishment cycle length is M.

Substituting $T = M$ into (10), we get

$$\Pi_2(n,p) \equiv \Pi_2(n,p,M) = -\frac{\bar{S}}{M} + ap^{-\delta}\left[p - c + (pI_{Be} - \upsilon I_{Vp})M - \frac{M}{2}(\upsilon r_B + pI_{Be} + \varphi)\right]. \tag{31}$$

Similar to the arguments of Case 1, we obtain the following result.

Result 2. For any given n,

(i) If there exists a value p_{21} which determined by solving the FONC of $\Pi_2(n,p)$ in (29), and satisfies the corresponding SOSC as well as $p_{21} \leqslant Min\{\hat{p}_{21}, \hat{p}_{22}\}$, then p_{21} is the optimal retail price.
(ii) If there exists a value p_{22} which determined by solving the FONC of $\Pi_2(n,p)$ in (30), and satisfies the corresponding SOSC as well as $p_{22} > \hat{p}_{21}$, then p_{22} is the optimal retail price.
(iii) If there exists a value p_{23} which determined by solving the FONC of $\Pi_2(n,p)$ in (31), and satisfies the corresponding SOSC as well as $p_{23} > \hat{p}_{22}$, then p_{23} is the optimal retail price.

Case 3. $T_d \leqslant M \leqslant T$

Working on $\Pi_3(n,p,T)$ (Eq. (11)), for fixed n and p, by solving $\partial\Pi_3(n,p,T)/\partial T = 0$, we obtain the value of T (denoted by $T_3(n,p)$) which maximizes $\Pi_3(n,p,T)$ as follows:

$$T_3(n,p) = \sqrt{\frac{2\bar{S} + D(\upsilon I_{Bp} - pI_{Be})M^2}{D[\upsilon(r_B + I_{Bp}) + \varphi]}} = \sqrt{\frac{2(S_V/n + S_B) + ap^{-\delta}(\upsilon I_{Bp} - pI_{Be})M^2}{ap^{-\delta}\{\upsilon(r_B + I_{Bp}) + c(r_V + I_{Vp})[(n-1)(1-\rho) + \rho]\}}}. \tag{32}$$

To ensure $T_d \leqslant M \leqslant T_3(n,p)$, substituting (32) into this inequality results in

$$\text{if} \quad 2\overline{S} \geqslant D(\upsilon r_B + pI_{Be} + \varphi)M^2 \quad \text{and} \quad M \geqslant T_d, \quad \text{then} \quad T_d \leqslant M \leqslant T_3(n,p). \tag{33}$$

Note that when $2\overline{S} \geqslant D(\upsilon r_B + pI_{Be} + \varphi)M^2$ holds, then $2\overline{S} + D(\upsilon I_{Bp} - pI_{Be})M^2 > 0$ holds, which implies (32) is well-defined and $\partial^2\Pi_3(n,p,T)/\partial T^2 < 0$ (see Appendix A for the proof). As $T_d = Q_d/(ap^{-\delta})$, substituting it into $M \geqslant T_d$, we have $p \leqslant (aM/Q_d)^{1/\delta}$.

Furthermore, from the discussions in Case 2, we see that $f_{22}(p) = D(\upsilon r_B + pI_{Be} + \varphi)M^2$ is strictly decreasing in p, and a unique value $\hat{p}_{22}$ exists such that $f_{22}(\hat{p}_{22}) = 2\overline{S}$. Thus, we have

$$2\overline{S} \geqslant D(\upsilon r_B + pI_{Be} + \varphi)M^2 \quad \text{and} \quad M \geqslant T_d \quad \text{if and only if} \quad \hat{p}_{22} \leqslant p \leqslant \left(\frac{aM}{Q_d}\right)^{\frac{1}{\delta}}. \tag{34}$$

Therefore, the following result is obtained.

Lemma 3. *For any given n and p,*

(i) If $2\overline{S} \geqslant D(\upsilon r_B + pI_{Be} + \varphi)M^2$ and $M \geqslant T_d$ (or equivalently, $\hat{p}_{22} \leqslant p \leqslant (aM/Q_d)^{1/\delta}$), then the optimal replenishment cycle length is $T_3(n,p)$.
(ii) If $2\overline{S} < D(\upsilon r_B + pI_{Be} + \varphi)M^2$ and $M \geqslant T_d$ (or equivalently, $p < \hat{p}_{22}$ and $p \leqslant (aM/Q_d)^{1/\delta}$), then the optimal replenishment cycle length is M.

Proof. The proof is similar to Lemma 1, we omit it here. □

Alternative 3-1. The buyer's optimal replenishment cycle length is $T_3(n,p)$.
Substituting (32) into (11), we can obtain

$$\begin{aligned}\Pi_3(n,p) &\equiv \Pi_3(n,p,T_3(n,p)) \\ &= ap^{-\delta}[p - c + \upsilon(I_{Bp} - I_{Vp})M] - \sqrt{ap^{-\delta}\{\upsilon(r_B + I_{Bp}) + c(r_V + I_{Vp})[(n-1)(1-\rho) + \rho]q\}} \\ &\quad \times \sqrt{2\left(\frac{S_V}{n} + S_B\right) + D(\upsilon I_{Bp} - pI_{Be})M^2}.\end{aligned} \tag{35}$$

Alternative 3-2. The buyer's optimal replenishment cycle length is M.
Substituting $T = M$ into (11), we have

$$\Pi_3(n,p) \equiv \Pi_3(n,p,M) = -\frac{\overline{S}}{M} + ap^{-\delta}[p - c + \upsilon(I_{Bp} - I_{Vp})M] - \frac{ap^{-\delta}M}{2}(2\upsilon I_{Bp} - pI_{Be} + \upsilon r_B + \varphi). \tag{36}$$

Similar to the arguments of Case 1, we obtain the following result.

Result 3. For any given n,

(i) If there exists a value p_{31} which determined by solving the FONC of $\Pi_3(n,p)$ in (35), and satisfies the corresponding SOSC as well as $\hat{p}_{22} \leqslant p_{31} \leqslant (aM/Q_d)^{1/\delta}$, then p_{31} is the optimal retail price.
(ii) If there exists a value p_{32} which determined by solving the FONC of $\Pi_3(n,p)$ in (36), and satisfies the corresponding SOSC, $p_{32} \leqslant (aM/Q_d)^{1/\delta}$ and $p_{32} < \hat{p}_{22}$, then p_{32} is the optimal retail price.

Case 4. $M \leqslant T_d \leqslant T$

Since the total profit per unit time in Case 4 is the same as that in Case 3, the optimal value of T (denoted as $T_4(n,p)$) for Case 4 is hence the same as $T_3(n,p)$, i.e., $T_4(n,p) = T_3(n,p)$, as showed in (32). To ensure $M \leqslant T_d \leqslant T_4(n,p)$, substituting (32) for $T_4(n,p)$ into this inequality, we have

$$\text{if} \quad 2\overline{S} \geqslant D[\upsilon(r_B + I_{Bp}) + \varphi]T_d^2 + DM^2(pI_{Be} - \upsilon I_{Bp}) \quad \text{and} \quad T_d \geqslant M, \quad \text{then} \quad M \leqslant T_d \leqslant T_4(n,p). \tag{37}$$

Since $T_d = Q_d/(ap^{-\delta})$, substituting it into $M \leqslant T_d$, we have $p \geqslant (aM/Q_d)^{1/\delta}$. Let

$$f_4(p) = D[\upsilon(r_B + I_{Bp}) + \varphi]T_d^2 + DM^2(pI_{Be} - \upsilon I_{Bp}) = [\upsilon(r_B + I_{Bp}) + \varphi]\frac{Q_d^2}{ap^{-\delta}} + ap^{-\delta}M^2(pI_{Be} - \upsilon I_{Bp}). \tag{38}$$

$$\text{Then, } f_4'(p) \equiv \frac{df_4(p)}{dp} = \delta[\upsilon(r_B + I_{Bp}) + \varphi]\frac{Q_d^2}{Dp} + M^2D\left[I_{Be} + \frac{\delta}{p}(\upsilon I_{Bp} - pI_{Be})\right]. \tag{39}$$

Likewise, we can find a unique value $\hat{p}_4$ such that $f_4(\hat{p}_4) = 2\overline{S}$. That is

$$2\overline{S} = a\hat{p}_4^{-\delta}[\upsilon(r_B + I_{Bp}) + \varphi]T_d^2 + a\hat{p}_4^{-\delta}M^2(\hat{p}_4 I_{Be} - \upsilon I_{Bp}). \tag{40}$$

Thus, we know that

$$\begin{aligned}&\text{(i) when } f_4'(p)>0,\ \text{then } 2\overline{S}\geqslant D[v(r_B+I_{Bp})+\varphi]T_d^2+DM^2(pI_{Be}-vI_{Bp}) \text{ if and only if } p\leqslant \hat{p}_4;\\&\text{(ii) when } f_4'(p)<0,\ \text{then } 2\overline{S}\geqslant D[v(r_B+I_{Bp})+\varphi]T_d^2+DM^2(pI_{Be}-vI_{Bp}) \text{ if and only if } p\geqslant \hat{p}_4.\end{aligned} \tag{41}$$

Therefore, we have the following result.

Lemma 4. *For any given n and p,*

(i) *If* $2\overline{S}\geqslant D[v(r_B+I_{Bp})+\varphi]T_d^2+DM^2(pI_{Be}-vI_{Bp})$ *and* $T_d\geqslant M$ *(or equivalently,* $(aM/Q_d)^{1/\delta}\leqslant p\leqslant \hat{p}_4$ *when* $f_4'(p)>0$*, or* $p\geqslant Max\{\hat{p}_4,(aM/Q_d)^{1/\delta}\}$ *when* $f_4'(p)<0$*), then the optimal replenishment cycle length is* $T_4(n,p)$*.*
(ii) *If* $2\overline{S}<D[v(r_B+I_{Bp})+\varphi]T_d^2+DM^2(pI_{Be}-vI_{Bp})$ *and* $T_d\geqslant M$ *(or equivalently,* $p>\hat{p}_4$ *and* $p\geqslant (aM/Q_d)^{1/\delta}$ *when* $f_4'(p)>0$*, or* $(aM/Q_d)^{1/\delta}\leqslant p\leqslant \hat{p}_4$ *when* $f_4'(p)<0$*), then the optimal replenishment cycle length is* T_d*.*

Proof. The proof is similar to Lemma 1, we omit it here. □

Alternative 4-1. The buyer's optimal replenishment cycle length is $T_4(n,p)$*.*

The profit function $\Pi_4(n,p)\equiv\Pi_4(n,p,T_4(n,p))=\Pi_3(n,p)$ is shown in (35).

Alternative 4-2. The buyer's optimal replenishment cycle length is T_d*.*

Substituting $T=T_d=Q_d/(ap^{-\delta})$ into (11), we have

$$\begin{aligned}\Pi_4(n,p)&\equiv\Pi_4(n,p,Q_d/(ap^{-\delta}))\\&=-\frac{ap^{-\delta}\overline{S}}{Q_d}+ap^{-\delta}[p-c+v(I_{Bp}-I_{Vp})M]-ap^{-\delta}\left\{\frac{(vI_{Bp}-pI_{Be})M^2ap^{-\delta}}{2Q_d}+\frac{Q_d}{2ap^{-\delta}}[v(r_B+I_{Bp})+\varphi]\right\}.\end{aligned} \tag{42}$$

Therefore, the following result is obtained.

Result 4. For any given n, if there exists a value p_{41} which satisfies the FONC and SOSC with (i) $(aM/Q_d)^{1/\delta}\leqslant p_{41}\leqslant\hat{p}_4$ when $f_4'(p)>0$, or with (ii) $p_{41}\geqslant Max\{\hat{p}_4,(aM/Q_d)^{1/\delta}\}$ when $f_4'(p)<0$, then p_{41} is the optimal retail price such that $\Pi_4(n,p)$ as shown in (35) is maximized. Otherwise, $\Pi_4(n,p)$ in (42) is maximized at $p=p_{42}$ which satisfies the FONC, SOSC and the inequality $p_{42}\geqslant(aM/Q_d)^{1/\delta}$ and $p_{42}>\hat{p}_4$ when $f_4'(p)>0$, or $(aM/Q_d)^{1/\delta}\leqslant p_{42}<\hat{p}_4$ when $f_4'(p)<0$.

Summarizing the above arguments, we now establish the following algorithm to obtain the optimal solution (n^*,p^*,T^*).

Algorithm

Step 1. Set $n=1$.
Step 2. Determine $\hat{p}_1$, $\hat{p}_{21}$, $\hat{p}_{22}$ and $\hat{p}_4$ from (16), (26), (27) and (40), respectively.
Step 3. Find p_{11} which satisfies the FONC and SOSC of $\Pi_1(n,p)$ (Eq. (18)) w.r.t. p.
  Step 3-1. If $p_{11}>\hat{p}_1$, then determine $T_1(n,p_{11})$ by (14) and calculate $\Pi_1(n,p_1,T_1)=\Pi_1(n,p_{11},T_1(n,p_{11}))$ by (18), go to Step 4; otherwise, perform Step 3-2.
  Step 3-2. Find p_{12} which satisfies the FONC and SOSC of $\Pi_1(n,p)$ (Eq. (21)) w.r.t. p. If $p_{12}\leqslant\hat{p}_1$, then take $T_1=Q_d/(ap_{12}^{-\delta})$ and calculate $\Pi_1(n,p_1,T_1)=\Pi_1(n,p_{12},Q_d/(ap_{12}^{-\delta}))$ by (21); otherwise, set $\Pi_1(n,p_1,T_1)=0$.
Step 4. Find p_{21} which satisfies the FONC and SOSC of $\Pi_2(n,p)$ (Eq. (29)) w.r.t. p.
  Step 4-1. If $p_{21}\leqslant Min\{\hat{p}_{21},\hat{p}_{22}\}$, then determine $T_2(n,p_{21})$ by (24) and calculate $\Pi_2(n,p_2,T_2)=\Pi_2(n,p_{21},T_2(n,p_{21}))$ by (29), go to Step 5; otherwise, perform Steps 4-2.
  Step 4-2. Find p_{22} which satisfies the FONC and SOSC of $\Pi_2(n,p)$ (Eq. (30)) w.r.t. p. If $p_{22}>\hat{p}_{21}$, then take $T_2=Q_d/(ap_{22}^{-\delta})$ and calculate $\Pi_2(n,p_2,T_2)=\Pi_2(n,p_{22},Q_d/(ap_{22}^{-\delta}))$ by (30), go to Step 5; otherwise, perform Steps 4-3.
  Step 4-3. Find p_{23} which satisfies the FONC and SOSC of $\Pi_2(n,p)$ (Eq. (31)) w.r.t. p. If $p_{23}>\hat{p}_{22}$, then take $T_2=M$ and calculate $\Pi_2(n,p_2,T_2)=\Pi_2(n,p_{23},M)$ by (31); otherwise, set $\Pi_2(n,p_2,T_2)=0$.
Step 5. Find p_{31} which satisfies the FONC and SOSC of $\Pi_3(n,p)$ (Eq. (35)) w.r.t. p.
  Step 5-1. If $\hat{p}_{22}\leqslant p_{31}\leqslant(aM/Q_d)^{1/\delta}$, then determine $T_3(n,p_{31})$ by (32) and calculate $\Pi_3(n,p_3,T_3)=\Pi_3(n,p_{31},T_3(n,p_{31}))$ by (35), go to Step 6; otherwise, perform Step 5-2.
  Step 5-2. Find p_{32} which satisfies the FONC and SOSC of $\Pi_3(n,p)$ (Eq. (36)) w.r.t. p. If $p_{32}\leqslant Min\{(aM/Q_d)^{1/\delta},\hat{p}_{22}\}$, then take $T_3=M$ and calculate $\Pi_3(n,p_3,T_3)=\Pi_3(n,p_{32},M)$ by (36); otherwise, set $\Pi_3(n,p_3,T_3)=0$.
Step 6. Find p_{41} which satisfies the FONC and SOSC of $\Pi_4(n,p)\equiv\Pi_3(n,p)$ (Eq. (35)) w.r.t. p, and examine $f_4'(p)$ by (39). If $p_{41}\geqslant(aM/Q_d)^{1/\delta}$, perform Step 6-1; otherwise, perform Step 6-2.
  Step 6-1.
  (i) If $f_4'(p)>0$and $p_{41}\leqslant\hat{p}_4$, then determine $T_4(n,p_{41})=T_3(n,p_{41})$ by (32) and calculate $\Pi_4(n,p_4,T_4)=\Pi_3(n,p_{41},T_4(n,p_{41}))$ by (35). Go to Step 7.
  (ii) If $f_4'(p)<0$ and $p_{41}\geqslant\hat{p}_4$, then determine $T_4(n,p_{41})=T_3(n,p_{41})$ by (32) and calculate $\Pi_4(n,p_4,T_4)=\Pi_3(n,p_{41},T_4(n,p_{41}))$ by (35). Go to Step 7. Otherwise, perform Step 6-2.

Step 6-2. Find p_{42} which satisfies the FONC and SOSC of $\Pi_4(n,p)$ (Eq. (42)) w.r.t. p.
(i) If $f_4'(p) > 0$ and $p_{42} \geqslant Max\{(aM/Q_d)^{1/\delta}, \hat{p}_4\}$, then take $T_4 = Q_d/(ap_{42}^{-\delta})$ and calculate $\Pi_4(n,p_4,T_4) = \Pi_4(n,p_{42}, Q_d/(ap_{42}^{-\delta}))$ by (42). Go to Step 7.
(ii) If $f_4'(p) < 0$ and $(aM/Q_d)^{1/\delta} \leqslant p_{42} < \hat{p}_4$, then take $T_4 = Q_d/(ap_{42}^{-\delta})$ and calculate $\Pi_4(n,p_4,T_4) = \Pi_4(n,p_{42}, Q_d/(ap_{42}^{-\delta}))$ by (42). Go to Step 7. Otherwise, set $\Pi_4(n,p_4,T_4) = 0$.

Step 7. Find $Max_{i=1,2,3,4}\{\Pi_i(n,p_i,T_i)\}$. Set $\Pi^{(n)}(n,p^{(n)},T^{(n)}) = Max_{i=1,2,3,4}\{\Pi_i(n,p_i,T_i)\}$. Then $(p^{(n)},T^{(n)})$ is the optimal solution for this given n.
Step 8. Set $n = n + 1$. Repeat Steps 2–7 to find $\Pi^{(n)}(n,p^{(n)},T^{(n)})$.
Step 9. If $\Pi^{(n)}(n,p^{(n)},T^{(n)}) \geqslant \Pi^{(n-1)}(n-1,p^{(n-1)},T^{(n-1)})$, go to Step 8. Otherwise, go to Step 10.
Step 10. Set $\Pi^*(n^*,p^*,T^*) = \Pi^{(n-1)}(n-1,p^{(n-1)},T^{(n-1)})$. (n^*,p^*,T^*) is the optimal solution.

Once the optimal solution (p^*,T^*) is obtained, the optimal order quantity $Q^* = D(p^*)T^*$ follows.

5. Numerical examples and discussion

Example 1. An inventory system with the following data is considered: $a = 1.0 \times 10^5$, $\delta = 1.5$, $\rho = 0.7$, $c = \$5$/unit, $v = \$10$/unit, $S_V = \$400$/setup, $S_B = \$50$/order, $r_V = 0.1$, $r_B = 0.1$, $I_{Be} = 0.05$, $I_{Bp} = 0.08$, $I_{Vp} = 0.02$ and $M = 30 days$. The optimal pricing, ordering and delivery policies for various Q_d are shown in Table 1.

From the results shown in Table 1, we find that as the value of Q_d increases, the annual total profit for the integrated system decreases, in which the buyer's total profit decreases first and then increases. The vendor's total profit increases first and then decreases. This consequence shows that setting the minimum order quantity (i.e., threshold) for permitting delay payments could be a successful strategy for the vendor to increase buyer's order quantity. We can see that the buyer's optimal order quantity, Q^*, is equal to Q_d when $Q_d = 300$ and 400, and less than Q_d when $Q_d \geqslant 500$. It reveals that the vendor should set the minimum order quantity carefully to make sure that this threshold is effective. If the threshold set by the vendor is too high, the buyer may decide not to order a quantity greater than the threshold to obtain delayed payments. This may work against the buyer or the entire system setting a lower retail price for the end demands. As a result, the effect of stimulating the demands from the buyer turns negative when the vendor adopts a policy to increase the threshold Q_d over some limit.

Example 2. To illustrate the effects of credit terms on performance, the optimal solutions of various credit periods are listed in Table 2. The data is same as Example 1 except we set $Q_d = 300$ here. Comparing the performance among the different credit terms, it is observed that longer trade credit can increase the buyer's total profit and the vendor's total profit. This result identifies that trade credit is an effective strategy for supply chain systems. The reasons are somewhat similar to that illustrated in Example 1. With an appropriate threshold for permitting delaying payments is set by the vendor, the buyer may thus gain some profit or reduce cost, then in turn setting a lower retail price to entice the end demands, so as to increase the profit for the entire system.

Table 1
Optimal solutions under different Q_d.

Q_d	Q^*	n^*	p^*	T^* (days)	$D(p^*)$	Profit ($)		
						Buyer	Vendor	System
0	272.27	10	15.88	62.86	1580.90	8857.12	7368.57	16225.69
100	272.27	10	15.88	62.86	1580.90	8857.12	7368.57	16225.69
200	272.27	10	15.88	62.86	1580.90	8857.12	7368.57	16225.69
300	300.00	9	15.83	68.94	1588.38	8822.94	7401.47	16224.41
400	400.00	7	15.68	90.64	1610.84	8690.99	7497.62	16188.61
500	270.66	10	16.00	63.21	1562.86	8841.02	7307.25	16148.27
600	270.66	10	16.00	63.21	1562.86	8841.02	7307.25	16148.27

Table 2
Optimal solutions under different $M(Q_d = 300)$.

M (days)	Q^*	n^*	p^*	T^* (days)	$D(p^*)$	Profit ($)		
						Buyer	Vendor	System
20	300.00	9	15.86	69.15	1583.43	8812.26	7386.20	16198.46
30	300.00	9	15.83	68.94	1588.38	8822.94	7401.47	16224.41
40	300.00	9	15.80	68.77	1592.34	8838.43	7411.91	16250.34
50	300.00	9	15.78	68.64	1595.31	8858.76	7417.48	16276.24
60	300.00	9	15.77	68.55	1597.26	8883.92	7418.17	16302.09

Example 3. In this example, we perform the sensitivity analysis to show the changes in the optimal solutions Q^*, n^*, p^*, and T^* when the value of parameters varies. All the parameter values are identical to those in Example 1 and $Q_d = 300$, except the given parameter. The influences of these various values on the optimal solutions are provided in Table 3. The results show that as the value of ρ, I_{Be} and I_{Bp} increases, or S_V, S_B and I_{Vp} decreases, the optimal order quantity Q^* remains at the threshold, the optimal retail price p^* and the optimal replenishment cycle length T^* decrease. Besides, as ρ and S_V increases, the optimal number of shipments from the vendor to the buyer per production run n^* increases. Furthermore, some reasonable results with regard to the profit are observed. When the retail price decreases, the total profit obtained at the buyer decreases. At the same time, for the situation where the unit price charged by the vendor to the buyer remains, since the market demand increases (due to lower retail price), the total profit obtained at the vendor increases.

Example 4. To see the efficiency of a vendor–buyer integrated system, the optimal integrated policy and independent policy solutions are listed in Table 4. The parameter values are identical to those in Example 1 and $Q_d = 300$. Table 4 shows that the total annual profit under the integrated policy, \$16224.41 (=\$8822.94 + 7401.47), is greater than the total annual profit under the independent policy, \$14405.20 (=\$11843.43 + 2561.77). This result reveals the benefit of integration between ven-

Table 3
Sensitivity analysis of some parameters.

Parameter		Q^*	n^*	p^*	T^* (days)	$D(p^*)$	Profit (\$)		
							Buyer	Vendor	System
ρ	0.6	300.00	8	15.88	69.30	1580.10	8862.88	7305.19	16168.07
	0.7	300.00	9	15.83	68.94	1588.38	8822.94	7401.47	16224.41
	0.8	300.00	11	15.75	68.41	1600.54	8763.95	7530.41	16294.36
	0.9	300.00	15	15.65	67.79	1615.35	8691.73	7699.60	16391.33
S_V	200	300.00	6	15.72	68.22	1605.15	8741.55	7623.00	16364.55
	300	300.00	8	15.76	68.49	1598.82	8772.31	7515.99	16288.30
	400	300.00	9	15.83	68.94	1588.38	8822.94	7401.47	16224.41
	500	300.00	10	15.88	69.30	1580.10	8862.88	7305.19	16168.07
S_B	30	300.00	9	15.63	67.64	1618.77	8782.88	7548.43	16331.31
	40	300.00	9	15.73	68.29	1603.46	8803.23	7474.37	16277.61
	50	300.00	9	15.83	68.94	1588.38	8822.94	7401.47	16224.41
	60	300.00	9	15.93	69.59	1573.53	8842.02	7329.69	16171.71
I_{Be}	0.040	300.00	9	15.84	69.01	1586.69	8826.63	7393.29	16219.92
	0.045	300.00	9	15.83	68.97	1587.53	8824.79	7397.38	16222.16
	0.050	300.00	9	15.83	68.94	1588.38	8822.94	7401.47	16224.41
	0.055	300.00	9	15.82	68.90	1589.23	8821.09	7405.57	16226.66
I_{Bp}	0.06	300.00	9	15.85	69.12	1584.21	8852.67	7381.33	16234.00
	0.07	300.00	9	15.84	69.03	1586.30	8837.81	7391.39	16229.20
	0.08	300.00	9	15.83	68.94	1588.38	8822.94	7401.47	16224.41
	0.09	300.00	9	15.81	68.85	1590.46	8808.08	7411.54	16219.63
I_{Vp}	0.010	300.00	9	15.80	68.78	1592.07	8805.05	7455.68	16260.73
	0.015	300.00	9	15.81	68.86	1590.23	8814.01	7428.56	16242.57
	0.020	300.00	9	15.83	68.94	1588.38	8822.94	7401.47	16224.41
	0.025	300.00	9	15.84	69.02	1586.54	8831.85	7374.41	16206.26

Table 4
Optimal solutions of independent and integrated models.

Model type	Buyer		Vendor	
Independent	Ordering quantity	178.60	Production quantity	1607.40
	Retail price	31.19		
	Annual demand	574.15		
	Total annual profit	\$11843.43	Total annual profit	\$2561.77
Integrated	Ordering quantity	300.00	Production quantity	2700.00
	Retail price	15.83		
	Annual demand	1588.38		
	Total annual profit	\$8822.94	Total annual profit	\$7401.47
	Allocated total annual profit	\$13339.12[1]	Allocated total annual profit	\$2885.29[2]

Note: 1. $(8822.94 + 7401.47) \times \frac{11843.43}{11843.43+2561.77} = 13339.12$. 2. $(8822.94 + 7401.47) \times \frac{2561.77}{11843.43+2561.77} = 2885.29$.

dor and buyer. From the vendor's perspective, the integrated policy is much more advantageous than the independent policy. To encourage the buyer to cooperate with the vendor, Goyal [20] suggested that the vendor should pay compensation to the buyer for the loss in profit. Following Goyal's suggestion, we evaluate the allocated total annual profit for the buyer and the supplier in the last row of Table 4.

6. Conclusion

We formulate an integrated vendor–buyer inventory model in this paper with the assumptions that the market demand is sensitive to the retail price and the credit terms are linked to the order quantity. By analyzing the total channel profit function, we develop a solution algorithm to simultaneously determine the optimal retail price, order quantity and number of shipments per production run from the vendor to the buyer. Finally, numerical examples are presented to illustrate the solution procedure, and sensitivity analysis of the optimal solution is also indicated.

Based on our analysis, it was found that a longer trade credit term can increase profits for the entire supply chain. However, the vendor should determine the minimum order quantity (i.e., threshold) for allowing delay in payments comprehensively to ensure the greatest benefit for both parties. We assumed that the threshold is a fixed constant and studied its effect on the integrated system. Treating the threshold as a decision variable from vendor's perspective or a negotiable factor from the view point of both parties would be of interest for future research. Our model can be extended to more general supply chain networks, for example, multi-echelon or assembly supply chains. Deteriorating items and the order quantity as a function of credit period will be considered in the proposed model in our future work.

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Appendix A

Note that $\frac{\partial^2 \Pi_3(n,p,T)}{\partial T^2} = -\frac{2\overline{S}}{T^3} - \frac{D(vI_{Bp}-pI_{Be})M^2}{T^3}$. From $2\overline{S} \geqslant D(vr_B + pI_{Be} + \varphi)M^2$, we have

$$2\overline{S} + D(vI_{Bp} - pI_{Be})M^2 \geqslant D(vr_B + pI_{Be} + \varphi)M^2 + D(vI_{Bp} - pI_{Be})M^2 = DM^2[v(r_B + I_{Bp}) + \varphi] > 0. \quad \text{(A.1)}$$

Thus, $\frac{\partial^2 \Pi_3(n,p,T)}{\partial T^2} = -\frac{1}{T^3}[2\overline{S} + D(vI_{Bp} - pI_{Be})M^2] < 0$. The proof is completed.

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