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Optimal Lot Size under Trade Credit Financing When Demand and Deterioration Are Fluctuating with Time

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Abstract

This paper considers a generalized lot size inventory model with deteriorated products under trade credit financing. The demand and deterioration rates are known, continuous, and differentiable functions of time. In addition, the trade credit policy adopted in this paper assumes that the supplier offers the retailer the permissible delay period M, and the retailer also offer the trade credit period N to his/her customer, where $M \geq N$. Under these general assumptions, the inventory model is formulated to attain the minimum total cost per unit time. For each circumstance, sufficient condition which lead to a minimal solution of the considered problem is also derived. Then, a rigorous mathematical analysis is used to prove that such a minimal solution is unique and global-optimal. Finally, some numerical examples are used to illustrate the model.

Keywords: Inventory, Time-Varying Demand, Deteriorating Items, Trade Credit.

1. Introduction

In the literature of inventory theory, the effect of trade credit on inventory control decisions has been continually modified so as to accommodate more practical features of the real inventory systems. Goyal (1985) studied an EOQ model under the conditions of permissible delay in payments. However, in real situations, ”time” is a significant key concept and plays an important role in inventory models. Certain types of commodities deteriorate in the course of time and hence are unstable. To accommodate more practical features of the real inventory systems, Aggarwal and Jaggi (1995) and Hwang and Shinn

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(1997) extended Goyal's (1985) model to consider the deterministic inventory model with a constant deterioration rate. Since the occurrence of shortages in inventory is a very nature phenomenon in real situations, Jamal et al. (1997), Sarker et al. (2000), Chang and Dye (2000) and Chang et al. (2002) extended Aggarwal and Jaggi's (1995) model to allow for shortages and makes it more applicable in real world. Teng (2002) provided an alternative conclusion from Goyal (1985), and mathematically proved that it makes economic sense for a well-established buyer to order less quantity and take the benefits of the permissible delay more frequently. Chang et al. (2003) then extended Teng's model, and established an EOQ model for deteriorating items in which the supplier provides a permissible delay to the purchaser if the order quantity is greater than or equal to a predetermined quantity. Huang (2003) developed an EOQ model in which a supplier offers a retailer the permissible delay period M, and the retailer in turn provides the trade credit period N (with $N \leq M$) to his/her customers. He then obtained the closed-form optimal solution for the problem. The shortcomings of Huang's (2003) model are the assumptions that unit purchase cost is the same as the selling price per unit, and the interest rate I_k charged by the supplier is equal or larger than the interest rate earned on investment I_e. By considering the difference between unit selling price and unit purchasing cost, Ouyang et al. (2004) developed an EOQ model with noninstantaneous receipt under conditions of permissible delay in payments. Recently, Teng et al. (2007) extended Huang's (2003) model by considering the difference between unit price and unit cost and relaxing the unnecessary condition of $I_k \geq I_e$.

However, all the above models make an implicit assumption that the demand rate is constant over an infinite planning horizon. This assumption is only valid during the maturity phase of a product life cycle. In the introduction and growth phase of a product life cycle, the firms face increasing demand with little competition. Some researchers (Resh et al. (1976), Donaldson (1977), Dave and Patel (1981), Sachan (1984), Goswami and Chaudhuri (1991), Goyal et al. (1992) and Chakrabarty (1998)) suggest that the demand rate can be well approximated by a linear form. A linear trend demand implies an uniform change in the demand rate of the product per unit time. This is a fairly unrealistic phenomenon and it seldom occurs in the real market. One can usually observe in the electronic market that the sales of items increase rapidly in the introduction and growth phase of the life cycle because there are few competitors in market. Recently, Khanra and Chaudhuri (2003) advise that the demand rate should be represented by a continuous quadratic function of time in the growth stage of a product life cycle. They also provide a heuristic algorithm to solve the problem when the planning horizon is finite.

In the present paper, we attempt to develop an inventory model for deteriorating items with time varying demand and deterioration rates under the conditions of permissible delay in payments and our study differs from the existing literature from the following three aspects. Firstly, the demand rate is a continuous function of time and increases at an increasing rate. This represents a rapidly expanding market. Secondly, the items deteriorate at an increasing varying rate of deterioration. Thirdly, a supplier offers a retailer the permissible delay period M, and the retailer in turn provides the trade credit period N (with $N \leq M$) to his/her customers.

2. Notations and Assumptions

Notations:

$f(t) =$ the demand rate is a continuous function of time t and increases at a increasing rate, i.e. $f(t)$ satisfies $f'(t) > 0$ and $f''(t) > 0$.

$A =$ ordering cost per order.

$c =$ unit purchasing cost.

$p =$ unit selling price, with $p > c$.

$h =$ holding cost excluding interest charges, \$/per unit/year.

$I_e =$ interest earned per \$ per year.

$I_k =$ interest charged per \$ in stocks per year by the supplier.

$M =$ the retailer's trade credit period offered by supplier in years.

$N =$ the customer's trade credit period offered by retailer in years, where $N \leq M$.

$T =$ replenishment time interval, where $T \geq 0$.

$AC\,(T) =$ the average total inventory cost per unit time.

$AC_1\,(T) =$ the average total inventory cost per unit time when $M \leq T$.

$AC_2\,(T) =$ the average total inventory cost per unit time when $N \leq T < M$.

$AC_3\,(T) =$ the average total inventory cost per unit time when $T < N$.

Assumptions:

1. The inventory system involves only one item.
2. The replenishment occurs instantaneously at an infinite rate.
3. The items deteriorate at a varying rate of deterioration $\theta(t)$, where $\theta'(t) \geq 0$ and $0 < \theta(t) \ll 1$. Here $\theta'(t)$ denotes the first derivative of $\theta'(t)$ with respect to t. Note that $\theta'(t) \geq 0$ means that the deterioration rate is nondecreasing over time.
4. When $T \geq M$, the account is settled at $T = M$ and the retailer starts paying for the interest charges on the items in stock with rate I_k. When $T < M$, the account is settled at $T = M$ and the retailer does not need to pay any interest charge.

5. The retailer can accumulate revenue and earn interest after his/her customer pays for the amount of purchasing cost to the retailer until the end of the trade credit period offered by the supplier. That is, the retailer can accumulate revenue and earn interest during the period N to M with rate I_e under the condition of trade credit.

3. Model Formulation

The depletion of the inventory occurs due to the combined effects of the demand and deterioration in the interval $(0, T)$. As a result, the rate of change in inventory over time, $\mathrm{d}I(t)/\mathrm{d}t$, may be written as

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -f(t) - \theta(t) I(t), \ 0 < t < T, \tag{1}$$

with boundary condition $I(T) = 0$. From (1), the inventory level at time t may be expressed as

$$I(t) = e^{-\int_0^t \theta(s)\mathrm{d}s} \int_t^T e^{\int_0^u \theta(s)\mathrm{d}s} f(u)\,\mathrm{d}u, \ 0 \le t \le T. \tag{2}$$

For notational convenience, let $g(t) = \int_0^t \theta(x)\mathrm{d}x$. Then applying (2), we can obtain the holding cost over the replenishment period as

$$h\int_0^T I(t)\,\mathrm{d}t = h\int_0^T e^{-g(t)} \int_t^T e^{g(u)} f(u)\,\mathrm{d}u\mathrm{d}t. \tag{3}$$

And, the deterioration cost per cycle is

$$c\int_0^T \left[e^{g(t)} - 1\right] f(t)\,\mathrm{d}t. \tag{4}$$

Regarding the exogenous variables, three possibilities may arise: (1) $M \le T$ (2) $N \le T < M$ and (3) $T < N$. The Graphical Representation for Case 1-3 are shown in Figure 1. We construct them as follows:

Case 1: $M \le T$

In this case, since the length of replenishment period is greater than the credit period, the retailer can use the sale revenue to earn an interest of $pI_e \int_N^M (M-t) f(t)\mathrm{d}t$ during the period of the permissible delay M. After the time that account is settled, the retailer starts to pay for the interest charges on the items in stocks with an annual rate I_k and thus the interest charges is $cI_k \int_M^T e^{-g(t)} \int_t^T e^{g(u)} f(t)\mathrm{d}u\mathrm{d}t$. The average total inventory cost $AC_1(T)$, which includes the cost of ordering, holding inventory cost, deterioration cost, interest charges and interest earn then becomes

$$\begin{aligned} AC_1(T) = \frac{1}{T}\Bigg\{ & A + c\int_0^T [e^{g(t)} - 1] f(t)\,\mathrm{d}t + h\int_0^T e^{-g(t)} \int_t^T e^{g(u)} f(u)\,\mathrm{d}u\mathrm{d}t \\ & + cI_k \int_M^T e^{-g(t)} \int_t^T e^{g(u)} f(u)\,\mathrm{d}u\mathrm{d}t - pI_e \int_N^M (M-t) f(t)\,\mathrm{d}t \Bigg\}. \end{aligned} \tag{5}$$

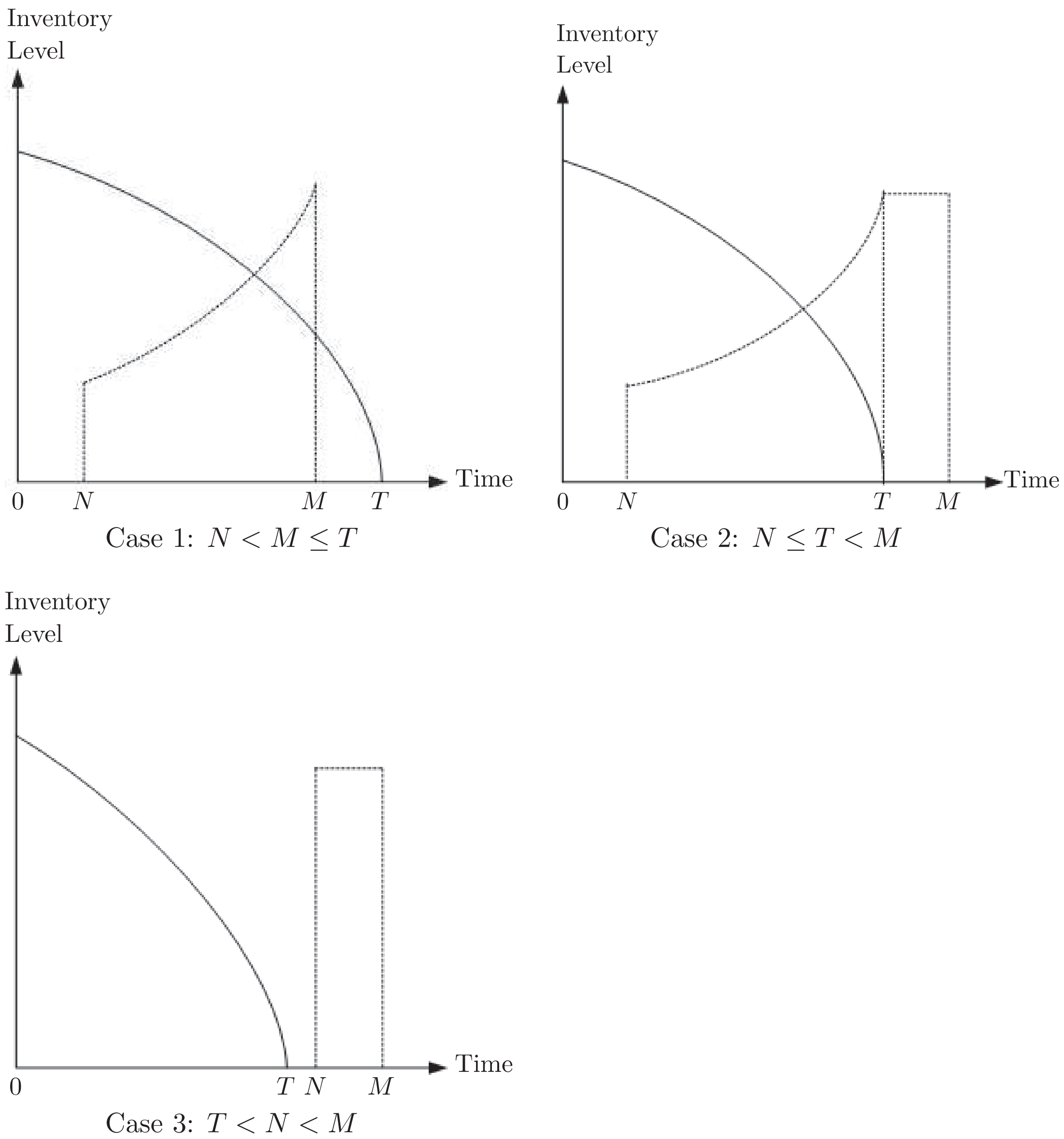

Figure 1. Graphical Representation of Inventory System

The objective of this problem is finding the minimum of $AC_1(T)$ in the interval $[M, \infty)$. To this end, we take the first and second order derivatives of $AC_1(T)$ with respect to T and obtain

$$\frac{\mathrm{d}AC_1(T)}{\mathrm{d}T} = -\frac{A}{T^2} + \frac{c\left[Tf(T)\left(e^{g(T)}-1\right) - \int_0^T \left(e^{g(t)}-1\right) f(t)\,\mathrm{d}t\right]}{T^2} + \frac{h\left[Tf(T)\int_0^T e^{g(T)-g(t)}\mathrm{d}t - \int_0^T e^{-g(t)}\int_t^T e^{g(u)} f(u)\,\mathrm{d}u\mathrm{d}t\right]}{T^2}$$

$$+\frac{cI_kTf(T)\int_M^T e^{g(T)-g(t)}\mathrm{d}t - cI_k\int_M^T e^{-g(t)}\int_t^T e^{g(u)}f(u)\,\mathrm{d}u\mathrm{d}t}{T^2}$$
$$+\frac{pI_e\int_N^M (M-t)f(t)\,\mathrm{d}t}{T^2} \tag{6}$$

and

$$\frac{\mathrm{d}^2AC_1(T)}{\mathrm{d}T^2} = \frac{2A}{T^3} + \frac{ck_1(T)}{T^3} + \frac{hk_2(T)}{T^3} + \frac{k_3(T)}{T^3}, \tag{7}$$

where

$$k_1(T) = 2\int_0^T \left(e^{g(t)}-1\right)f(t)\,\mathrm{d}t + \left(e^{g(T)}-1\right)T^2f'(T) + Tf(T)\left[2+e^{g(T)}\left(-2+Tg'(T)\right)\right], \tag{8}$$

$$k_2(T) = 2\int_0^T e^{-g(t)}\int_t^T e^{g(u)}f(u)\,\mathrm{d}u\mathrm{d}t + T^2f'(T)\int_0^T e^{g(T)-g(t)}\mathrm{d}t + Tf(T)\left[T+\left(Tg'(T)-2\right)\int_0^T e^{g(T)-g(t)}\mathrm{d}t\right] \tag{9}$$

and

$$k_3(T) = cI_k\left\{2\int_M^T e^{-g(t)}\int_t^T e^{g(u)}f(u)\,\mathrm{d}u\mathrm{d}t + T^2f'(T)\int_M^T e^{g(T)-g(t)}\mathrm{d}t + Tf(T)\left[T+\left(Tg'(T)-2\right)\int_0^T e^{g(T)-g(t)}\mathrm{d}t\right]\right\} - 2pI_e\int_N^M (M-t)f(t)\,\mathrm{d}t. \tag{10}$$

Under the condition, $2A > \max\left\{(pI_e - cI_k)M^2f(M), pI_e\left[N^2f(M)+M^3f'(M)\right]\right\}$, we have the following proposition:

Proposition 1. *If*

$$2A > \max\left\{(pI_e - cI_k)M^2f(M), pI_e\left[N^2f(M)+M^3f'(M)\right]\right\},$$

then $\mathrm{d}AC_1(T)/\mathrm{d}T$ *is a strictly increasing function of* T *and there exists a unique real solution* $T^* \in [M,\infty)$ *such that* $AC_1(T)$ *is minimum.*

Proof. We first take the first order derivative of $k_1(T)$ with respect to T and obtain

$$\frac{\mathrm{d}k_1(T)}{\mathrm{d}T} = T^2\left\{\left(e^{g(T)}-1\right)f''(T) + e^{g(T)}\left[2f'(T)g'(T) + f(T)\left(g'(T)^2+g''(T)\right)\right]\right\}.$$

Since $0 < g'(T) = \theta(T)$ and $g''(T) = \theta'(T) \geq 0$, it is clear to see that $\mathrm{d}k_1(T)/\mathrm{d}T > 0$. Therefore, $k_1(T)$ is a strictly increasing function of T. Further, due to $k_1(0) = 0$, it is obvious that $k_1(T) \geq k_1(M) > k_1(0) = 0$ for $0 < M \leq T$.

Next, differentiating $k_2(T)$ with respect to T yields

$$\begin{aligned}\frac{\mathrm{d}k_2\left(T\right)}{\mathrm{d}T} &= T^2\left\{f\left(T\right)g'\left(T\right)+2f'\left(T\right)\left[1+g'\left(T\right)\right]\int_0^T e^{g(T)-g(t)}\mathrm{d}t\right.\\ &\quad \left.+\left[f''\left(T\right)+f\left(T\right)\left(g'\left(T\right)^2+g''\left(T\right)\right)\right]\int_0^T e^{g(T)-g(t)}\mathrm{d}t\right\}\\ &> 0.\end{aligned}$$

Because $k_2(0)=0$, we obtain $k_2(T)\geq k_2(M)>k_2(0)=0$. By the same arguments, we have

$$\begin{aligned}\frac{\mathrm{d}k_3\left(T\right)}{\mathrm{d}T} &= cI_kT^2\left\{f\left(T\right)g'\left(T\right)+2f'\left(T\right)\left[1+g'\left(T\right)\int_M^T e^{g(T)-g(t)}\mathrm{d}t\right]\right.\\ &\quad \left.+\left[f''\left(T\right)+f\left(T\right)\left(g'\left(T\right)^2+g''\left(T\right)\right)\int_M^T e^{g(T)-g(t)}\mathrm{d}t\right]\right\}\\ &> 0\end{aligned}$$

and

$$\begin{aligned}k_3\left(M\right) &= cI_kM^2f\left(M\right)-2pI_e\int_N^M\left(M-t\right)f\left(t\right)\mathrm{d}t\\ &> cI_kM^2f\left(M\right)-2pI_e\int_N^M\left(M-t\right)f\left(M\right)\mathrm{d}t\\ &= cI_kM^2f\left(M\right)-pI_e\left(M-N\right)^2f\left(M\right)\\ &> cI_kM^2f\left(M\right)-pI_eM^2f\left(M\right).\end{aligned}$$

Under the condition, $2A>\max\left\{(pI_e-cI_k)M^2f\left(M\right),pI_e\left[N^2f\left(M\right)+M^3f'\left(M\right)\right]\right\}$, it is easy to show that $2A+k_3\left(T\right)\geq 2A+k_3\left(M\right)>0$. Clearly, by Eq. (7), $AC_1\left(T\right)$ is convex in T, and consequently, there exists a unique $T^*\in[M,\infty)$ such that $AC_1\left(T\right)$ is minimum.

Case 2: $N\leq T<M$

In this case, it is assumed that the length of replenishment period is shorter than the credit period, thus the retailer pays no interest charges and earns the interest during the period (N,M). The interest earned here is $pI_e\left[\int_N^T\left(T-t\right)f\left(t\right)\mathrm{d}t+\left(M-T\right)\int_0^T f\left(t\right)\mathrm{d}t\right]$. From this, the average total inventory cost per unit time can be formulated as

$$\begin{aligned}AC_2\left(T\right) = \frac{1}{T}&\left\{A+c\int_0^T[e^{g(t)}-1]f\left(t\right)\mathrm{d}t+h\int_0^T e^{-g(t)}\int_t^T e^{g(u)}f\left(u\right)\mathrm{d}u\mathrm{d}t\right.\\ &\left.-pI_e\left[\int_N^T\left(T-t\right)f\left(t\right)\mathrm{d}t+\left(M-T\right)\int_0^T f\left(t\right)\mathrm{d}t\right]\right\}.\end{aligned}\tag{11}$$

Taking the first and second order derivatives of $AC_2(T)$ with respect to T yields

$$\frac{\mathrm{d}AC_2\left(T\right)}{\mathrm{d}T}=-\frac{A}{T^2}+\frac{c\left[Tf\left(T\right)\left(e^{g(T)}-1\right)-\int_0^T\left(e^{g(t)}-1\right)f\left(t\right)\mathrm{d}t\right]}{T^2}$$

$$+\frac{h\left[Tf\left(T\right)\int_0^T e^{g(T)-g(t)}\mathrm{d}t-\int_0^T e^{-g(t)}\int_t^T e^{g(u)}f\left(u\right)\mathrm{d}u\mathrm{d}t\right]}{T^2}$$
$$-\frac{pI_e\left(M-T\right)\left(Tf\left(T\right)-\int_0^T f\left(t\right)\mathrm{d}t\right)}{T^2}$$
$$+\frac{pI_e\left[\int_N^T\left(T-t\right)f\left(t\right)\mathrm{d}t-T\int_0^N f\left(t\right)\mathrm{d}t\right]}{T^2} \tag{12}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2 AC_2\left(T\right)}{\mathrm{d}T^2}=&\frac{2A}{T^3}+\frac{ck_1\left(T\right)}{T^3}+\frac{hk_2\left(T\right)}{T^3}\\&+\frac{pI_e}{T^3}\left\{T\left(2M-T\right)f\left(T\right)-2\int_N^T\left(T-t\right)f\left(t\right)\mathrm{d}t\right.\\&\left.-\left(M-T\right)\left[2\int_0^T f\left(t\right)\mathrm{d}t+T^2 f'\left(T\right)\right]-2T\int_0^N f\left(t\right)\mathrm{d}t\right\}.\end{aligned}$$

respectively. We then have the following proposition:

Proposition 2. *If*

$$2A>\max\left\{(pI_e-cI_k)M^2 f\left(M\right),pI_e\left[N^2 f\left(M\right)+M^3 f'\left(M\right)\right]\right\},$$

then $\mathrm{d}AC_2\left(T\right)/\mathrm{d}T$ *is a strictly increasing function of* T *and there exists a unique real solution* $T^*\in[N,M)$ *such that* $AC_2\left(T\right)$ *is minimum.*

Proof. Since $N\le T\le M$, $f'\left(t\right)>0$ and $f''\left(t\right)>0$, we obtain

$$\begin{aligned}&T\left(2M-T\right)f\left(T\right)-2\int_N^T\left(T-t\right)f\left(t\right)\mathrm{d}t\\&-\left(M-T\right)\left[2\int_0^T f\left(t\right)\mathrm{d}t+T^2 f'\left(T\right)\right]-2T\int_0^N f\left(t\right)\mathrm{d}t\\>&T\left(2M-T\right)f\left(T\right)-2\int_N^T\left(T-t\right)f\left(T\right)\mathrm{d}t\\&-\left(M-T\right)\left[2\int_0^T f\left(T\right)\mathrm{d}t+T^2 f'\left(T\right)\right]-2T\int_0^N f\left(T\right)\mathrm{d}t\\=&T^3 f'\left(T\right)-N^2 f\left(T\right)-T^2 M f'\left(T\right)\\>&-N^2 f\left(M\right)-M^3 f'\left(M\right).\end{aligned}$$

Now we apply Proposition 1 and above finding with the assumption, $2A>\max\{(pI_e-cI_k)M^2 f\left(M\right),pI_e\left[N^2 f\left(M\right)+M^3 f'\left(M\right)\right]\}$, we can conclude that $\mathrm{d}^2 AC_2(T)/\mathrm{d}T^2>0$ for all $T\in[N,M)$. In other words, $AC_2\left(T\right)$ is convex in T. Consequently, there exists a unique $T^*\in[N,M)$ such that $AC_2\left(T\right)$ is minimum.

Case 3: $\boldsymbol{T<N}$

In this case, the length of replenishment period is shorter than the credit periods M and N , thus the retailer pays no interest charges and earns the interest during the period (N, M). The interest earned here is $pI_e\left(M-N\right)\int_0^T f\left(t\right)\mathrm{d}t$. The average total inventory cost per unit time, thus, becomes

$$\begin{aligned} AC_3\left(T\right) = \frac{1}{T}\bigg\{ & A + c\int_0^T [e^{g(t)}-1]f\left(t\right)\mathrm{d}t + h\int_0^T e^{-g(t)}\int_t^T e^{g(u)}f\left(u\right)\mathrm{d}u\mathrm{d}t \\ & - pI_e\left(M-N\right)\int_0^T f\left(t\right)\mathrm{d}t\bigg\}, \end{aligned} \tag{13}$$

and the first and second order derivatives of $AC_3(T)$ with respect to T is

$$\begin{aligned} \frac{\mathrm{d}AC_3\left(T\right)}{\mathrm{d}T} = & -\frac{A}{T^2} + \frac{c\left[Tf\left(T\right)\left(e^{g(T)}-1\right) - \int_0^T\left(e^{g(t)}-1\right)f\left(t\right)\mathrm{d}t\right]}{T^2} \\ & + \frac{h\left[Tf\left(T\right)\int_0^T e^{g(T)-g(t)}\mathrm{d}t - \int_0^T e^{-g(t)}\int_t^T e^{g(u)}f\left(u\right)\mathrm{d}u\mathrm{d}t\right]}{T^2} \\ & - \frac{pI_e\left(M-N\right)\left(Tf\left(T\right) - \int_0^T f\left(t\right)\mathrm{d}t\right)}{T^2} \end{aligned} \tag{14}$$

and

$$\begin{aligned} \frac{\mathrm{d}^2AC_3\left(T\right)}{\mathrm{d}T^2} = & \frac{2A}{T^3} + \frac{ck_1\left(T\right)}{T^3} + \frac{hk_2\left(T\right)}{T^3} \\ & + \frac{pI_e\left(M-N\right)}{T^3}\left\{2\left(Tf\left(T\right) - \int_0^T f\left(t\right)\mathrm{d}t\right) - T^2f'\left(T\right)\right\}, \end{aligned}$$

respectively. Since

$$\begin{aligned} & \left(M-N\right)\left[2\left(Tf\left(T\right) - \int_0^T f\left(t\right)\mathrm{d}t\right) - T^2f'\left(T\right)\right] \\ & > \left(M-N\right)\left[2\left(Tf\left(T\right) - \int_0^T f\left(T\right)\mathrm{d}t\right) - M^2f'\left(M\right)\right] \\ & > -\left(M-N\right)M^2f'\left(M\right) \\ & > -N^2f\left(M\right) - M^3f'\left(M\right), \end{aligned}$$

we have following result.

Proposition 3. *If*

$$2A > \max\left\{(pI_e - cI_k)M^2f\left(M\right), pI_e\left[N^2f\left(M\right) + M^3f'\left(M\right)\right]\right\},$$

then $\mathrm{d}AC_3\left(T\right)/\mathrm{d}T$ *is a strictly increasing function of* T *and there exists a unique real solution* $T^* \in (0, N)$ *such that* $AC_3\left(T\right)$ *is minimum.*

Proof. Follows immediately from Proposition 1.

4. Solution Procedures

In Case 1, we had showed that $\mathrm{d}AC_1(T)/\mathrm{d}T$ is a strictly increasing function in T. From Eq. (6), since

$$\lim_{T\to\infty}\Big\{-A+c\Big[Tf(T)\Big(e^{g(T)}-1\Big)-\int_0^T\Big(e^{g(t)}-1\Big)f(t)\,\mathrm{d}t\Big]$$
$$+h\Big[Tf(T)\int_0^T e^{g(T)-g(t)}\mathrm{d}t-\int_0^T e^{-g(t)}\int_t^T e^{g(u)}f(u)\,\mathrm{d}u\mathrm{d}t\Big]$$
$$+cI_kTf(T)\int_M^T e^{g(T)-g(t)}\mathrm{d}t-cI_k\int_M^T e^{-g(t)}\int_t^T e^{g(u)}f(u)\,\mathrm{d}u\mathrm{d}t$$
$$+pI_e\int_N^M(M-t)f(t)\,\mathrm{d}t\Big\}=\infty,$$

by L'Hospital rule, we have

$$\begin{aligned}\lim_{T\to\infty}AC_1'(T)=&\frac{c\left[f'(T)\left(e^{g(T)}-1\right)+f(T)g'(T)e^{g(T)}\right]}{2}\\&+\frac{h\left[f(T)+f'(T)\int_0^T e^{g(T)-g(t)}\mathrm{d}t+f(T)g'(T)\int_0^T e^{g(T)-g(t)}\mathrm{d}t\right]}{2}\\&+\frac{cI_k\left[f(T)+f'(T)\int_M^T e^{g(T)-g(t)}\mathrm{d}t+f(T)g'(T)\int_M^T e^{g(T)-g(t)}\mathrm{d}t\right]}{2}\\=&\infty.\end{aligned}$$

Therefore, there exits a unique value of T that minimizes $AC_1(T)$. The minimum value of $AC_1(T)$ will occur at the point T^* which satisfies

$$\frac{\mathrm{d}AC_1(T)}{\mathrm{d}T}=0,\text{ otherwise }T^*=M,\text{ if }\lim_{T\to M^+}\frac{\mathrm{d}AC_1(T)}{\mathrm{d}T}>0.$$

Similarly, since $\mathrm{d}AC_2(T)/\mathrm{d}T$ and $\mathrm{d}AC_3(T)/\mathrm{d}T$ are also strictly increasing in T, the minimum value of $AC_2(T)$ and $AC_3(T)$ will occur at the point T^* which satisfies

$$\frac{\mathrm{d}AC_2(T)}{\mathrm{d}T}=0,\text{ otherwise }T^*=\begin{cases}N,\text{ if }\lim_{T\to N^+}\frac{\mathrm{d}AC_2(T)}{\mathrm{d}T}>0\\M,\text{ if }\lim_{T\to M^-}\frac{\mathrm{d}AC_2(T)}{\mathrm{d}T}<0\end{cases}$$

and

$$\frac{\mathrm{d}AC_3(T)}{\mathrm{d}T}=0,\text{ otherwise }T^*=N,\text{ if }\lim_{T\to N^-}\frac{\mathrm{d}AC_3(T)}{\mathrm{d}T}<0,$$

respectively. Besides, it is not difficult to check from Eqs. (6), (12) and (14) that

$$\lim_{T\to M^+}AC_1'(T)=\lim_{T\to M^-}AC_2'(T),$$
$$\lim_{T\to N^+}AC_2'(T)=\lim_{T\to N^-}AC_3'(T)$$

and

$$\lim_{T\to 0^+} AC_3'(T) = -\infty.$$

Hence, if the exogenous values are known, then the values of $\lim_{T\to N^-} \mathrm{d}AC_3(T)/\mathrm{d}T$ and $\lim_{T\to M^-} \mathrm{d}AC_2(T)/\mathrm{d}T$ can be calculated. Using the relationships among the two auxiliary values, we can find that there is only one case of $AC_i(T)$ has a solution to

$$\frac{\mathrm{d}AC_i(T)}{\mathrm{d}T} = 0.$$

We then obtain the desired results.

Proposition 4. *For* $2A > \max\left\{(pI_e - cI_k)M^2 f(M), pI_e\left[N^2 f(M) + M^3 f'(M)\right]\right\}$, *we have:*

(1) *If* $\lim_{T\to N^-} AC_3'(T) < 0$ *and* $\lim_{T\to M^-} AC_2'(T) < 0$, *then*

$$AC(T^*) = \min\{AC_1(T_1^*), AC_2(T_2^*), AC_3(T_3^*)\} = AC_1(T_1^*)$$

(2) *If* $\lim_{T\to N^-} AC_3'(T) < 0$ *and* $\lim_{T\to M^-} AC_2'(T) > 0$, *then*

$$AC(T^*) = \min\{AC_1(T_1^*), AC_2(T_2^*), AC_3(T_3^*)\} = AC_2(T_2^*)$$

(3) *If* $\lim_{T\to N^-} AC_3'(T) > 0$ *and* $\lim_{T\to M^-} AC_2'(T) > 0$, *then*

$$AC(T^*) = \min\{AC_1(T_1^*), AC_2(T_2^*), AC_3(T_3^*)\} = AC_3(T_3^*)$$

Proof.

(1) From Proposition 1, 2 and 3, we obtain $AC_3(T^*) = AC_3(N)$ and $AC_2(T^*) = AC_2(M)$. Since $\lim_{T\to M^-} AC_2'(T) = \lim_{T\to M^+} AC_1'(T) < 0$ and $\lim_{T\to\infty} AC_1'(T) = \infty$, the Intermediate Value Theorem implies that the root of $\mathrm{d}AC_1(T)/\mathrm{d}T = 0$ is unique, which implies that $AC_1(T^*) < AC_1(M) = AC_2(M) < AC_3(N)$. Therefore, we have

$$AC(T^*) = \min\{AC_1(T^*), AC_2(T^*), AC_3(T^*)\} = AC_1(T^*).$$

(2) From Proposition 1, 2 and 3, we obtain $AC_3(T^*) = AC_3(N)$ and $AC_1(T^*) = AC_1(M)$. Since $\lim_{T\to N^+} AC_2'(T) < 0$ and $\lim_{T\to M^-} AC_2'(T) > 0$, the Intermediate Value Theorem implies that the root of $\mathrm{d}AC_2(T)/\mathrm{d}T = 0$ is unique, which implies that $AC_2(T^*) < AC_2(M) = AC_1(M)$ and $AC_2(T^*) < AC_2(N) = AC_3(N)$. Therefore, we have

$$AC(T^*) = \min\{AC_1(T^*), AC_2(T^*), AC_3(T^*)\} = AC_2(T^*).$$

(3) From Proposition 1, 2 and 3, we obtain $AC_1(T^*) = AC_1(M)$ and $AC_2(T^*) = AC_2(N)$. Since $\lim_{T\to 0^+} AC_3'(T) = -\infty$ and $\lim_{T\to N^-} AC_2'(T) > 0$, the Intermediate Value Theorem implies that the root of $\mathrm{d}AC_3(T)/\mathrm{d}T = 0$ is unique, which implies that $AC_3(T^*) < AC_2(N) < AC_1(M)$. Therefore, we have

$$AC(T^*) = \min\{AC_1(T^*), AC_2(T^*), AC_3(T^*)\} = AC_3(T^*).$$

5. Numerical Examples

Example 1. Let us take the parameter values of the inventory system as $A = 200$, $f(t) = 1000 + 100t + 20t^2$, $c = 30$, $p = 50$, $h = 6$, $I_k = 0.15$, $I_e = 0.12$, $\theta(t) = \alpha\beta t^{\beta-1} = 0.08 \times 1.5 \times t^{0.5}$ (eg. Weibull deterioration rate, where α is scale parameter and β is shape parameter.), $M = 45/365$ and $N = 15/365$. Since $\max\{(pI_e - cI_k)M^2 f(M), pI_e[N^2 f(M) + M^3 f'(M)]\} = \max\{23.2283, 11.4411\} = 23.2283 < 400 = 2A$, there exists a unique optimal T such that $AC_i(T)$ is a minimum.

The values of $AC_3'(N)$ and $AC_2'(M)$ are -115135 and -6237, respectively. By Proposition 4-1, we observe that $AC(T^*) = AC_1(T^*)$. Solving Eq. (6), the optimal value of T equals 0.1885. Then, taking the value into Eq. (5), the optimal value of $AC(T)$ is 1658.35.

Example 2. $A = 200$, $f(t) = 1000 + 100t + 20t^2$, $c = 30$, $p = 50$, $h = 6$, $I_k = 0.15$, $I_e = 0.12$, $\theta(t) = \alpha\beta t^{\beta-1} = 0.08 \times 1.5 \times t^{0.5}$, $M = 60/365$ and $N = 30/365$. Since $\max\{(pI_e - cI_k)M^2 f(M), pI_e[N^2 f(M) + M^3 f'(M)]\} = \max\{41.2211, 44.0616\} = 44.0616 < 400 = 2A$, there exists a unique optimal T such that $AC_i(T)$ is a minimum. Furthermore, the values of $AC_3'(N)$ and $AC_2'(M)$ are -26174.9 and 49.3172, respectively. By Proposition 4-2, the optimal value of $AC(T)$ is $AC_2(0.1638) = 1656.45$.

6. Concluding Remarks

In this paper, an inventory model for deteriorating items with time-varying demand and deterioration rates is studied when the credit periods are offered. The analytical formulations of the problem on the general framework described have been given. Since the demand rate increases rapidly in the introduction and growth phase of a product life cycle, the assumptions of $f'(t) > 0$ and $f''(t) > 0$ are very realistic. Under the specific circumstance, $2A > \max\left\{(pI_e - cI_k)M^2 f(M), pI_e\left[N^2 f(M) + M^3 f'(M)\right]\right\}$, we derive results which ensure the existence of a unique optimal solution for each case. We also establish Proposition 4, which provide us a simple way to obtain the global minimum. By our method, we can easily obtain the optimal replenishment policy among those cases with the help of two auxiliary values.

Furthermore, we can also see that any non-decreasing deterioration rate can be applied to this model such as the three-parameter Weibull deterioration rate (eg. Philip (1974) and Chakrabarty et al. (1998)) and Gamma deterioration rate (eg. Tadikamalla (1978)). Hence, the model can be also easily applied to the problem with a finite planning horizon by using the algorithm of Khanra and Chaudhuri (2003). In contrast to previous models, the utilization of general time varying demand and deterioration rates make the scope of the application broader.

The proposed model can be extended in several ways. For instance, we may consider finite rate of replenishment. Also, we could extend the deterministic demand function to stochastic fluctuating demand patterns. Finally, we could generalize the model to allow for shortages, inflation and others.

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References

[1] Aggarwal, S. P. and Jaggi, C. K. (1995), Ordering policies of deteriorating items under permissible delay in payments, *Journal of Operational Research Society*, Vol. 46, pp.658-662.

[2] Chang, C.T., Ouyang, L.Y. and Teng, J.T. (2003), *An EOQ model for deteriorating items under supplier credits linked to ordering quantity*, Applied Mathematical Modelling, Vol. 27, pp.983-996.

[3] Chang, H. J. and Dye, C. Y. (2000), *An inventory model for deteriorating items with partial backlogging and permissible delay in payments*, International Journal of Systems Science, Vol. 32, pp.345-352.

[4] Chang, H. J., Hung, C. H. and Dye, C. Y. (2002), *A finite time horizon inventory model with deterioration and time-value of money under the conditions of permissible delay in payments*, International Journal of Systems Science, Vol. 33, pp.141-151.

[5] Chakrabarty, T., Giri, B. C. and Chaudhuri, K. S. (1998), *An EOQ model for items with Weibull distribution deterioration, shortages and trended demand: An extension of Philip's model*, Computers & Operations Research, Vol. 25, pp.649-657.

[6] Dave, U. and Patel, L. K. (1981), *(T, S_i) policy inventory model for deteriorating items with time proportional demand*, Journal of the Operational Research Society, Vol.32, pp.137-142.

[7] Donaldson, W. A. (1977), *Inventory replenishment policy for a linear trend in demand: An analytical solution*, Operational Research Quarterly, Vol. 28, pp.663-670.

[8] Goswami, A. and Chaudhuri, K. S. (1991), *An EOQ model for deteriorating items with shortages and a linear trend in demand*, Journal of Operational Research Society Vol. 42, pp.1105-1110.

[9] Goyal, S. K. (1985), *Economic order quantity under conditions of permissible delay in payments*, Journal of Operational Research Society, Vol. 36, pp.335-338.

[10] Goyal, S. K., Morin, D. and Nebebe, F. (1992), *The finite horizon trended inventory replenishment problem with shortages*, Journal of the Operational Research Society, Vol. 43, pp.1173-1178.

[11] Hwang, H. and Shinn, S. W. (1997), *Retailer's pricing and lot sizing policy for exponentially deteriorating product under the condition of permissible delay in payments*, Computers & Operations Research, Vol. 24, pp.539-547.

[12] Huang, Y. F. (2003), *Optimal retailer's ordering policies in the EOQ model under trade credit financing*, Journal of the Operational Research Society, Vol. 54, pp.1011-1015.

[13] Jamal, A. M, Sarker, B. R. and Wang, S. (1997), *An ordering policy for deteriorating items with allowable shortage and permissible delay in payment*, Journal of Operational Research Society, Vol. 48, pp.826-833.

[14] Khanra, S. and Chaudhuri, K. S. (2003), *A note on an order-level inventory model for a deteriorating item with time-dependent quadratic demand*, Computers & Operations Research, Vol. 30, pp.1901-1916.

[15] Ouyang, L. Y., Chuang, K. W. and Chuang, B. R. (2004), *An inventory model with noninstantaneous receipt and permissible delay in payments*, International Journal of Information and Management Sciences, Vol. 15, pp.1-11.

[16] Philip, G. C. (1974), *A generalized EOQ model for items with Weibull distribution deterioration*, AIIE Transactions, Vol. 6, pp.159-162.

[17] Resh, M., Friedman, M. and Barbosa, L. C. (1976), *On a general solution of the deterministic lot size problem with time-proportional demand*, Operations Research, Vol. 24, pp.718-725.

[18] Sachan, R. S. (1984), *On* (T, S_i) *policy inventory model for deteriorating items with time proportional demand*, Journal of the Operational Research Society, Vol. 35, pp.1013-1019.

[19] Sarker, B. R., Jamal, A. M. and Wang, S. (2000), *Supply chain models for perishable products under inflation and permissible delay in payment*, Computers & Operations Research, Vol. 27, pp.59-75.

[20] Shinn S. W., Hwang H. P. and Sung S. (1996), *Joint price and lot size determination under conditions of permissible delay in payments and quantity discounts for freight cost*, European Journal of Operational Research, Vol. 91, pp.528-542.

[21] Tadikamalla, P. R. (1978), *An EOQ inventory model for items with gamma distribution*, AIIE Transactions, Vol. 10, pp.108-112.

[22] Teng, J. T. (2002), *On the economic order quantity under conditions of permissible delay in payments*, Journal of the Operational Research Society, Vol. 53, pp.915-918.

[23] Teng, J. T., Chang C. T., Chern, M. S. and Chan, Y. L. (2007), *Retailers optimal ordering policies with trade credit financing*, International Journal of Systems Science, Vol. 38, pp.269-278.

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