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A SIMPLE SOLUTION METHOD FOR THE FINITE HORIZON EOQ MODEL FOR DETERIORATING ITEMS WITH COST CHANGES

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In this article, we generalize Lev and Weiss's (1990) finite horizon economic order quantity (EOQ) model with cost change to the inventory system with deterioration. Supplier announces some or all of cost parameters may change after a decided time. Depending on whether the inventory is depleted at the time of the last opportunity to purchase before some or all of the cost parameters may change, there are two types of inventory models to be discussed. The main objective of this paper is to identify the optimal ordering policy of the inventory system by comparing the minimum cost of the two types of models. We suggest a finite horizon EOQ model to combine the above two types and propose a theorem that can quickly identify the optimal policy of the suggested model. In considering temporary price discount problem and discrete-time EOQ problem, in general, there are integer operators in mathematical models, but our approach offers a closed-form solution to these kinds of problems. Numerical examples are presented to demonstrate the results of the proposed properties and theorem.

Keywords: Inventory; EOQ; deterioration; cost change; finite horizon.

1. Introduction

The determination of economic order quantity (EOQ) is a popular issue in supply chain management. Inventory theory literature contains many interesting EOQ models. The traditional EOQ inventory model assumes that the inventory parameters (e.g., per unit cost, demand rate, setup cost or holding cost) are constant during the sale period. In reality, there are many reasons for a supplier to change some or all of cost parameters to buyers. If the supplier announces an increase in product or raw material costs, the buyer generally responds to the price-increase by engaging in forward-buying. They purchase a lot before the price increases. Naddor (1966) developed an EOQ inventory model that addressed the situation when there is an

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announced price increase. Schwarz (1972) discussed a finite horizon EOQ model, where the costs of the model were static and the optimal number of orders for a finite horizon needed to be determined. Lev and Weiss (1990) considered the case where the cost parameters may change, and the horizon length may be finite as well as infinite. They suggested some methods to solve the inventory problem, but there are many calculation steps in their algorithms and there are infeasible solutions in their results. Gascon (1995) indicated that the lower and upper bounds on order number in Lev and Weiss's (1990) *Type* 2 model do not guarantee feasibility of the solution. Luo and Huang (2003) showed that some of the results in Lev and Weiss (1990) are not necessarily optimal. If the length of the finite horizon is deterministic, the same size order number can be decided by Schwarz's (1972) result. But in Lev and Weiss (1990), the length of the finite horizon after special order is unknown and it is dependent on order number. The obtained order number may fail to be the solution of the minimum sum of costs.

A common situation of cost parameter change is when suppliers offer temporary reduction in selling price to buyers. Buyers typically respond by placing a special order for a large lot. Tersine (1994) proposed a temporary price discount model, where the optimal ordering policy is obtained by maximizing the difference between regular EOQ cost and special order quantity cost during the sale period. Martin (1994) revealed that Tersine's (1994) representation of average inventory in the total cost is flawed, and suggested the true representation of average inventory. Wee and Yu (1997) assumed that the items deteriorated exponentially with time when temporary price discount purchase occurs at the regular and non-regular replenishment time. The above articles solve the temporary price discount problems by numeric method instead of closed form solution. Since there are integer operators in their objective functions, it is hard to solve this kind of problem. The related analysis on inventory systems of these problems have been performed by Abad (2003), Saker and Al Kindi (2006), Hsu and Yu (2009), C'ardenas-Barr'on (2009a, 2009b), C'ardenas-Barr'on *et al.* (2010), etc.

Since many products like machine parts are not continuously divisible, some authors have investigated the EOQ problems when the lot size should be an integer quantity. Kovalev and Ng (2008) proposed a model where the planning horizon is finite and demand is constant during discrete time periods. They minimized the objective function to one variable and developed an algorithm to find the optimal number of orders, which takes $O(\log n)$ time. Li (2009) modified Kovalev and Ng's (2008) method and presented a solution method which can determine the optimal solution without need of such a search. García-Laguna *et al.* (2010) presented methods to obtain the solutions of the EOQ and EPQ models when the lot sizes are integer variables to be determined. The suggested approach allows obtaining a rule to discriminate between the situations in which the optimal solution is unique and when there are two solutions. The related analysis on classical EOQ model with integer variable quantity problems have been performed by Bertazzi and Speranza (2005), Lodree (2007), Ng *et al.* (2009), etc.

We present an EOQ model for deteriorating inventory with cost changes over a finite horizon and suggest an easy approach to solve the models with integer operators. The remainder of this paper is organized as follows. In the next section, we describe the notation and assumptions, which are used throughout this paper. In Sec. 3, we describe the mathematical models and suggest properties and theorem to determine the optimal ordering policy. Numerical examples are provided in Sec. 4 to support our proposed theorem. Conclusions are given in the last section.

2. Notation and Assumptions

In developing the inventory model to determine the optimal ordering policy, the following notation and assumptions are used.

Notation:

$\lceil\rceil$: Integer operator, integer value equal to or greater than its argument
$\lfloor\rfloor$: Integer operator, integer value equal to or less than its argument
Q : Inventory level of the system
θ : Constant deterioration rate
D : Constant demand rate
A_i : Ordering cost per order in period $i(i = 1, 2)$
P_i : Per unit cost for items brought into stock in period $i(i = 1, 2)$
h_i : Per unit per cycle holding cost rate for all items brought into stock in period $i(i = 1, 2)$
T : Time of the last opportunity to purchase before some or all of the cost parameters may change
H : Horizon length in type 1 and type 2 models
T' : Horizon length in type 3 model
n_i : Number of o equal order sizes in the interval $[0, T)$ for type $i(i = 1, 2)$ model
K : Number of order size Q_K in type 3 model
T_s : Depletion time which corresponds to order size Q_s in type 1 model
T_e : Depletion time which corresponds to order size Q_K in type 3 model
t_e : Depletion time which corresponds to order size Q_m in type 3 model
$FS_1(T_e)$: Total cost of type 1 model
$FS_2(T_e)$: Total cost of type 2 model
$TC(T_e)$: Total cost of type 3 model
$F_1(n, m)$: Average cost of type 1 model in Lev and Weiss (1990)
* : The superscript representing optimal value

Assumptions:

(1) The initial inventory is zero and that delivery is instantaneous.
(2) Shortages are not allowed.
(3) The demand rate is constant and deterministic.
(4) There is no constraint in space or capacity.

(5) The time horizon length is finite.
(6) The holding cost is charged against the purchase price.

3. Mathematical Model

In this article, the mathematical models of the inventory problem with constant deterioration rate and constant demand rate are discussed. In general, an inventory level Q of the system with constant deterioration rate θ and constant demand rate D at time t over period τ can be described by the following equation:

$$dQ(t)/dt + \theta Q(t) = -D \quad 0 \leq t \leq \tau \tag{1}$$

The solution of the above equation with boundary condition $Q(\tau) = 0$ is

$$Q(t) = D[\exp\theta(\tau - t) - 1]/\theta \quad 0 \leq t \leq \tau \tag{2}$$

The initial inventory at $t = 0$ is

$$Q(0) = D[\exp(\theta\tau) - 1]/\theta \tag{3}$$

Ordering cost is A per order, purchasing price is P per unit and holding cost rate is h per unit per cycle. The total cost during the cycle is composed of ordering cost, purchasing cost and holding cost, that is

$$TC(\tau) = A + PQ(0) + h\int_0^\tau Q(t)dt \tag{4}$$

$$TC(\tau) = A + PD[\exp(\theta\tau) - 1]/\theta + hD[\exp(\theta\tau) - \theta\tau - 1]/\theta^2 \tag{5}$$

For $\theta\tau \ll 1$, using Taylor series approximation, $\exp(\theta\tau) \approx 1 + \theta\tau + \theta^2\tau^2/2$. The approximate total inventory cost is

$$TC(\tau) = A + PD\tau + D(P\theta + h)\tau^2/2 \tag{6}$$

The total finite time horizon H is partitioned into two periods. Period 1 is the interval $[0, T]$ and period 2 is the interval $(T, H]$, where T is the time of the last opportunity to purchase before some or all of the cost parameters may change. In period 1, items for ordering cost is A_1 per order, purchasing price is P_1 per unit and holding cost rate is h_1 per unit per cycle. In period 2, items for ordering cost is A_2 per order, purchasing price is P_2 per unit and holding cost rate is h_2 per unit per cycle. Initial and final inventories are both zero. In order to describe the behavior of optimal policy in the intervals $[0, T)$ and $(T, H]$, the following property investigates the consecutive order cycles in the interval $[0, T)$ and $(T, H]$.

Property 1. The optimal policy for the consecutive order cycles are equal as orders are placed in the intervals $[0, T)$ and $(T, H]$, respectively.

Proof of property 1 is given in the Appendix.

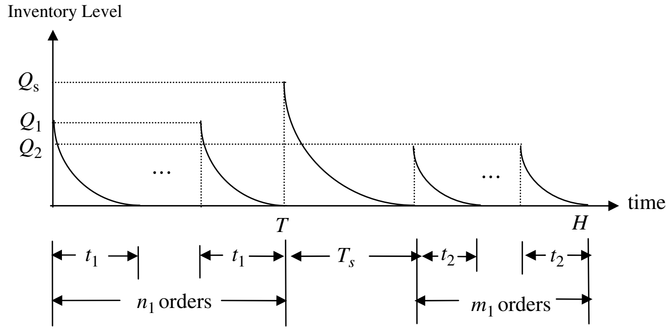

Fig. 1. Inventory is depleted at time T (*Type* 1 model).

Depending on whether the inventory is depleted at time T or not, there are two possible types of inventory models to be discussed. *Type* 1 model: the inventory is depleted at time T, as shown in Fig. 1, where T is the time of the last opportunity to purchase before some or all of the cost parameters may change. At time T, the buyer takes a special order. *Type* 2 model: the inventory is not depleted at time T, as shown in Fig. 2. Because the inventory is not depleted at time T, there is no chance to take a special order at time T.

From property 1, it implies that the optimal policy for the consecutive order sizes are equal as orders are placed in the intervals $[0, T)$ and $(T, H]$, respectively. For *Type* 1 model, the optimal policy is to place n_1 equal order sizes $Q_1 = D[\exp(\theta T/n_1) - 1]/\theta$ in the interval $[0, T)$, then to place a special order of size $Q_s = D[\exp(\theta T_s) - 1]/\theta$ at time T and, finally, to place m_1 equal order sizes $Q_2 = D\{\exp[\theta(H - T - T_s)/m_1] - 1\}/\theta$ in the interval $(T + T_s, H]$. For *Type* 2 model, the optimal policy is to place n_2 equal order sizes $Q_3 = D[\exp(\theta t_3) - 1]/\theta$ before T and then m_2 equal orders of size $Q_4 = D\{\exp[\theta(H - nt_3)/m_2] - 1\}/\theta$ after T. The following property describes the optimal order number n_1 in the interval $[0, T)$ for *Type* 1 model.

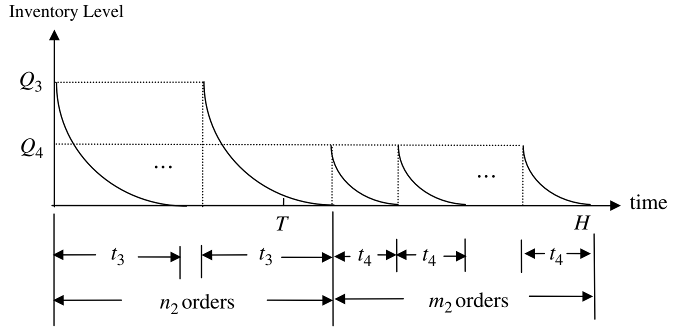

Fig. 2. Inventory is not depleted at time T (*Type* 2 model).

Property 2. There are n_1 consecutive orders placed within the interval $[0, T)$, and the inventory is zero at time T. Let

$$x = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{(P_1\theta + h_1)DT^2}{2A_1}} \tag{7}$$

If x is not an integer, the optimal order number n_1^* for the minimum cost within the interval $[0, T)$ is $n_1^* = \lceil x \rceil$. Otherwise, both $n_1^* = \lceil x \rceil$ and $n_1^* = \lceil x \rceil + 1$ are optimal integer solutions of n_1 consecutive orders.

Proof of property 2 is given in the Appendix.

For *Type* 2 model, the optimal order number n_2^* for the minimum cost within the interval $[0, T)$ is given by Theorem 2 proposed in Lev and Weiss (1990). That is, $n_2^* = n_1^*$ or $n_2^* = n_1^* + 1$. *Type* 2 model must satisfy the condition $(n_2^* - 1)t_1 < T < n_2^* t_1$.

Now we propose a general *Type* 3 model as shown in Fig. 3, which can be applied to both *Type* 1 and *Type* 2 models. For *Type* 3 model, the finite horizon of length is T', the optimal policy is to place K equal order sizes Q_K and place m equal order sizes Q_m in the interval $[0, T']$. T_e is the depletion time which corresponds to order size Q_K and t_e is the depletion time which corresponds to order size Q_m.

Comparing *Type* 3 model with *Type* 1 model (We take $K = 1$, $T_e = T_s$ and $t_e = t_2$), the time during 0 and T' in *Type* 3 model is the same as the time during T and H in *Type* 1 model. Comparing *Type* 3 model with *Type* 2 model (We take $K = n_2^*$, $T_e = t_3$ and $t_e = t_4$), the time during 0 and T' in *Type* 3 model is the same as the time during 0 and H in *Type* 2 model. In the interval $[0, KT_e)$, items for ordering cost is A_1 per order, purchasing price is P_1 per unit and holding cost rate is h_1 per unit per cycle. In the interval $[KT_e, T']$, items for ordering cost is A_2 per order, purchasing price is P_2 per unit and holding cost rate is h_2 per unit per cycle. If K is given, the total cost of *Type* 3 model is

$$\begin{aligned} TC(T_e, m) = {} & K[A_1 + P_1DT_e + D(P_1\theta + h_1)T_e^2/2] + mA_2 + P_2D(T' - KT_e) \\ & + D(P_2\theta + h_2)(T' - KT_e)^2/2m \quad \text{if } m > 0 \end{aligned} \tag{8}$$

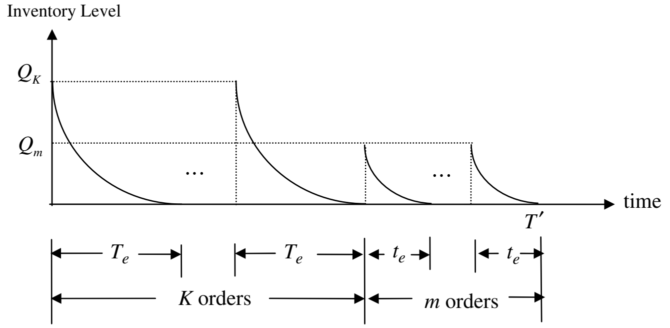

Fig. 3. Graphical representation of the *Type* 3 model.

$$TC(T_e) = K[A_1 + P_1 DT_e + D(P_1\theta + h_1)T_e^2/2] \quad \text{if } m = 0 \tag{9}$$

Let

$$y = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{(p_2\theta + h_2)D(T' - KT_e)^2}{2A_2}} \tag{10}$$

Because the inventory is zero at T', from property 2, the relation of T_e and m is $m = \lceil y \rceil$ for y is not an integer number. Otherwise, $m = \lceil y \rceil$ and $m = \lceil y \rceil + 1$ for y is an integer number. For $m > 0$, $TC(T_e, m)$ reduces to

$$\begin{aligned} TC(T_e) = {} & K[A_1 + P_1 DT_e + \frac{1}{2}D(P_1\theta + h_1)T_e^2] + P_2 D(T' - KT_e) \\ & + \left\lceil -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{(p_2\theta + h_2)D(T' - KT_e)^2}{2A_2}} \right\rceil A_2 \\ & + \frac{D(P_2\theta + h_2)(T' - KT_e)^2}{2\lceil -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{(p_2\theta + h_2)D(T' - KT_e)^2}{2A_2}} \rceil} \end{aligned} \tag{11}$$

Since the time during 0 and T' in *Type* 3 model is the same as the time during T and H in *Type* 1 model, applying *Type* 3 model to *Type* 1 model in the interval $[T, H]$ and taking $K = 1$, the total cost $FS_1(T_e)$ for *Type* 1 model in the interval $[0, H]$ is

$$FS_1(T_e) = n_1^* A_1 + P_1 DT + D(P_1\theta + h_1)T^2/2n_1^* + TC(T_e) \tag{12}$$

Applying *Type* 3 model to *Type* 2 model, taking $K = n_2^*$, the total cost $FS_2(T_e)$ for *Type* 2 model is

$$FS_2(T_e) = TC(T_e) \tag{13}$$

Type 2 model must satisfy the condition $(K-1)T_e < T < KT_e$. Comparing $FS_1(T_e)$ with $FS_2(T_e)$, we can decide the optimal ordering policy of the inventory system. We propose some properties below that can help us find quickly the minimum value of total cost $TC(T_e)$.

Property 3. $TC(T_e)$ is a continuous piecewise quadratic function of T_e.

Proof of proposition 3 is given in the Appendix.

Property 4. m is a positive integer. In order to check the property of $TC(T_e)$, let

$$t_m = [T' - \sqrt{2A_2[(m + 1/2)^2 - 1/4]/D(P_2\theta + h_2)}]/K$$

$$\begin{aligned} S_+(m) = {} & 2A_2(P_1\theta + h_1)^2 m^3 + 2A_2(P_1\theta + h_1)[2(P_2\theta + h_2)K - (P_1\theta + h_1)]m^2 \\ & + (P_2\theta + h_2)\{2A_2K[(P_2\theta + h_2)K - 2(P_1\theta + h_1)] \\ & - D[(P_1\theta + h_1)T' - K(P_2 - P_1)]^2\}m - 2A_2(P_2\theta + h_2)^2K^2 \end{aligned} \tag{14}$$

$$\begin{aligned}S_-(m) &= 2A_2(P_1\theta+h_1)^2m^3 + 2A_2(P_1\theta+h_1)[2(P_2\theta+h_2)K+(P_1\theta+h_1)]m^2\\ &\quad + (P_2\theta+h_2)\{2A_2K[(P_2\theta+h_2)K+2(P_1\theta+h_1)]\\ &\quad - D[(P_1\theta+h_1)T' - K(P_2-P_1)]^2\}m + 2A_2(P_2\theta+h_2)^2K^2 \qquad (15)\end{aligned}$$

The intersection region of $S_+(m) < 0$ and $S_-(m) > 0$ is $m_R < m < m_L$.

(i) If $m < \lfloor m_R \rfloor$, $TC(T_e)$ is an increasing function of T_e within the interval $[t_m, t_{m-1})$.
(ii) If $m > \lfloor m_L \rfloor$, $TC(T_e)$ is a decreasing function of T_e within the interval $[t_m, t_{m-1})$.
(iii) If $\lfloor m_R \rfloor \le m \le \lfloor m_L \rfloor$, $TC(T_e)$ is a convex function of T_e within the interval $[t_m, t_{m-1})$.

Proof of proposition 4 is given in the Appendix.

Proposition 4 implies that $TC(T_e)$ has a local minimum value within the interval $[t_m, t_{m-1})$ in the range $\lfloor m_R \rfloor \le m \le \lfloor m_L \rfloor$. From Eq. (A.6), let $dTC(T_e)/dT_e = 0$, the local minimum value within the interval $[t_m, t_{m-1})$ for a given m is located at

$$T_e(m) = \frac{m(P_2-P_1)+(P_2\theta+h_2)T'}{m(P_1\theta+h_1)+(P_2\theta+h_2)K} \qquad (16)$$

Taking $T_e(m)$ into $TC(T_e)$, the minimum total cost of *Type* 3 model is

$$\begin{aligned}TM(m) &= \frac{-m(P_1-P_2)^2K + [2K(P_1-P_2)+(P_1\theta+h_1)T'](P_2\theta+h_2)T'}{[m(P_1\theta+h_1)+K(P_2\theta+h_2)]}\frac{D}{2}\\ &\quad + P_2DT' + KA_1 + mA_2 \qquad (17)\end{aligned}$$

Now we propose Theorem 1 that can quickly and directly find the global minimum total cost $TC(T_e)$ in *Type* 3 model.

Theorem 1. *The minimum total cost of $TC(T_e)$ is $TM(m)$, as shown in Eq.* (17), *m is a non-negative integer. T_e is the depletion time corresponds to order sizes Q_K, as shown in Eq.* (16). *$S_+(m)$ and $S_-(m)$ are defined in property* 4. *The intersection region of $S_+(m) < 0$ and $S_-(m) > 0$ is $m_R < m < m_L$. Let*

$$z = -\left[\frac{1}{2} + \frac{K(P_2\theta+h_2)}{(P_1\theta+h_1)}\right] + \sqrt{\frac{1}{4} + \frac{D(P_2\theta+h_2)[K(P_2-P_1)-(P_1\theta+h_1)T']^2}{2(P_1\theta+h_1)^2A_2}}$$

When z is not an integer, taking $m^* = \lceil z \rceil$. Otherwise, taking $m^* = \lceil z \rceil$ and $m^* = \lceil z \rceil + 1$.

(i) *If $m^* \le \lfloor m_R \rfloor$, the minimum value of $TC(T_e)$ can be founded by $TM(\lfloor m_R \rfloor)$, and $T_e = T_e(\lfloor m_R \rfloor)$.*
(ii) *If $m^* \ge \lfloor m_L \rfloor$, the minimum value of $TC(T_e)$ can be founded by $TM(\lfloor m_L \rfloor)$ and $T_e = T_e(\lfloor m_L \rfloor)$.*

(iii) *If* $\lfloor m_R \rfloor < m^* < \lfloor m_L \rfloor$, *the minimum value of* $TC(T_e)$ *can be founded by* $TM(m^*)$ *and* $T_e = T_e(m^*)$.

Proof of Theorem 1 is given in the Appendix.

Theorem 1 reveals that: (1) If $m^* \leq \lfloor m_R \rfloor$, the optimal policy for type 3 is to place K equal order sizes Q_K and place $\lfloor m_R \rfloor$ equal order sizes Q_m in the interval $[0, T']$. The minimum total cost of $TC(T_e)$ is $TM(\lfloor m_R \rfloor)$ and the depletion time corresponds to order sizes Q_K is $T_e(\lfloor m_R \rfloor)$. (2) If $m^* \geq \lfloor m_L \rfloor$, the optimal policy for type 3 is to place K equal order sizes Q_K and place $\lfloor m_L \rfloor$ equal order sizes Q_m in the interval $[0, T']$. The minimum total cost of $TC(T_e)$ is $TM(\lfloor m_L \rfloor)$ and the depletion time corresponds to order sizes Q_K is $T_e(\lfloor m_L \rfloor)$. (3) If $\lfloor m_R \rfloor < m^* < \lfloor m_L \rfloor$, the optimal policy for type 3 is to place K equal order sizes Q_K and place m^* equal order sizes Q_m in the interval $[0, T']$. The minimum total cost of $TC(T_e)$ is $TM(m^*)$ and the depletion time corresponds to order sizes Q_K is $T_e(m^*)$.

Applying Theorem 1 to *Type* 1 model, we take n_1^* by property 2 and $K = 1$, the total cost of *Type* 1 is

$$FS_1(T_e) = n_1^* A_1 + P_1 DT + D(P_1\theta + h_1)T^2/2n_1^* + TC(T_e)$$

Applying Theorem 1 to *Type* 2 model, we take $K = n_1^*$ or $K = n_1^* + 1$, the total cost of *Type* 2 model is

$$FS_2(T_e) = TC(T_e)$$

Type 2 model must satisfy the condition $(K-1)T_e < T < KT_e$. Taking the minimum total cost of the two type models, we can decide the optimal ordering policy of the inventory system. If the deterioration rate $\theta = 0$, this is the special case of Lev and Weiss (1990) finite horizon model.

4. Numerical Examples

We use the same cost parameters of Lev and Weiss (1990) except $\theta = 0.001$/year to illustrate the properties and theorem we proposed: $D = 120$ units/year, $A_1 = A_2 = 80$/order, $P_1 = 100$/unit, $P_2 = 101$/unit, $h_1 = 12$/unit/year, $h_2 = 13.2$/unit/year, $T = 0.5$ year, H varies from 1 to 3 years. Figures 4 and 5 show the diagram of $TC(T_e)$ and $TM(m)$ respectively when H equals to 1.5. From properties 4, the intersection of $S_+(m) < 0$ and $S_-(m) > 0$ is $0.90 < m < 2.55$. In Fig. 4, $TC(T_e)$ is a convex function of T_e in the range $0 \leq m \leq 1$ and $1 \leq m \leq 2$, and $TC(T_e)$is a decreasing function of T_e when $m > 2$. In Fig. 5, owing to $m^* = 2$, $\lfloor m_R \rfloor = 0$ and $\lfloor m_L \rfloor = 2$, $TM(m)$ decreases as integer m increases in the range $\lfloor m_R \rfloor \leq m \leq \lfloor m_L \rfloor$. From theorem 1, we an get $n_1^* = 2$, $t_1 = 0.25$, $m_1^* = 2$, $T_s = 0.39$ and total cost is $FS_1(T_e) = 18810.8$ for *Type* 1 model. Also we get $n_2^* = 2$, $t_3 = 0.42$, $m_2^* = 2$, and total cost is $FS_2(T_e) = 18815.0$ for *Type* 2 model. Comparing *Type* 1 model with *Type* 2 model, the optimal ordering policy is *Type* 1 model.

Table 1 shows the optimal ordering policy of *Type* 1 and *Type* 2 models for a finite horizon that varies from 1 to 3 years. If we take $\theta = 0$, there is no deterioration,

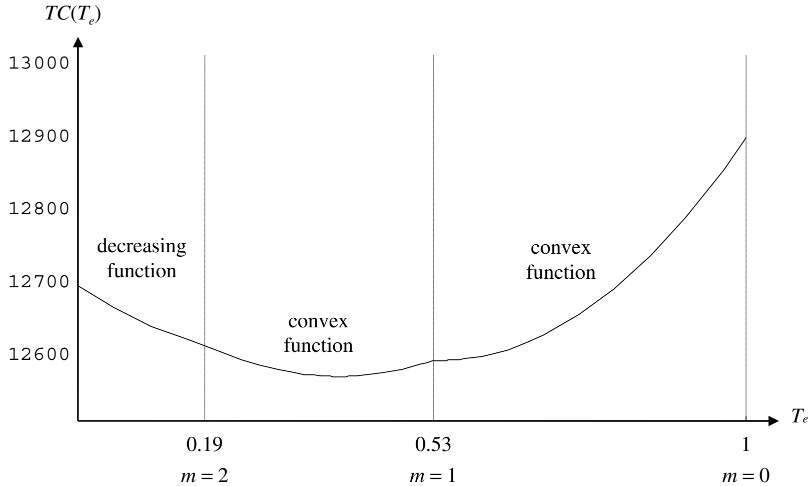

Fig. 4. Diagram of $TC(T_e)$ when $H = 1.5$.

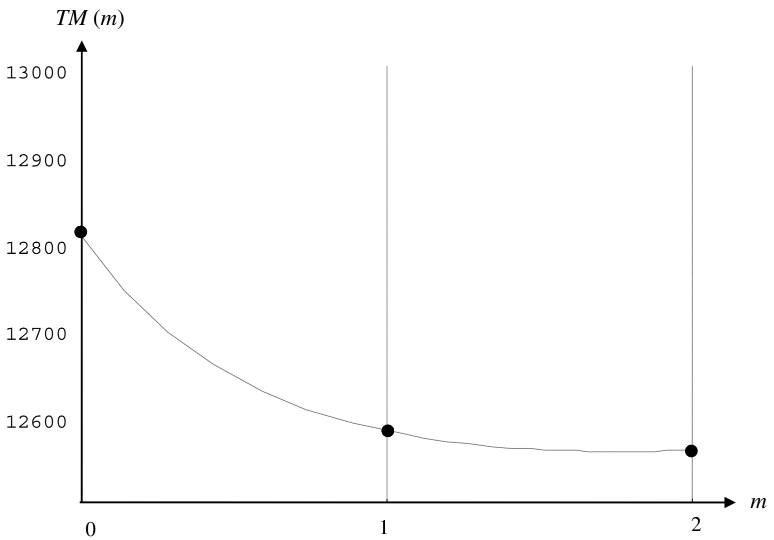

Fig. 5. Diagram of $TM(m)$ when $H = 1.5$.

this is the special case of Lev and Weiss (1990) finite horizon model. Considering the average cost of their model, we modify some results of their *Type* 1 model in Table 2. When the finite horizon of length H equals to 12, 18 and 21, the optimal policy in decision-making is *Type* 1 model rather than *Type* 2 model.

Luo and Huang (2003) extend Lev and Weiss's (1990) expressions of the optimal order number. Applying our suggested *Type* 3 model and using the same cost

Table 1. Total cost and optimal policy of *Type* 1 and *Type* 2 models for different finite horizons.

$D = 120$ units/year, $A_1 = A_2 = 80$/order, $P_1 = 100$/unit, $P_2 = 101$/unit, $h_1 = 12$/unit/year, $h_2 = 13.2$/unit/year, $T = 0.5$ year, $\theta = 0.001$/year									Optimal policy
	Type 1 policy				*Type* 2 policy				
H	n_1^*	m_1^*	T_s	$FS_1(T_e)$	n_2^*	m_2^*	t_3	$FS_2(T_e)$	
1.0	2	0	0.5	12512.3	2	1	0.36	12519.3	1
13/12	2	1	0.33	13567.2	2	1	0.39	13564.7	2
1.5	2	2	0.39	18810.8	2	2	0.42	18815.0	1
2.0	2	3	0.44	25113.1	2	4	0.39	25120.1	1
2.5	2	5	0.40	31414.3	2	5	0.42	31419.4	1
3.0	2	6	0.43	37717.0	2	7	0.40	37723.3	1

Table 2. Average cost and optimal policy of *Type* 1 model in Lev and Weiss.

$D = 10$ unit/month, $A_1 = A_2 = 80$/order, $P_1 = 100$/unit, $P_2 = 101$/unit, $h_1 = 1$/unit/month, $h_2 = 1.01$/unit/month, $T = 6$ month								
	Lev and Weiss's origin results				Lev and Weiss's modified results			
H	n	m	T_s	$F_1(n,m)$	n	m	T_s	$F_1(n,m)$
12	2	1	3.51	1043.98	2	0	6	1042.50
17	2	1	6.02	1045.07	2	2	4.36	1044.87
18	2	1	6.52	1046.05	2	2	4.69	1044.89
21	2	2	5.69	1045.89	2	3	4.53	1045.80
25	2	3	5.53	1046.52	2	4	4.63	1046.46
29	2	4	5.43	1046.99	2	5	4.70	1046.95
33	2	5	5.36	1047.36	2	6	4.75	1047.33

Table 3. Comparing Luo and Huang's result to our result.

$D = 12$ units/year, $H = 2.25$ year, $T = 0.25$ year, $A_1 = 10$/order, $A_2 = 50$/order, $P_1 = 2$/unit, $P_2 = 12$/unit, $h_1 = 60$/unit/year, $h_2 = 600$/unit/year, $\theta = 0$				
	n	m	T_s	$FS_1(T_e)$
Luo and Huang	2	6	1.31	1381.5
Our method	2	6	1.31	1381.5

parameters of Luo and Huang (2003), we obtain the same total cost and optimal policy of Luo and Huang's (2003) *Type* 1 model. The results are shown in Table 3.

5. Conclusion

This article deals with an approach to decide the optimal ordering policy for deteriorating inventory when some or all of the cost parameters may change over a finite horizon. We propose a closed-form solution to solve this problem. Because there are integer operators in some inventory models which have several local minima, it is hard to find closed form solution for those models. A distinguishing feature of the proposed approach is that it can be applied to solve this kind of problem, especially, in temporary price discount problem and discrete-time EOQ problem.

By using the proposed theorem, we can obtain the minimum solutions more easily and simply than the algorithm proposed by Lev and Weiss (1990). We also modify some results of their *Type* 1 model and correct some optimal ordering policies in decision-making.

The further advanced research will extend the proposed method to handle the EOQ inventory model with shortage is allowed, either the shortage cost is fixed backorder cost or linear backorder cost.

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Appendix

Proof of Property 1. Considering the deteriorating inventory system, suppose there are n consecutive orders placed in the interval $[0, T)$. Each order cycle is t_i, $i = 1, 2, \ldots, n$. $\sum_{i=1}^{n} t_i = t$, t may be smaller, equal or larger than T. The total cost of these n orders is

$$TC(t_i) = nA_1 + P_1 D \sum_{i=1}^{n} t_i + D(P_1\theta + h_1) \sum_{i=1}^{n} t_i^2/2$$

then

$$TC(t_i) = nA_1 + P_1 Dt + D(P_1\theta + h_1) \sum_{i=1}^{n} t_i^2/2 \tag{A.1}$$

From Chaucy–Scharz inequality, $TC(t_i)$ has minimum value in the condition $t_i = t/n$. Hence, the optimal policy for the order cycles are equal of n consecutive orders placed in $[0, T)$. The proof can be applied equally to the consecutive order cycles in the interval $(T, H]$. □

Proof of Property 2. In the interval $[0, T]$, items for ordering cost is A_1 per order, purchasing price is P_1 per unit and holding cost rate is h_1 per unit per cycle. Because the inventory is zero at T, the total cost of these n_1 orders is

$$TC(n_1) = n_1 A_1 + P_1 DT + D(P_1\theta + h_1)T^2/2n_1 \tag{A.2}$$

Since $TC(n_1)$ is a convex function over the set of the positive integer numbers, by $TC(n_1 + 1) - TC(n_1) \geq 0$, let $x = -0.5 + \sqrt{0.25 + (P_1\theta + h_1)DT^2/2A_1}$, we get the optimal lower solution

$$n_1^* = \lceil x \rceil \tag{A.3}$$

Similarly, by $TC(n_1 - 1) - TC(n_1) \geq 0$, we get the optimal upper solution

$$n_1^* = \lfloor x + 1 \rfloor \tag{A.4}$$

If x is not an integer number, the unique optimal solution for the minimum cost within the interval $[0, T)$ is $n_1^* = \lceil x \rceil = \lfloor x+1 \rfloor$. Otherwise, there are two optimal solutions $n_1^* = \lceil x \rceil$ and $n_1^* = \lceil x \rceil + 1$. □

Proof of Property 3. Let $t_m = [T' - \sqrt{2A_2[(m+1/2)^2 - 1/4]/D(P_2\theta + h_2)}]/K$, m is a positive integer. For $t_m \leq T_e < t_{m-1}$

$$TC(T_e) = [(P_1\theta + h_1) + K(P_2\theta + h_2)m]KDT_e^2/2 + [P_1 - P_2 - (P_2\theta + h_2)T'/m]KDT_e + P_2DT' + KA_1 + mA_2 + D(P_2\theta + h_2)T'^2/2m \quad (A.5)$$

$$dTC(T_e)/dT_e = [(P_1\theta + h_1) + K(P_2\theta + h_2)m]KDT_e - [P_2 - P_1 + (P_2\theta + h_2)T'/m]KD \quad (A.6)$$

$$d^2TC(T_e)/dT_e^2 = [(P_1\theta + h_1) + K(P_2\theta + h_2)m]KD > 0 \quad (A.7)$$

Hence, $TC(T_e)$ is a quadratic function of T_e within the interval $[t_m, t_{m-1})$. In addition

$$\lim_{\alpha \to 0^+} TC(t_{m+\alpha}) = (P_1\theta + h_1)DT'^2/2K + P_1DT' + KA_1 + (2m+1)A_2 + (P_1\theta + h_1)A_2m(m+1)/[(P_2\theta + h_2)K] + [P_2 - P_1 - (P_1\theta + h_1)T'K]\sqrt{2A_2D/(P_2\theta + h_2)}\sqrt{m(m+1)} \quad (A.8)$$

$$\lim_{\beta \to 0^+} TC(t_{m-\beta}) = (P_1\theta + h_1)DT'^2/2K + P_1DT' + KA_1 + (2m+1)A_2 + (P_1\theta + h_1)A_2m(m+1)/[(P_2\theta + h_2)K] + [P_2 - P_1 - (P_1\theta + h_1)T'K]\sqrt{2A_2D/(P_2\theta + h_2)}\sqrt{m(m+1)} \quad (A.9)$$

$$\lim_{\alpha \to 0^+} TC(t_{m+\alpha}) = \lim_{\beta \to 0^+} TC(t_{m-\beta})$$

Therefore, $TC(T_e)$ is a continuous piecewise quadratic function of T_e. □

Proof of Property 4. From Eqs (A.5)–(A.7), $TC(T_e)$ is a quadratic function of T_e within the interval $[t_m, t_{m-1})$. If

$$T_e(m) = \frac{m(P_2 - P_1) + (P_2\theta + h_2)T'}{m(P_1\theta + h_1) + (P_2\theta + h_2)K} \quad (A.10)$$

then $dTC(T_e)/dT_e = 0$. For a given m, if $dTC(T_e)/dT_e > 0$, it would be $T_e > T_e(m)$. The necessary condition for $dTC(T_e)/dT_e > 0$ within the interval $[t_m, t_{m-1})$ is $T_e(m) < t_{m-1}$. By calculating, m is found by $S_+(m) < 0$. If $dTC(T_e)/dT_e < 0$, it would be $T_e < T_e(m)$. The necessary condition for $dTC(T_e)/dT_e < 0$ within the interval $[t_m, t_{m-1})$ is $t_m < T_e(m)$. By calculating, m is found by $S_-(m) > 0$. The intersection region of $S_+(m) < 0$ and $S_-(m) > 0$ is $m_R < m < m_L$. Because m is a integer, we take $\lfloor m_R \rfloor \leq m \leq \lfloor m_L \rfloor$. We conclude as follows: (i) If $m < \lfloor m_R \rfloor$, $TC(T_e)$ has $dTC(T_e)/dT_e > 0$ property, also $d^2TC(T_e)/dT_e^2 > 0$, it makes sense that $TC(T_e)$ is an increasing function of T_e within the interval $[t_m, t_{m-1})$. (ii) If $m > \lceil m_L \rceil$, $TC(T_e)$ has $dTC(T_e)/dT_e < 0$ property, also $d^2TC(T_e)/dT_e^2 > 0$, it makes sense that $TC(T_e)$ is a decreasing function of T_e within the interval $[t_m, t_{m-1})$. (iii) If $\lfloor m_R \rfloor \leq m \leq \lfloor m_L \rfloor$, from Eq (A.6) and Eq (A.7), $dTC(T_e)/dT_e$ is a continuous function and has $dTC(T_e)/dT_e = 0$ property, also $d^2TC(T_e)/dT_e^2 > 0$, it makes sense that $TC(T_e)$ is a convex function of T_e within the interval $[t_m, t_{m-1})$. □

Proof of Theorem 1. Considering m as a continuous variable, from Eq (17)

$$\frac{dTM(m)}{dm} = -\frac{[K(P_2 - P_1) - (P_1\theta + h_1)T']^2 D(P_2\theta + h_2)}{2[m(P_1\theta + h_1) + K(P_2\theta + h_2)]^2} + A_2 \tag{A.11}$$

$$\frac{d^2 TM(m)}{dm^2} = \frac{[K(P_2 - P_1) - (P_1\theta + h_1)T']^2 D(P_2\theta + h_2)(P_1\theta + h_1)}{[m(P_1\theta + h_1) + K(P_2\theta + h_2)]^3} > 0 \tag{A.12}$$

It can be seen that $TM(m)$ is a convex function of m. This implies that $TM(m)$ has minimum value. Since m is an integer number, using the same proof of property 2, let

$$z = -\left[\frac{1}{2} + \frac{K(P_2\theta + h_2)}{(P_1\theta + h_1)}\right] + \sqrt{\frac{1}{4} + \frac{D(P_2\theta + h_2)[K(P_2 - P_1) - (P_1\theta + h_1)T']^2}{2(P_1\theta + h_1)^2 A_2}} \tag{A.13}$$

For z is not an integer, taking $m^* = \lceil z \rceil$. Otherwise, taking $m^* = \lceil z \rceil$ and $m^* = \lceil z \rceil + 1$.

(i) If $m^* \leq \lfloor m_R \rfloor$, owing $TM(m-1) \leq TM(m)$ in the condition $m \geq m^*$, $TM(m)$ increases as integer m increases in the range $\lfloor m_R \rfloor \leq m \leq \lfloor m_L \rfloor$. Hence, $TM(\lfloor m_R \rfloor)$ is the minimum value of $TM(m)$ in the range $\lfloor m_R \rfloor \leq m \leq \lfloor m_L \rfloor$. (ii) If $m^* \geq \lfloor m_L \rfloor$, owing $TM(m-1) \geq TM(m)$ in the condition $m \leq m^*$, $TM(m)$ decreases as integer m increases in the range $\lfloor m_R \rfloor \leq m \leq \lfloor m_L \rfloor$. Hence, $TM(\lfloor m_L \rfloor)$ is the minimum value of $TM(m)$ in the range $\lfloor m_R \rfloor \leq m \leq \lfloor m_L \rfloor$. (iii) If $\lfloor m_R \rfloor < m^* < \lfloor m_L \rfloor$, owing $TM(m-1) \geq TM(m)$ in the condition $m \leq m^*$ and $TM(m-1) \leq TM(m)$ in the condition $m \geq m^*$, $TM(m^*)$ is the minimum value of $TM(m)$ in the range $\lfloor m_R \rfloor \leq m \leq \lfloor m_L \rfloor$. □

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