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A note on "optimal replenishment policies for non-instantaneous deteriorating items with price and stock sensitive demand under permissible delay in payment"

Jiang Wu [a], Konstantina Skouri [b], Jinn-Tsair Teng [c], Liang-Yuh Ouyang [d,*]

[a] School of Statistics, Southwestern University of Finance & Economics, Chengdu 611130, China
[b] Department of Mathematics, University of Ioannina, Ioannina, Greece
[c] Department of Marketing and Management Sciences, Cotsakos College of Business, The William Paterson University of New Jersey, Wayne, NJ 07470, USA
[d] Department of Management Sciences, Tamkang University, Tamshui Dist., New Taipei City, Taiwan, ROC

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ABSTRACT

Soni 2013. Int. J. Prod. Econ., 146 (1), 259–268 proposed optimal replenishment policies for non-instantaneous deteriorating items (i.e., the product starts deteriorating after a period of no-deterioration) with price and stock sensitive demand. With a stock-dependent demand, it is desirable to have non-zero ending inventory due to potential profit resulting from the increased demand. However, Soni 2013. Int. J. Prod. Econ., 146 (1), 259–268 treated those ending inventory as fresh stocks to go through another period of non-deterioration again. Additionally, he assumed for simplicity that the replenishment cycle time T must be longer than the period of non-deterioration t_d(i.e., $T > t_d$). In reality, one should consider all possible replenishment cycle time to maximize the profit. In this note, we complement the shortcomings of his model by (i) selling those ending inventory as salvages, and (ii) considering all possible replenishment cycle time, which may be shorter than the period of non-deterioration. With these modifications the repeatability of the replenishment cycle is ensured and the applicability of Soni's model is strengthened. The numerical examples indicate that the global optimal solution is indeed possible in the case of $T \le t_d$.

1. Introduction

Trade credit arises when a seller allows a buyer to delay payment for purchased goods and services. Seifert et al. (2013) stated about 80% of United States as well as United Kingdom firms offer their products on trade credit. Hence, trade credit is increasingly recognized as an important mean to increase profitability in a supply chain. Trade credit reduced the buyer's inventory holding cost, and thus affects the buyer's economic order quantity (thereafter, EOQ). In recent years, extensive research on EOQ models under trade credit has been developed such as Chen et al. (2013a, 2013b), Chern et al. (2013), Lin et al. (2012), Musa and Sani (2012), Ouyang and Chang (2013), Sarkar (2012), Teng and Lou (2012), Teng et al. (2012, 2013), Thangam (2012), Tsoa (2012), Wang et al. (2014), Yu (2013) and others.

Since Harris (1913) developed the EOQ model, extensive studies on EOQ in operations management literature have been developed. Recently Soni (2013) presented an inventory model for non-instantaneous deteriorating items, in which (i) the on-hand inventory deteriorates at a constant rate after a fixed time period of non-deterioration, (ii) the demand rate is a function of the selling price and inventory level, (iii) the ending inventory level may be non-zero because inventory has a positive effect on demand, (iv) there is a maximum inventory level because the retailer has a limited shelf space, and (v) the supplier offers a certain credit period without charging any interest while the resulting revenue is deposited in an interest bearing account. Under these considerations a mathematical model was developed and an optimization procedure was presented for the determination of the optimal order quantity (which is equivalent to the optimal replenishment cycle, T) and the ending inventory level.

In this study we revisit the above mentioned work, and notice the following two serious deficiencies: (a) it inappropriately treats those ending-inventory items as fresh stocks to go through another period of non-deterioration again, and (b) it mistakenly assumes for simplicity that the replenishment cycle time must be longer than the period of non-deterioration, which implies its solution is only regionally optimal, not globally optimal. In this note, we complement the shortcomings of this model by (1) selling the ending inventory as salvages at the end of each replenishment

* Corresponding author. Tel.: +886 2 26215656.
E-mail address: liangyuh@mail.tku.edu.tw (L.-Y. Ouyang).

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cycle, and (2) taking all possible replenishment cycle time into consideration, which includes the case where cycle time may be shorter than the period of non-deterioration. With the above modifications, the repeatability of the replenishment cycle is logically developed, the optimization procedure is completed and the applicability of Soni's model is strengthened. The numerical examples using the same data as in Soni (2013) indicate that the global optimal solution is possible in the case of $T \le t_d$, which may improve the profit substantially.

2. Notation and assumptions

The following notation and assumptions are used in developing the model in the entire paper.

2.1. Notation

A	ordering cost per order in dollars.
c	purchase cost per unit in dollars.
D	the market annual demand rate in units which is a function of price and stock.
h	unit holding cost per year in dollars excluding interest charge.
I_c	interest charged per dollar per year.
I_e	interest earned per dollar per year.
$I=I(t)$	inventory level in units at time t.
M	up-stream credit period in years offered by the supplier.
θ	constant deterioration rate after time t_d.
p	selling price per unit in dollars, with $p>c$.
q	ending inventory level in units (a decision variable).
Q_i	order quantity in units for $i=1$ and 2.
s	salvage price per unit in dollars.
T	replenishment cycle time in years (a decision variable).
t_d	time period in years during which the product has no deterioration.
U	maximum inventory level in units.
$P_{ij}(q,T)$	profit per unit time (or per year) in dollars, for $i=1$ and 2, $j=1$, 2, and 3.
$P^*_{ij}(q,T)$	optimal value of $P_{ij}(q,T)$.
q^*	optimal ending inventory level in units.
T^*	optimal replenishment cycle time in years.
P^*_i	optimal profit in dollars, for $i=1$ and 2.

2.2. Assumptions

Next, the following assumptions are made to develop the mathematical inventory model.

1. Demand rate D is a function of price and stock level. We assume that $D(p,I)=\alpha(p)+\beta I$, where $\alpha(p)$ is positive and decreasing in p, and β is a positive parameter.
2. The product has no deterioration during $[0, t_d]$. Thereafter t_d, the on-hand stocks deteriorate with constant rate θ, where $0 \le \theta < 1$.
3. Most retailers have limited shelf spaces. Hence, we assume that there is a maximum inventory level, U.
4. At the end of replenishment cycle time T, the ending-inventory q items are sold for salvages.
5. The retailer deposits the sales revenue into an interest bearing account. If $M>T$, then the retailer receives all revenue and pays off the entire purchase cost at the end of the permissible delay M. Otherwise, the retailer pays the supplier the sum of all units sold by M, keeps the profit for the use of the other activities, and starts paying for the interest charges on the items sold after M.
6. Replenishment rate is instantaneous and lead time is negligible.
7. In today's time-based competition, we may assume that shortages are not allowed to occur.

Given the above notation and assumptions, it is possible to formulate the retailer's annual profit as a function of the ending-inventory level q and the replenishment cycle time T for non-instantaneous deteriorating items into a mathematical model.

3. Mathematical model

An order quantity of fresh Q units arrives at time $t=0$. The inventory level is depleted only due to demand rate over time interval $[0, t_d]$. Thereafter the inventory level is depleted to q units at time $t=T$ due to the combination of demand and deterioration. At the end of the replenishment cycle T, the retailer sells those ending-inventory q units for salvage value, and the new replenishment cycle is repeated as mentioned above. Notice that Soni (2013) inappropriately assumed that the retailer receives only fresh $Q-q$ units, and resells those unsold ending-inventory q units (which have gone through the non-deterioration period already) so that the retailer has fresh Q units at time $t=0$ to go through another non-deterioration period $[0, t_d]$.

The replenishment cycle time T is a decision variable, which may be in one of two possible options: either $T \le t_d$ or $T > t_d$. We discuss the option of $T \le t_d$ first and then the other option of $T > t_d$ accordingly. Notice that Soni (2013) ignored the first option of $T \le t_d$, and discussed only the second option of $T > t_d$ (please see Fig. 1 and Eqs. (1)–(6) in Soni (2013)). As a result, he obtained only a regionally optimal solution for $T > t_d$, not a globally optimal solution for any $T \ge 0$.

Option 1. $T \le t_d$

If $T \le t_d$, then the product has no deterioration during the entire replenishment cycle, and the differential equation which describes the variation of the inventory level is:

$$\frac{dI_1(t)}{dt} = -\alpha(p) - \beta I_1(t), \quad 0 \le t \le T, \tag{1}$$

with boundary condition $I_1(T)=q \ge 0$. Hence, its solution is

$$I_1(t) = \left(q + \frac{\alpha(p)}{\beta}\right) e^{\beta(T-t)} - \frac{\alpha(p)}{\beta}, \quad 0 \le t \le T. \tag{2}$$

Thus, the ordering quantity for Case 1 is

$$Q_1 = I_1(0) = \left(q + \frac{\alpha(p)}{\beta}\right) e^{\beta T} - \frac{\alpha(p)}{\beta} \le U \tag{3}$$

Option 2. $T > t_d$

In this case, the product has no deterioration during $[0, t_d]$, while has deterioration during $[t_d, T]$. Hence, the inventory level

$$I(t) = \begin{cases} I_{2a}(t), & 0 \le t \le t_d \\ I_{2b}(t), & t_d \le t \le T \end{cases} \tag{4}$$

is governed by the following two differential equations:

$$\frac{dI_{2a}(t)}{dt} = -\alpha(p) - \beta I_{2a}(t), \quad 0 \le t \le t_d, \text{ with boundary condition}$$

$$I_{2a}(0) = Q_2 \tag{5}$$

and

$$\frac{dI_{2b}(t)}{dt} = -\alpha(p) - (\beta+\theta)I_{2b}(t),\quad t_d \le t \le T, \text{with boundary condition } I_{2b}(T) = q. \tag{6}$$

The solutions of the above differential equations are:

$$I_{2a}(t) = \left(Q_2 + \frac{\alpha(p)}{\beta}\right)e^{-\beta t} - \frac{\alpha(p)}{\beta},\quad 0 \le t \le t_d; \tag{7}$$

and

$$I_{2b}(t) = \left(q + \frac{\alpha(p)}{\beta+\theta}\right)e^{(\beta+\theta)(T-t)} - \frac{\alpha(p)}{\beta+\theta},\quad t_d \le t \le T; \tag{8}$$

respectively. Therefore the corresponding demand functions are

$$D_{2a}(p,I) = \alpha(p) + \beta I_{2a}(t) = [\beta Q_2 + \alpha(p)]e^{-\beta t},\quad 0 \le t \le t_d; \tag{9}$$

and

$$D_{2b}(p,I) = \alpha(p) + \beta I_{2b}(t) = \left(\beta q + \frac{\beta\alpha(p)}{\beta+\theta}\right)e^{(\beta+\theta)(T-t)} + \alpha(p)\left(1 - \frac{\beta}{\beta+\theta}\right),\quad t_d \le t \le T; \tag{10}$$

respectively. By the continuity of inventory level at t_d(i.e., $I_{2a}(t_d) = I_{2b}(t_d)$), and simplifying terms, we obtain the ordering quantity for Case 2 as

$$Q_2 = e^{\beta t_d}\left[\left(q + \frac{\alpha(p)}{\beta+\theta}\right)e^{(\beta+\theta)(T-t_d)} - \frac{\alpha(p)}{\beta+\theta}\right] + \frac{\alpha(p)(e^{\beta t_d}-1)}{\beta} \le U. \tag{11}$$

Our aim is to derive the total profit per unit time and then to maximize it with respect to q and T. The profit is given as

$$\begin{aligned}\text{Profit} = {} & \text{revenue received from sales} + \text{salvage value of ending inventory at time } T + \text{interest earned from sales} \\ & + \text{interest earned from salvage value} - \text{interest charged} \\ & - \text{ordering cost} - \text{holding cost} - \text{purchase cost}\end{aligned} \tag{12}$$

According to the values of the parameters M and t_d the following cases are considered, which actually consist two different cases: either $M \le t_d$ or $M > t_d$.

3.1. The case of $M \le t_d$

In this case, the decision variable T has three alternatives: (a) $T \le M \le t_d$, (b) $M < T \le t_d$, and (c) $T > t_d \ge M$. We discuss them in the same order.

3.1.1. Sub-case $T \le M \le t_d$

In this sub-case, the relevant costs and revenues for each replenishment cycle are as follows:

(i) The ordering cost is A.

(ii) The inventory holding cost excluding interest charges is

$$h\int_0^T I_1(t)dt = h\int_0^T\left[\left(q + \frac{\alpha(p)}{\beta}\right)e^{\beta(T-t)} - \frac{\alpha(p)}{\beta}\right]dt = h\left(\frac{q}{\beta} + \frac{\alpha(p)}{\beta^2}\right)(e^{\beta T}-1) - h\frac{\alpha(p)}{\beta}T \tag{13}$$

(iii) The purchase cost including the deterioration cost is

$$cQ_1 = c\left(q + \frac{\alpha(p)}{\beta}\right)e^{\beta T} - c\frac{\alpha(p)}{\beta}. \tag{14}$$

(iv) The revenue received from sales is

$$p\int_0^T[\alpha(p) + \beta I_1(t)]dt = p\left(q + \frac{\alpha(p)}{\beta}\right)(e^{\beta T}-1) \tag{15}$$

(v) The salvage value of ending-inventory q units sold at time $t = T$ is sq.

(vi) The capital opportunity cost consists of interest charged and interest earned. In this sub-case, there is no interest charged, and the interest earned per cycle has two parts. The interest earned from sales during the credit period is

$$\begin{aligned} & pI_e\left[\int_0^T\int_0^t[\alpha(p) + \beta I_1(x)]dxdt + (M-T)\int_0^T[\alpha(p) + \beta I_1(t)]dt\right] \\ & = pI_e\left[\left(q + \frac{\alpha(p)}{\beta}\right)Te^{\beta T} + \left(\frac{q}{\beta} + \frac{\alpha(p)}{\beta^2}\right)(1 - e^{\beta T}) + (M-T)\left(q + \frac{\alpha(p)}{\beta}\right)(e^{\beta T}-1)\right]. \end{aligned} \tag{16}$$

The interest earned from salvage value of ending-inventory q units during the credit period is $sqI_e(M-T)$.

Consequently, if $T \le M \le t_d$, then we know from (12) that the profit per unit time is

$$\begin{aligned} P_{11}(q,T) = \frac{1}{T}\Bigg\{ & p\left(q + \frac{\alpha(p)}{\beta}\right)(e^{\beta T}-1) + sq \\ & + pI_e\left[\left(q + \frac{\alpha(p)}{\beta}\right)Te^{\beta T} + \left(\frac{q}{\beta} + \frac{\alpha(p)}{\beta^2}\right)(1 - e^{\beta T}) + (M-T)\left(q + \frac{\alpha(p)}{\beta}\right)(e^{\beta T}-1)\right] \\ & + sqI_e(M-T) - A - h\left(\frac{q}{\beta} + \frac{\alpha(p)}{\beta^2}\right)(e^{\beta T}-1) + h\frac{\alpha(p)}{\beta}T \\ & - c\left(q + \frac{\alpha(p)}{\beta}\right)e^{\beta T} + c\frac{\alpha(p)}{\beta}\Bigg\} \end{aligned} \tag{17}$$

3.1.2. Sub-case $M < T \le t_d$

In this sub-case, the relevant costs and revenues for each replenishment cycle are similar to those in Section 3.1.1 except the interest earned and interest charged. Since $T > M$, the retailer must finance all items sold after M. Hence, there is an interest charge. Additionally, there is no interest earned from salvage value. Hence, we obtain the following capital opportunity costs:

(vi) The interest charged after the credit period is

$$cI_c\int_M^T I_1(t)dt = cI_c\left[\left(\frac{q}{\beta} + \frac{\alpha(p)}{\beta^2}\right)(e^{\beta(T-M)}-1) - \frac{\alpha(p)}{\beta}(T-M)\right]. \tag{18}$$

The interest earned from sales during the credit period is

$$pI_e\int_0^M\int_0^t[\alpha(p) + \beta I_1(x)]dxdt = pI_e\left[\left(q + \frac{\alpha(p)}{\beta}\right)Me^{\beta T} + \left(\frac{q}{\beta} + \frac{\alpha(p)}{\beta^2}\right)(e^{\beta(T-M)} - e^{\beta T})\right] \tag{19}$$

Consequently, if $M < T \le t_d$, then the profit per unit time is given by

$$\begin{aligned} P_{12}(q,T) = \frac{1}{T}\Bigg\{ & p\left(q + \frac{\alpha(p)}{\beta}\right)(e^{\beta T}-1) + sq \\ & + pI_e\left[\left(q + \frac{\alpha(p)}{\beta}\right)Me^{\beta T} + \left(\frac{q}{\beta} + \frac{\alpha(p)}{\beta^2}\right)(e^{\beta(T-M)} - e^{\beta T})\right] \\ & - cI_c\left[\left(\frac{q}{\beta} + \frac{\alpha(p)}{\beta^2}\right)(e^{\beta(T-M)}-1) - \frac{\alpha(p)}{\beta}(T-M)\right] - A \\ & - h\left(\frac{q}{\beta} + \frac{\alpha(p)}{\beta^2}\right)(e^{\beta T}-1) + h\frac{\alpha(p)}{\beta}T - c\left(q + \frac{\alpha(p)}{\beta}\right)e^{\beta T} + c\frac{\alpha(p)}{\beta}\Bigg\} \end{aligned} \tag{20}$$

Next, we discuss the last sub-case of Case 1.

3.1.3. Sub-case $M \le t_d < T$

If $M \le t_d < T$, using Eqs. (7), (8), and (11), we obtain the relevant costs and revenues for each replenishment cycle as follows:

(ii) The inventory holding cost excluding interest charges is

$$h\left[\int_0^{t_d} I_{2a}(t)dt+\int_{t_d}^{T} I_{2b}(t)dt\right]=h\left\{\left(\frac{Q_2}{\beta}+\frac{\alpha(p)}{\beta^2}\right)(1-e^{-\beta t_d})-\frac{\alpha(p)}{\beta}t_d + \left(\frac{q}{\beta+\theta}+\frac{\alpha(p)}{(\beta+\theta)^2}\right)\left[e^{(\beta+\theta)(T-t_d)}-1\right]-\frac{\alpha(p)}{\beta+\theta}(T-t_d)\right\} \quad (21)$$

(iii) The purchase cost including the deterioration cost is

$$cQ_2=ce^{\beta t_d}\left[\left(q+\frac{\alpha(p)}{\beta+\theta}\right)e^{(\beta+\theta)(T-t_d)}-\frac{\alpha(p)}{\beta+\theta}\right] +\frac{c\alpha(p)(e^{\beta t_d}-1)}{\beta}. \quad (22)$$

(iv) The revenue received from sales is

$$p\int_0^{t_d}[\alpha(p)+\beta I_{2a}(t)]dt+p\int_{t_d}^{T}[\alpha(p)+\beta I_{2b}(t)]dt$$
$$=p\left(Q_2+\frac{\alpha(p)}{\beta}\right)(1-e^{-\beta t_d}) +p\left[\left(\frac{\beta}{\beta+\theta}\right)\left(q+\frac{\alpha(p)}{\beta+\theta}\right)\left[e^{(\beta+\theta)(T-t_d)}-1\right] +\alpha(p)\left(1-\frac{\beta}{\beta+\theta}\right)(T-t_d)\right] \quad (23)$$

(v) The salvage value is sq.

(vi) The interest earned is given by

$$pI_e\int_0^{M}\int_0^{t}[\alpha(p)+\beta I_{2a}(x)]dxdt =pI_e\left[\left(Q_2+\frac{\alpha(p)}{\beta}\right)M+\left(\frac{Q_2}{\beta}+\frac{\alpha(p)}{\beta^2}\right)(e^{-\beta M}-1)\right], \quad (24)$$

The interest charged is

$$cI_c\left[\int_M^{t_d} I_{2a}(t)dt+\int_{t_d}^{T} I_{2b}(t)dt\right] =cI_c\left\{\left(\frac{Q_2}{\beta}+\frac{\alpha(p)}{\beta^2}\right)(e^{-\beta M}-e^{-\beta t_d})-\frac{\alpha(p)}{\beta}(t_d-M) +\left(\frac{q}{\beta+\theta}+\frac{\alpha(p)}{(\beta+\theta)^2}\right)[e^{(\beta+\theta)(T-t_d)}-1]-\frac{\alpha(p)}{\beta+\theta}(T-t_d)\right\} \quad (25)$$

As a result, if$M \le t_d < T$, then the profit per unit time is

$$P_{13}(q,T)=\frac{1}{T}\left\{p\left[\left(Q_2+\frac{\alpha(p)}{\beta}\right)(1-e^{-\beta t_d}) +\left(\frac{\beta}{\beta+\theta}\right)\left(q+\frac{\alpha(p)}{\beta+\theta}\right)(e^{(\beta+\theta)(T-t_d)}-1). +\alpha(p)\left(1-\frac{\beta}{\beta+\theta}\right)(T-t_d)\right]+sq \right.$$
$$+pI_e\left[\left(Q_2+\frac{\alpha(p)}{\beta}\right)M+\left(\frac{Q_2}{\beta}+\frac{\alpha(p)}{\beta^2}\right)(e^{-\beta M}-1)\right] -cI_c\left[\left(\frac{Q_2}{\beta}+\frac{\alpha(p)}{\beta^2}\right)(e^{-\beta M}-e^{-\beta t_d})-\frac{\alpha(p)}{\beta}(t_d-M) +\left(\frac{q}{\beta+\theta}+\frac{\alpha(p)}{(\beta+\theta)^2}\right)(e^{(\beta+\theta)(T-t_d)}-1)-\frac{\alpha(p)}{\beta+\theta}(T-t_d)\right]$$
$$-A-h\left[\left(\frac{Q_2}{\beta}+\frac{\alpha(p)}{\beta^2}\right)(1-e^{-\beta t_d})-\frac{\alpha(p)}{\beta}t_d +\left(\frac{q}{\beta+\theta}+\frac{\alpha(p)}{(\beta+\theta)^2}\right)(e^{(\beta+\theta)(T-t_d)}-1)-\frac{\alpha(p)}{\beta+\theta}(T-t_d)\right]$$
$$\left.-ce^{\beta t_d}\left[\left(q+\frac{\alpha(p)}{\beta+\theta}\right)e^{(\beta+\theta)(T-t_d)}-\frac{\alpha(p)}{\beta+\theta}\right]-\frac{c\alpha(p)(e^{\beta t_d}-1)}{\beta}.\right. \quad (26)$$

Notice that the warehouse capacity is U. Thus, if $T \le t_d$ then $Q_1 \le U$, which implies

$q \le [U+\alpha(p)/\beta]e^{-\beta T}-\alpha(p)/\beta$by (3). However, if $T > t_d$ then $Q_2 \le U$. Hence, from (11) we obtain the following inequality

$$q \le \left\{\left[U-\frac{\alpha(p)}{\beta}(e^{\beta t_d}-1)\right]e^{-\beta t_d}+\frac{\alpha(p)}{\beta+\theta}\right\}e^{-(\beta+\theta)(T-t_d)}-\frac{\alpha(p)}{\beta+\theta} \equiv U_2e^{-(\beta+\theta)(T-t_d)}-\frac{\alpha(p)}{\beta+\theta}$$

where

$$U_2=\left[U-\frac{\alpha(p)}{\beta}(e^{\beta t_d}-1)\right]e^{-\beta t_d}+\frac{\alpha(p)}{\beta+\theta}$$

Combining (17), (20), and (26), we know that if $M \le t_d$ then the profit per unit time is

$$P_1(q,T)=\begin{cases} P_{11}(q,T), & 0<T\le M\le t_d,\ 0\le q\le [U+\alpha(p)/\beta]e^{-\beta T}-\alpha(p)/\beta; \\ P_{12}(q,T), & M<T\le t_d,\ 0\le q\le [U+\alpha(p)/\beta]e^{-\beta T}-\alpha(p)/\beta; \\ P_{13}(q,T), & M\le t_d<T,\ 0\le q\le U_2e^{-(\beta+\theta)(T-t_d)}-\alpha(p)/(\beta+\theta). \end{cases} \quad (27)$$

To maximize $P_1(q,T)$requires the maximization of each sub-case separately and then the comparison of the respective results to obtain the maximum value of$P_1(q,T)$. Therefore, we get

$$\max P_1(q,T)=\max\left\{\max_{0<T\le M,\ 0\le q\le q_A(T)} P_{11}(q,T), \max_{M<T\le t_d,\ 0\le q\le q_A(T)} P_{12}(q,T), \max_{T\ge t_d,\ 0\le q\le q_B(T)} P_{13}(q,T)\right\} \quad (28)$$

where

$$q_A(T)=[U+\alpha(p)/\beta]e^{-\beta T}-\alpha(p)/\beta$$

and

$$q_B(T)=U_2e^{-(\beta+\theta)(T-t_d)}-\alpha(p)/(\beta+\theta)$$

3.2. The case of $M > t_d$

Again there are three possible alternatives for the decision variable T: (a) $T \le t_d < M$,(b) $t_d < T \le M$,and (c) $t_d < M < T$. We discuss them according to their order.

3.2.1. Sub-case $T \le t_d < M$

In this sub-case, the profit per unit time is exactly the same as that in Section 3.1.1. Hence,

$$P_{21}(q,T)=P_{11}(q,T) \quad (29)$$

3.2.2. Sub-case $t_d < T \le M$

If $t_d < T \le M$, then there is no interest charges, and the relevant costs and revenues per cycle are the same as those in Section 3.1.3. However, the interest earned is slightly different.

vi. The interest earned per cycle from sales is

$$pI_e\left[\int_0^{t_d}\int_0^{t} D_{2a}(p,x)dxdt+(M-t_d)\int_0^{t_d} D_{2a}(p,t)dt +\int_{t_d}^{T}\int_{t_d}^{t} D_{2b}(p,x)dxdt+(M-T)\int_{t_d}^{T} D_{2b}(p,t)dt\right]$$
$$=pI_e\left[\left(Q_2+\frac{\alpha(p)}{\beta}\right)[t_d+(1-e^{-\beta t_d})(M-t_d)]+\left(\frac{Q_2}{\beta}+\frac{\alpha(p)}{\beta^2}\right)(e^{-\beta t_d}-1)\right] +pI_e\left\{(T-t_d)\left[\frac{\beta}{\beta+\theta}\left(q+\frac{\alpha(p)}{\beta+\theta}\right)e^{(\beta+\theta)(T-t_d)}-\alpha(p)\left(1-\frac{\beta}{\beta+\theta}\right)t_d\right]\right.$$

$$+\frac{\alpha(p)}{2}\left(1-\frac{\beta}{\beta+\theta}\right)(T^2-t_d^2)+\frac{\beta}{(\beta+\theta)^2}\left(q+\frac{\alpha(p)}{\beta+\theta}\right)[1-e^{(\beta+\theta)(T-t_d)}]\Bigg\}$$
$$+pI_e(M-T)\left[\left(\frac{\beta}{\beta+\theta}\right)\left(q+\frac{\alpha(p)}{\beta+\theta}\right)[e^{(\beta+\theta)(T-t_d)}-1]\right.$$
$$\left.+\alpha(p)\left(1-\frac{\beta}{\beta+\theta}\right)(T-t_d)\right] \quad (30)$$

The interest earned from salvages is $sqI_e(M-T)$. Consequently, the profit per unit time is

$$P_{22}(q,T)=\frac{1}{T}\Bigg\{p\left[\left(Q_2+\frac{\alpha(p)}{\beta}\right)(1-e^{-\beta t_d})\right.$$
$$+\left(\frac{\beta}{\beta+\theta}\right)\left(q+\frac{\alpha(p)}{\beta+\theta}\right)(e^{(\beta+\theta)(T-t_d)}-1).$$
$$\left.+\alpha(p)\left(1-\frac{\beta}{\beta+\theta}\right)(T-t_d)\right]+sq$$
$$+pI_e\left[\left(Q_2+\frac{\alpha(p)}{\beta}\right)[t_d+(1-e^{-\beta t_d})(M-t_d)]\right.$$
$$\left.+\left(\frac{Q_2}{\beta}+\frac{\alpha(p)}{\beta^2}\right)(e^{-\beta t_d}-1)\right]$$
$$+pI_e\Bigg\{(T-t_d)\left[\frac{\beta}{\beta+\theta}\left(q+\frac{\alpha(p)}{\beta+\theta}\right)e^{(\beta+\theta)(T-t_d)}-\alpha(p)\left(1-\frac{\beta}{\beta+\theta}\right)t_d\right]$$
$$+\frac{\alpha(p)}{2}\left(1-\frac{\beta}{\beta+\theta}\right)(T^2-t_d^2)+\frac{\beta}{(\beta+\theta)^2}\left(q+\frac{\alpha(p)}{\beta+\theta}\right)[1-e^{(\beta+\theta)(T-t_d)}]\Bigg\}$$
$$+pI_e(M-T)\left[\left(\frac{\beta}{\beta+\theta}\right)\left(q+\frac{\alpha(p)}{\beta+\theta}\right)[e^{(\beta+\theta)(T-t_d)}-1]\right.$$
$$\left.+\alpha(p)\left(1-\frac{\beta}{\beta+\theta}\right)(T-t_d)\right]$$
$$+sqI_e(M-T)+-A-h\left[\left(\frac{Q_2}{\beta}+\frac{\alpha(p)}{\beta^2}\right)(1-e^{-\beta t_d})-\frac{\alpha(p)}{\beta}t_d\right.$$
$$\left.+\left(\frac{q}{\beta+\theta}+\frac{\alpha(p)}{(\beta+\theta)^2}\right)(e^{(\beta+\theta)(T-t_d)}-1)-\frac{\alpha(p)}{\beta+\theta}(T-t_d)\right]$$
$$-ce^{\beta t_d}\left[\left(q+\frac{\alpha(p)}{\beta+\theta}\right)e^{(\beta+\theta)(T-\ t_d)}-\frac{\alpha(p)}{\beta+\theta}\right]-\frac{c\alpha(p)(e^{\beta\ t_d}-1)}{\beta}. \quad (31)$$

3.2.3. Sub-case $t_d < M < T$

In this sub-case, the relevant costs and revenues per cycle are the same as those in Section 3.2.2 except the capital opportunity cost.

vi. The interest earned per cycle from sales is

$$pI_e\left[\int_0^{t_d}\int_0^t D_{2a}(p,x)dxdt+(M-t_d)\int_0^{t_d}D_{2a}(p,t)dt\right.$$
$$\left.+\int_{t_d}^M\int_{t_d}^t D_{2b}(p,x)dxdt\right]$$
$$=pI_e\left[\left(Q_2+\frac{\alpha(p)}{\beta}\right)[t_d+(1-e^{-\beta t_d})(M-t_d)]+\left(\frac{Q_2}{\beta}+\frac{\alpha(p)}{\beta^2}\right)(e^{-\beta t_d}-1)\right]$$
$$+pI_e\Bigg\{(M-t_d)\left[\frac{\beta}{\beta+\theta}\left(q+\frac{\alpha(p)}{\beta+\theta}\right)e^{(\beta+\theta)(T-t_d)}-\alpha(p)\left(1-\frac{\beta}{\beta+\theta}\right)t_d\right]$$
$$+\frac{\alpha(p)}{2}\left(1-\frac{\beta}{\beta+\theta}\right)(M^2-t_d^2)$$
$$+\frac{\beta}{(\beta+\theta)^2}\left(q+\frac{\alpha(p)}{\beta+\theta}\right)[e^{(\beta+\theta)(T-M)}-e^{(\beta+\theta)(T-t_d)}]\Bigg\} \quad (32)$$

The interest charged is

$$cI_c\int_M^T I_{2b}(t)dt=cI_c\left\{\left(\frac{q}{\beta+\theta}+\frac{\alpha(p)}{(\beta+\theta)^2}\right)[e^{(\beta+\theta)(T-M)}-1]-\frac{\alpha(p)}{\beta+\theta}(T-M)\right\} \quad (33)$$

Therefore, the profit per unit time is

$$P_{23}(q,T)=\frac{1}{T}\Bigg\{p\left[\left(Q_2+\frac{\alpha(p)}{\beta}\right)(1-e^{-\beta t_d})\right.$$
$$+\left(\frac{\beta}{\beta+\theta}\right)\left(q+\frac{\alpha(p)}{\beta+\theta}\right)(e^{(\beta+\theta)(T-t_d)}-1).$$
$$\left.+\alpha(p)\left(1-\frac{\beta}{\beta+\theta}\right)(T-t_d)\right]+sq$$
$$+pI_e\left[\left(Q_2+\frac{\alpha(p)}{\beta}\right)[t_d+(1-e^{-\beta t_d})(M-t_d)]\right.$$
$$\left.+\left(\frac{Q_2}{\beta}+\frac{\alpha(p)}{\beta^2}\right)(e^{-\beta t_d}-1)\right]$$
$$+pI_e\Bigg\{(M-t_d)\left[\frac{\beta}{\beta+\theta}\left(q+\frac{\alpha(p)}{\beta+\theta}\right)e^{(\beta+\theta)(T-t_d)}-\alpha(p)\left(1-\frac{\beta}{\beta+\theta}\right)t_d\right]$$
$$+\frac{\alpha(p)}{2}\left(1-\frac{\beta}{\beta+\theta}\right)(M^2-t_d^2)$$
$$+\frac{\beta}{(\beta+\theta)^2}\left(q+\frac{\alpha(p)}{\beta+\theta}\right)[e^{(\beta+\theta)(T-M)}-e^{(\beta+\theta)(T-t_d)}]\Bigg\}$$
$$-cI_c\left\{\left(\frac{q}{\beta+\theta}+\frac{\alpha(p)}{(\beta+\theta)^2}\right)[e^{(\beta+\theta)(T-M)}-1]-\frac{\alpha(p)}{\beta+\theta}(T-M)\right\}$$
$$-A-h\left[\left(\frac{Q_2}{\beta}+\frac{\alpha(p)}{\beta^2}\right)(1-e^{-\beta t_d})-\frac{\alpha(p)}{\beta}t_d\right.$$
$$\left.+\left(\frac{q}{\beta+\theta}+\frac{\alpha(p)}{(\beta+\theta)^2}\right)(e^{(\beta+\theta)(T-t_d)}-1)-\frac{\alpha(p)}{\beta+\theta}(T-t_d)\right]$$
$$-ce^{\beta\ t_d}\left[\left(q+\frac{\alpha(p)}{\beta+\theta}\right)e^{(\beta+\theta)(T-\ t_d)}-\frac{\alpha(p)}{\beta+\theta}\right]-\frac{c\alpha(p)(e^{\beta\ t_d}-1)}{\beta}. \quad (34)$$

Combining (29), (31), and (34), we know that if $M > t_d$ then the profit per unit time is

$$P_2(q,T)=\begin{cases} P_{21}(q,T), & T\le t_d<M,\quad 0\le q\le[U+\alpha(p)/\beta]e^{-\beta\ T}-\alpha(p)/\beta; \\ P_{22}(q,T), & t_d<T\le M,\quad 0\le q\le U_2e^{-(\beta+\theta)(T-t_d)}-\alpha(p)/(\beta+\theta); \\ P_{23}(q,T), & t_d<M<T,\quad 0\le q\le U_2e^{-(\beta+\theta)(T-t_d)}-\alpha(p)/(\beta+\theta). \end{cases} \quad (35)$$

To maximize $P_2(q,T)$ requires the maximization of each sub-case separately and then the comparison of the respective results to obtain the maximum value of$P_2(q,T)$.

Therefore, we have

$$\max P_2(q,T)=\max\{\max_{T\le t_d,\ 0\le q\le q_A(T)}P_{21}(q,T),$$
$$\max_{t_d<T\le M,\ 0\le q\le q_B(T)}P_{22}(q,T),\max_{T\ge M,\ 0\le q\le q_B(T)}P_{23}(q,T)\} \quad (36)$$

4. The optimal solution procedure

To find the optimal solution to $P_{11}(q,T)$, taking the first- and second-order derivatives of $P_{11}(q,T)$with respect to q, and rearranging terms, we get

$$\frac{\partial P_{11}(q,T)}{\partial q}=\frac{1}{T}\Big\{[p+pI_eM-c-\frac{1}{\beta}(pI_e+h)](e^{\beta T}-1)$$
$$-c+s+pI_eT+sI_e(M-T)\Big\} \quad (37)$$

and

$$\frac{\partial^2 P_{11}(q,T)}{\partial q^2}=0 \quad (38)$$

For simplicity, let's define

$$\Delta_{11}=[p+pI_eM-c-\frac{1}{\beta}(pI_e+h)](e^{\beta T}-1)-c+s+pI_eT+sI_e(M-T) \quad (39)$$

If $\Delta_{11}>0$, then $P_{11}(q,T)$ is an increasing function of q. From (3), we have

$$0\le q\le\left[U+\frac{\alpha(p)}{\beta}\right]e^{-\beta T}-\frac{\alpha(p)}{\beta} \quad (40)$$

Table 1
Optimal solution when $M \le t_d$.

Parameters		Soni's solution				Our solution			
p	s	q^*	T^*	Q^*	P^*	q^*	T^*	Q^*	P^*
35	31	0.00	0.327	186.92	844.21	281.18	0.027(=M)	300	5,410.19
40	35	0.00	0.444	257.84	3,914.99	281.52	0.027(=M)	300	49,838.91
45	39	0.00	1.175	873.60	7,087.98	281.86	0.027(=M)	300	94,241.68

Hence, if $\Delta_{11} > 0$ then $P_{11}(q, T)$ is maximized at

$$q^* = \left[U + \frac{\alpha(p)}{\beta}\right] e^{-\beta T} - \frac{\alpha(p)}{\beta} \tag{41}$$

On the other hand, if $\Delta_{11} < 0$ then $P_{11}(q, T)$ is a decreasing function of q, and hence is maximized at

$$q^* = 0 \tag{42}$$

We then substitute q^* into $P_{11}(q, T)$ such that $P_{11}(q^*, T) = P_{11}(T)$ becomes a function of T only. By applying Calculus, we can obtain the optimal solution T^* to $P_{11}(T)$. Finally, substituting T^* into (41) and (17) we obtain q^*, and $P_{11}^*(q, T)$.

Checking all $P_{ij}(q, T)$ for $i = 1$, and 2, $j = 1$, 2, and 3, one can see every $P_{ij}(q, T)$ is linear in q. Hence, the optimal solution q^* is either its upper or lower limit. By using the analogous argument, one can easily solve the optimal value $P_{ij}^*(q, T)$ for $j = 1$–3. Then comparing $P_{ij}^*(q, T)$ for $j = 1$–3, one can find the maximum value among them, which is the optimal $P_i^*(q, T)$.

5. Numerical examples

For comparison reason, let's use the same data as in Example 1 of Soni (2013).

Example 1. A=\$200/order, $\alpha(p) = 600 - 2.5p$, β=0.6, c=\$30/unit, $D(p, I) = \alpha(p) + \beta I(t)$, h=\$3.5/unit/year, I_c=0.12/\$/year, I_e=0.09/\$/year, M=10/365 years, θ=0.08, t_d=20/365 year, and U=300 units. The optimal solutions of Soni (2013) and our proposed model for different values of selling price p and salvage value s are shown in Table 1.

Since the profit function in Soni (2013) is different from ours, we cannot compare which model is better simply based on profit per unit time. However, Table 1 reveals that our proposed model obtains the global optimal solution in the region of $T \le t_d$. By contrast, Soni (2013) considered only the case in which $T > t_d$ while the global optimal solution is actually in the region of $T \le t_d$. Notice that in the third instance Soni got the initial inventory level Q^*=873.60 is significantly greater than the maximum inventory level U=300, which is clearly incorrect.

6. Conclusion

Recently, Soni (2013) has formulated an interesting and relevant inventory model for non-instantaneous deteriorating items in which (i) the on-hand inventory goes through a fixed time period of non-deterioration and then deteriorates at a constant rate, (ii) the demand rate is sensitive to the selling price and inventory level, (iii) the ending inventory level may be non-zero in order to increase demand, (iv) the shelf space is limited, and (v) the supplier offers a trade credit period without charging any interest. In this note, we have pointed out two serious shortcomings in his model: (a) it considers those ending inventory as fresh stocks to go through another period of non-deterioration again, and (b) it assumes that the replenishment cycle time must be longer than the period of non-deterioration, which ignores potential optimal solution at which the replenishment time is less than the period of non-deterioration. We then have complemented the shortcomings of his model by (1) selling those ending inventory as salvages at the end of each replenishment cycle, and (2) taking all possible replenishment cycle time into consideration. Finally, we have provided some numerical examples to show that the global optimal solution is indeed in the region of $T \le t_d$, while Soni (2013) considered only the region in which $T > t_d$.

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