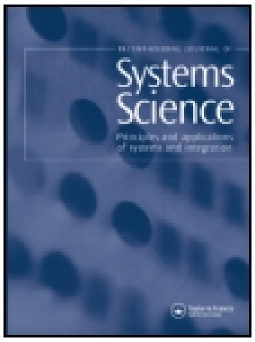

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Fuzzy portfolio model with fuzzy-input return rates and fuzzy-output proportions

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In the finance market, a short-term investment strategy is usually applied in portfolio selection in order to reduce investment risk; however, the economy is uncertain and the investment period is short. Further, an investor has incomplete information for selecting a portfolio with crisp proportions for each chosen security. In this paper we present a new method of constructing fuzzy portfolio model for the parameters of fuzzy-input return rates and fuzzy-output proportions, based on possibilistic mean–standard deviation models. Furthermore, we consider both excess or shortage of investment in different economic periods by using fuzzy constraint for the sum of the fuzzy proportions, and we also refer to risks of securities investment and vagueness of incomplete information during the period of depression economics for the portfolio selection. Finally, we present a numerical example of a portfolio selection problem to illustrate the proposed model and a sensitivity analysis is realised based on the results.

Keywords: fuzzy portfolio model; incomplete information; fuzzy proportion; fuzzy mean–variance model

1. Introduction

In finance, there are many models presenting uncertainty due to fluctuation in financial market, including financial engineering, portfolio management, derivatives pricing, financial management, and investment appraisal models (Thavaneswaran, Singh, and Appadoo 2006; Chrysafis and Papadopoulos 2009; Xu, Wu, Xu, and Li 2009; Guerra, Sorini, and Stefanini 2011; Chrysafis 2012; Guo, Li, Zou, Guo, and Yan 2012; Suganya and Vijayalakshmi Pai 2012; Yan 2012). There have been a number of research works that have focused on portfolio management, selecting a combination of securities among portfolios in order to reach the following investment goals: (1) maximum investment return; (2) minimum investment risk. One of the most popular models in portfolio management is the mean–variance portfolio model proposed by Markowitz (1952), which has played an important role in the development of portfolio theory based on probability theory (Sharpe 1970; Merton 1972; Pang 1980; Perold 1984; Vörös 1986; Best and Grauer 1990; Best and Hlouskova 2000).

In contrast to probability theory, fuzzy set theory and possibility theory (Dubois and Prade 1988; Fullér and Majlender 2003) are employed by several researchers to manage portfolios in a fuzzy environment (Ramaswamy 1998; Tsaur 2013). For example, Tanaka and Guo (1999), and Tanaka, Guo, and Türksen (2000) converted the portfolio problems into quadratic programming problems. Watada (1997) proposed a fuzzy portfolio in which the expected return rate and risk are vague targets. León, Liem, and Vercher (2002) applied modified fuzzy linear programming techniques to help investors manage a portfolio. Huang (2008a) proposed two fuzzy mean–semi-variance models to obtain high returns while avoiding risk. Inuiguchi and Tanino (2000) addressed the problem of imprecision and uncertainty in the portfolio-selection and investment-decision problems. Giove, Funari, and Nardelli (2006) formulated a minimax regret portfolio problem and transformed the initial interval problem into a set of optimisation problems for the portfolio. Carlsson and Fullér (2001), Carlsson, Fullér, and Majlender (2002), Chen and Tan (2009), Lacagnina and Pecorella (2006), and Lai, Wang, Xu, Zhu, and Fang (2002) extended mean–variance models to help investors find optimal investment strategies under complicated financial situations. Zhang and Nie (2003, 2004) presented notions of lower and upper possibilistic variances and covariances of fuzzy numbers in fuzzy portfolio analysis. Zhang (2007) solved portfolio selection problems for bounded assets under possibility distributions. Huang (2008b) proposed a new definition of risk for portfolio selection in fuzzy environment, and designed a hybrid intelligent algorithm to improve the effectiveness of the portfolio model. Li, Qin, and Kar (2010) proposed a concept of skewness, defined as the third central moment, and extended the fuzzy mean–variance model to a mean–variance–skewness model.

Even though many researchers have devoted themselves in the field of fuzzy portfolio models, no researcher has considered the relation between economic conditions and portfolio selection. The better the economic condition, the

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easier the portfolio selection; in contrast, the worse the economic condition, the more difficult the portfolio selection. Therefore, in order to cope with the above problems and enlarge the range of applications in the field of portfolio selections, two topics should be extended to provide robustness to fuzzy portfolio models. First, in the fuzzy portfolio model, under a depression period with unknown investment risks, if an investor has incomplete knowledge or information about securities investments, it is difficult to express the invested proportions in each security with crisp values. Second, the sum of the proportions invested in securities is set as 1, which might be violated when the economic environment fluctuates dramatically. Therefore, an excess investment might be considered when an investor would like to stand higher investment risk; otherwise, a shortage investment can be considered if an investor would like to avoid higher investment risks during the depression period. Consequently, this study adopts the advantages of possibilistic mean–standard deviation models which can transfer a fuzzy number into a crisp value, and relaxes the invested proportion of each security to be a fuzzy number when the investor has incomplete information during a fluctuating economy. In addition, when the economy is prospering, an excess proportion, higher than the fuzzy value of 1, is considered; otherwise, a smaller proportion, less than the fuzzy value of 1, is considered. The merits of this study are the following:

(1) The proposed model is based on fuzzy-input and fuzzy-output data, which can be biased from past financial conditions.
(2) An investor can take into account the market in different economic conditions.
(3) An investor can express their investment risks for a period of time according to their information, experience or perception.
(4) Especially, an investor can derive a fuzzy-output proportion for each security in a portfolio selection during a period with higher investment risk.
(5) They also have the option to consider both excess or shortage investment in different economic periods by using fuzzy constraint for the sum of the fuzzy proportions.

2. Lower and upper possibilistic means and variances

Let $\tilde{A}$ be a fuzzy number with a normal, convex, and continuous membership function. Carlsson and Fullér (2001) defined the lower and upper possibilistic mean values of a fuzzy number $\tilde{A}$ with α-levels as shown in Equation (1).

$$\tilde{A}^{\alpha} = [a_1(\alpha),\ a_2(\alpha)](\alpha > 0). \tag{1}$$

Then, the lower possibilistic mean value can be defined as in Equation (2).

$$M_*(\tilde{A}) = \frac{\int_0^1 Pos[\tilde{A} \le a_1(\alpha)]a_1(\alpha)d\alpha}{\int_0^1 Pos[\tilde{A} \le a_1(\alpha)]d\alpha} = 2\int_0^1 \alpha \cdot a_1(\alpha)d\alpha. \tag{2}$$

The upper possibilistic mean value can be defined as in Equation (3).

$$M^*(\tilde{A}) = \frac{\int_0^1 Pos[\tilde{A} \ge a_2(\alpha)]a_2(\alpha)d\alpha}{\int_0^1 Pos[\tilde{A} \ge a_2(\alpha)]d\alpha} = 2\int_0^1 \alpha \cdot a_2(\alpha)d\alpha, \tag{3}$$

where *Pos* denotes the possibility. Let $\tilde{A}$, $\tilde{B}$ be fuzzy numbers. Then,

$$\mathrm{M}_*(\tilde{A} + \tilde{B}) = \mathrm{M}_*(\tilde{A}) + \mathrm{M}_*(\tilde{B}) \tag{4}$$

$$M^*(\tilde{A} + \tilde{B}) = M^*(\tilde{A}) + M^*(\tilde{B}). \tag{5}$$

Thus, the possibilistic mean value of $\tilde{A} + \tilde{B}$ can be obtained as

$$M(\tilde{A} + \tilde{B}) = (1/2)[M_*(\tilde{A} + \tilde{B}) + M^*(\tilde{A} + \tilde{B})]. \tag{6}$$

Corresponding to the lower and upper possibilistic means, Zhang and Nie (2003) introduced the lower and upper possibilistic variances and possibilistic covariances of fuzzy numbers. The lower and upper possibilistic variances of fuzzy number $\tilde{A}$, with $\tilde{A}^{\alpha} = [a_1(\alpha), a_2(\alpha)](\alpha > 0)$ are defined in Equation (7) and Equation (8), respectively.

$$Var_*(\tilde{A}) = \frac{\int_0^1 Pos[\tilde{A} \le a_1(\alpha)][M_*(\tilde{A}) - a_1(\alpha)]^2 d\alpha}{\int_0^1 Pos[\tilde{A} \le a_1(\alpha)]d\alpha} = 2\int_0^1 \alpha \cdot [M_*(\tilde{A}) - a_1(\alpha)]^2 d\alpha, \tag{7}$$

$$Var^*(\tilde{A}) = \frac{\int_0^1 Pos[\tilde{A} \ge a_2(\alpha)][M^*(\tilde{A}) - a_2(\alpha)]^2 d\alpha}{\int_0^1 Pos[\tilde{A} \ge a_2(\alpha)]d\alpha} = 2\int_0^1 \alpha \cdot [M^*(\tilde{A}) - a_2(\alpha)]^2 d\alpha. \tag{8}$$

Then, the possibilistic standard deviation value of $\tilde{A}$, $SD(\tilde{A})$, can be defined as

$$SD(\tilde{A}) = ½\{[Var_*(\tilde{A})]^{0.5} + [Var^*(\tilde{A})]^{0.5}\}. \tag{9}$$

The lower and upper possibilistic covariances between fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are defined as

$$Cov_*(\tilde{A},\tilde{B}) = 2\int_0^1 \alpha\cdot[M_*(\tilde{A}) - a_1(\alpha)] \times [M_*(\tilde{B}) - b_1(\alpha)]d\alpha, \quad (10)$$

$$Cov^*(\tilde{A},\tilde{B}) = 2\int_0^1 \alpha\cdot[M^*(\tilde{A}) - a_2(\alpha)] \times [M^*(\tilde{B}) - b_2(\alpha)]d\alpha \quad (11)$$

respectively, where $\tilde{A}^\alpha = [a_1(\alpha), a_2(\alpha)]$ and $\tilde{B}^\alpha = [b_1(\alpha), b_2(\alpha)]$ $\forall\alpha \in [0,1]$. If ρ and μ are any numbers, then the lower and upper possibilistic variances of the fuzzy number $\rho\tilde{A} + \mu\tilde{B}$ are derived as follows:

$$Var_*(\rho\tilde{A} + \mu\tilde{B}) = 2\int_0^1 \alpha\cdot[M_*(\rho\tilde{A} + \mu\tilde{B}) - (\rho a_1(\alpha) + \mu b_1(\alpha)]^2 d\alpha, \quad (12)$$

$$Var^*(\rho\tilde{A} + \mu\tilde{B}) = 2\int_0^1 \alpha\cdot[M^*(\rho\tilde{A} + \mu\tilde{B}) - (\rho a_2(\alpha) + \mu b_2(\alpha)]^2 d\alpha. \quad (13)$$

3. Fuzzy portfolio selection with incomplete information

In portfolio selection, it is usually difficult to decide which securities should be determined, because of the existence of uncertainty on their return rates. Some researchers use fuzzy numbers to represent the uncertainty of the future returns on assets, and they cast the selection of portfolio as a problem of mathematical programming for selecting the best investment. By contrast, in a period of financial crisis with incomplete information about the future economy, it may be difficult for investors to make a crisp decision about the selection of the portfolio, and thus, the proportion for each asset in the composition of the portfolio is assumed to be a fuzzy number. In fuzzy portfolio analysis, let the fuzzy return rate and fuzzy proportion be symmetric triangular fuzzy numbers defined as $\tilde{r}_i = (r_i, c_i)$ and $\tilde{x}_i = (x_i, d_i)$, respectively, where r_i and x_i are the central values, and c_i, d_i are the spread values of $\tilde{r}_i$ and $\tilde{x}_i$, respectively. Then, the α-cut of the fuzzy return rate $\tilde{r}_i$, and the fuzzy proportion $\tilde{x}_i$ for $i = 1, 2$ can be derived as

$$(\tilde{r}_1)^\alpha = [r_1 - (1-\alpha)c_1, r_1 + (1-\alpha)c_1], \quad (14)$$

$$(\tilde{r}_2)^\alpha = [r_2 - (1-\alpha)c_2, r_2 + (1-\alpha)c_2], \quad (15)$$

$$(\tilde{x}_1)^\alpha = [x_1 - (1-\alpha)d_1, x_1 + (1-\alpha)d_1], \quad (16)$$

$$(\tilde{x}_2)^\alpha = [x_2 - (1-\alpha)d_2, x_{21} + (1-\alpha)d_2], \quad (17)$$

where r_1, r_2, x_1, x_2 are the central values, and c_1, c_2, d_1, d_2 are the spread values of the fuzzy numbers $\tilde{r}_1$,$\tilde{r}_2$, $\tilde{x}_1$, $\tilde{x}_2$, besides, $r_i - (1-\alpha)c_i$ and $x_i - (1-\alpha)d_i$ are defined as positive values, $\forall i = 1, 2$. Then, the α-cut of the fuzzy returns on a portfolio $\{\tilde{x}_1, \tilde{x}_2\}$ can be obtained, where the lower fuzzy return for the first asset is defined in Equation (18), and the upper fuzzy return for the first asset is defined in Equation (19).

$$\begin{aligned}(\tilde{r}_1\tilde{x}_1)_*^\alpha &= [r_1 - (1-\alpha)c_1] \times [x_1 - (1-\alpha)d_1] \\ &= r_1x_1 - r_1d_1(1-\alpha) - x_1c_1(1-\alpha) + c_1d_1 \times (1-\alpha)^2 \\ &= \alpha^2 c_1d_1 + \alpha(r_1d_1 + x_1c_1 - 2c_1d_1) + (r_1x_1 - r_1d_1 - x_1c_1 + c_1d_1)\end{aligned} \quad (18)$$

$$\begin{aligned}(\tilde{r}_1\tilde{x}_1)^{*\alpha} &= [r_1 + (1-\alpha)c_1] \times [x_1 + (1-\alpha)d_1] \\ &= r_1x_1 + r_1d_1(1-\alpha) + x_1c_1(1-\alpha) + c_1d_1 \times (1-\alpha)^2 \\ &= \alpha^2 c_1d_1 - \alpha(r_1d_1 + x_1c_1 + 2c_1d_1) + (r_1x_1 + r_1d_1 + x_1c_1 + c_1d_1)\end{aligned} \quad (19)$$

Next, the α-cut of the lower fuzzy return for the second asset is defined in Equation (20), and the upper fuzzy return for the first asset is defined as in Equation (21).

$$\begin{aligned}(\tilde{r}_2\tilde{x}_2)_*^\alpha &= [r_2 - (1-\alpha)c_2] \times [x_2 - (1-\alpha)d_2] \\ &= r_2x_2 - r_2d_2(1-\alpha) - x_2c_2(1-\alpha) + c_2d_2 \times (1-\alpha)^2 \\ &= \alpha^2 c_2d_2 + \alpha(r_2d_2 + x_2c_2 - 2c_2d_2) + (r_2x_2 - r_2d_2 - x_2c_2 + c_2d_2)\end{aligned} \quad (20)$$

$$\begin{aligned}(\tilde{r}_2\tilde{x}_2)^{*\alpha} &= [r_2 + (1-\alpha)c_2] \times [x_2 + (1-\alpha)d_2] \\ &= r_2x_2 + r_2d_2(1-\alpha) + x_2c_2(1-\alpha) + c_2d_2 \times (1-\alpha)^2 \\ &= \alpha^2 c_2d_2 - \alpha(r_2d_2 + x_2c_2 + 2c_2d_2) + (r_2x_2 + r_2d_2 + x_2c_2 + c_2d_2)\end{aligned} \quad (21)$$

Then, the lower possibilistic mean value can be obtained as

$$\begin{aligned}M_*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2) &= M_*(\tilde{r}_1\tilde{x}_1) + M_*(\tilde{r}_2\tilde{x}_2) = 2\int_0^1 \alpha^* \times [(\tilde{r}_1\tilde{x}_1)_*^\alpha + (\tilde{r}_2\tilde{x}_2)_*^\alpha]d\alpha \\ &= 2\int_0^1 \alpha^3 c_1d_1 + \alpha^2(r_1d_1 + x_1c_1 - 2c_1d_1) \\ &\quad + \alpha(r_1x_1 - r_1d_1 - x_1c_1 + c_1d_1)d\alpha + 2\int_0^1 \alpha^3 c_2d_2 \\ &\quad + \alpha^2(r_2d_2 + x_2c_2 - 2c_2d_2) + \alpha(r_2x_2 - r_2d_2 \\ &\quad - x_2c_2 + c_2d_2)d\alpha\end{aligned}$$

$$= r_1x_1 + r_2x_2 - \frac{1}{3}(r_1d_1 + r_2d_2 + x_1c_1 + x_2c_2) + \frac{1}{6}(c_1d_1 + c_2d_2). \quad (22)$$

Next, the upper possibilistic mean value of the return associated with the portfolio can be obtained as follows:

$$M^*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2) = M^*(\tilde{r}_1\tilde{x}_1) + M^*(\tilde{r}_2\tilde{x}_2) = 2\int_0^1 \alpha^* \times [(\tilde{r}_1\tilde{x}_1)^{*\alpha} + (\tilde{r}_2\tilde{x}_2)^{*\alpha}]d\alpha$$
$$= 2\int_0^1 \alpha^3c_1d_1 - \alpha^2(r_1d_1 + x_1c_1 + 2c_1d_1) + \alpha(r_1x_1 + r_1d_1 + x_1c_1 + c_1d_1)d\alpha + 2\int_0^1 \alpha^3c_2d_2 - \alpha^2 \times (r_2d_2 + x_2c_2 + 2c_2d_2) + \alpha(r_2x_2 + r_2d_2 + x_2c_2 + c_2d_2)d\alpha = r_1x_1 + r_2x_2 + \frac{1}{3}(r_1d_1 + r_2d_2 + x_1c_1 + x_2c_2) + \frac{1}{6}(c_1d_1 + c_2d_2). \quad (23)$$

Based on Equations (19) and (20), the upper and lower possibilistic variances associated with the portfolio can be obtained as shown in Equations (24) and (25) below.

$$Var^*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2) = 2\int_0^1 \alpha \times \{M^*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2) - [(\tilde{r}_1\tilde{x}_1)^{*\alpha} + (\tilde{r}_2\tilde{x}_2)^{*\alpha}]\}^2 d\alpha$$
$$= 2\int_0^1 \alpha\Big[r_1x_1 + r_2x_2 + \frac{1}{3}(r_1d_1 + r_2d_2 + x_1c_1 + x_2c_2) + \frac{1}{6}(c_1d_1 + c_2d_2) - \alpha^2(c_1d_1 + c_2d_2) + \alpha(r_1d_1 + r_2d_2 + x_1c_1 + x_2c_2 - 2c_1d_1 - 2c_2d_2) - (r_1x_1 + r_2x_2 + r_1d_1 + r_2d_2 + x_1c_1 + x_2c_2 + c_1d_1 + c_2d_2)\Big]^2 d\alpha$$
$$= \int_0^1 2\alpha\Big[\Big(\alpha - \frac{2}{3}\Big)(r_1d_1 + r_2d_2 + x_1c_1 + x_2c_2) + \Big(-\alpha^2 + 2\alpha - \frac{5}{6}\Big)(c_1d_1 + c_2d_2)\Big]^2 d\alpha$$
$$= \frac{1}{18}\Big[(r_1d_1 + r_2d_2 + x_1c_1 + x_2c_2) + \frac{4}{5}(c_1d_1 + c_2d_2)\Big]^2 + \frac{1}{300}(c_1d_1 + c_2d_2)^2 \quad (24)$$

$$Var_*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2) = 2\int_0^1 \alpha \times \{M_*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2) - [(\tilde{r}_1\tilde{x}_1)_*^{\alpha} + (\tilde{r}_2\tilde{x}_2)_*^{\alpha}]\}^2 d\alpha$$
$$= 2\int_0^1 \alpha\Big[r_1x_1 + r_2x_2 - \frac{1}{3}(r_1d_1 + r_2d_2 + x_1c_1 + x_2c_2) + \frac{1}{6}(c_1d_1 + c_2d_2) - \alpha^2(c_1d_1 + c_2d_2) - \alpha(r_1d_1 + r_2d_2 + x_1c_1 + x_2c_2 - 2c_1d_1 - 2c_2d_2) - (r_1x_1 + r_2x_2 - r_1d_1 - r_2d_2 - x_1c_1 - x_2c_2 + c_1d_1 + c_2d_2)\Big]^2 d\alpha$$
$$= \int_0^1 2\alpha\Big[\Big(\frac{2}{3} - \alpha\Big)(r_1d_1 + r_2d_2 + x_1c_1 + x_2c_2) + \Big(-\alpha^2 + 2\alpha - \frac{5}{6}\Big)(c_1d_1 + c_2d_2)\Big]^2 d\alpha$$
$$= \frac{1}{18}\Big[(r_1d_1 + r_2d_2 + x_1c_1 + x_2c_2) - \frac{4}{5}(c_1d_1 + c_2d_2)\Big]^2 + \frac{1}{300}(c_1d_1 + c_2d_2)^2 \quad (25)$$

For n securities, the lower possibilistic mean of the fuzzy return rates associated with the portfolio can be induced according to the following theorem:

Theorem 1: *Let the fuzzy return and fuzzy proportion be symmetric triangular fuzzy numbers defined as $\tilde{r}_i = (r_i, c_i)$ and $\tilde{x}_i = (x_i, d_i)$, respectively, where r_i and x_i are the central values, and c_i, d_i are the spread values of $\tilde{r}_i$ and $\tilde{x}_i$, for $i = 1, 2, \ldots, n$. Then, the lower and upper possibilistic means of the fuzzy return associated with the fuzzy portfolio are obtained by Equation (26) and (27), respectively.*

$$M_*\left(\sum_{i=1}^n \tilde{r}_i\tilde{x}_i\right) = \sum_{i=1}^n r_ix_{i2} - \frac{1}{3}\sum_{i=1}^n (r_id_i + x_ic_i) + \frac{1}{6}\sum_{i=1}^n (c_id_i) \quad (26)$$

$$M^*\left(\sum_{i=1}^n \tilde{r}_i\tilde{x}_i\right) = \sum_{i=1}^n r_ix_{i2} + \frac{1}{3}\sum_{i=1}^n (r_id_i + x_ic_i) + \frac{1}{6}\sum_{i=1}^n (c_id_i) \quad (27)$$

Proof: (1) The lower possibilistic mean of n securities can be denoted as

$$M_*\left(\sum_{i=1}^n \tilde{r}_i\tilde{x}_i\right) = M_*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2 + \cdots + \tilde{r}_n\tilde{x}_n)$$
$$= M_*(\tilde{r}_1\tilde{x}_1) + M_*(\tilde{r}_2\tilde{x}_2) + \cdots + M_*(\tilde{r}_n\tilde{x}_n)$$
$$= r_1x_1 - \frac{1}{3}(r_1d_1 + x_1c_1) + \frac{1}{6}(c_1d_1) + \cdots + r_nx_n - \frac{1}{3}(r_nd_n + x_nc_n) + \frac{1}{6}(c_nd_n)$$
$$= \sum_{i=1}^n r_ix_i - \frac{1}{3}\sum_{i=1}^n (r_id_i + x_ic_i) + \frac{1}{6}\sum_{i=1}^n (c_id_i).$$

(2) The upper possibilistic mean of n securities can be derived as

$$M^*\left(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i\right) = M^*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2 + \cdots + \tilde{r}_n\tilde{x}_n)$$
$$= M^*(\tilde{r}_1\tilde{x}_1) + M^*(\tilde{r}_2\tilde{x}_2) + \cdots + M^*(\tilde{r}_n\tilde{x}_n)$$
$$= r_1x_1 + \frac{1}{3}(r_1d_1 + x_1c_1) + \frac{1}{6}(c_1d_1) + \cdots + r_nx_n + \frac{1}{3}(r_nd_n + x_nc_n) + \frac{1}{6}(c_nd_n)$$
$$= \sum_{i=1}^{n} r_ix_i + \frac{1}{3}\sum_{i=1}^{n}(r_id_i + x_ic_i) + \frac{1}{6}\sum_{i=1}^{n}(c_id_i).$$

Then, the possibilistic mean of the return associated with the portfolio, as given by Equations (26) and (27) is

$$M\left(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i\right) = \sum_{i=1}^{n} r_ix_{i2} + \frac{1}{6}\sum_{i=1}^{n}(c_id_i). \tag{28}$$

Theorem 2: *Let the fuzzy return and fuzzy proportion be symmetric triangular fuzzy numbers defined as $\tilde{r}_i = (r_i, c_i)$ and $\tilde{x}_i = (x_i, d_i)$, respectively, where r_i and x_i are the central values, and c_i, d_i are the spread values of $\tilde{r}_i$ and $\tilde{x}_i$, for $i = 1, 2, \ldots, n$. Then, the lower and upper possibilistic variances of the fuzzy return associated with the fuzzy portfolio, defined by Equations (29) and (30), are, respectively,*

$$Var_*\left(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i\right) = \frac{1}{18}\left[\sum_{i=1}^{n}(r_id_i + x_ic_i) - \frac{4}{5}\sum_{i=1}^{n}(c_id_i)\right]^2 + \frac{1}{300}\sum_{i=1}^{n}(c_id_i)^2, \tag{29}$$

$$Var^*\left(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i\right) = \frac{1}{18}\left[\sum_{i=1}^{n}(r_id_i + x_ic_i) + \frac{4}{5}\sum_{i=1}^{n}(c_id_i)\right]^2 + \frac{1}{300}\sum_{i=1}^{n}(c_id_i)^2. \tag{30}$$

Proof: (1) The lower possibilistic variance of n securities can be denoted as shown below:

$$Var_*\left(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i\right) = 2\int_0^1 \alpha * \left[M_*\left(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i\right) - \left(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i\right)_*^{\alpha}\right]^2 d\alpha = 2\int_0^1 \alpha\left[\sum_{i=1}^{n} r_ix_i\right.$$
$$- \frac{1}{3}\left(\sum_{i=1}^{n}(r_id_i + x_ic_i)\right) + \frac{1}{6}\left(\sum_{i=1}^{n}c_id_i\right) - \alpha^2\left(\sum_{i=1}^{n}c_id_i\right) - \alpha\left(\sum_{i=1}^{n}(r_id_i + x_ic_i) - 2\sum_{i=1}^{n}c_id_i\right)$$
$$\left. - \left(\sum_{i=1}^{n}r_ix_i - \sum_{i=1}^{n}(r_id_i + x_ic_i) + \sum_{i=1}^{n}c_id_i\right)\right]^2 d\alpha$$
$$= \int_0^1 2\alpha\left[\left(\frac{2}{3} - \alpha\right)\sum_{i=1}^{n}(r_id_i + x_ic_i) + \left(-\alpha^2 + 2\alpha - \frac{5}{6}\right) \times \left(\sum_{i=1}^{n}c_id_i\right)\right]^2 d\alpha = \frac{1}{18}\left[\sum_{i=1}^{n}(r_id_i + x_ic_i) - \frac{4}{5}\left(\sum_{i=1}^{n}c_id_i\right)\right]^2 + \frac{1}{300}\left(\sum_{i=1}^{n}c_id_i\right)^2$$

(2) The upper possibilistic variance of n securities can be denoted as shown below:

$$Var^*\left(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i\right)$$
$$= 2\int_0^1 \alpha^*\left\{M^*\left(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i\right) - \left[\left(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i\right)^{*\alpha}\right]\right\}^2 d\alpha$$
$$= 2\int_0^1 \alpha\left[\sum_{i=1}^{n}r_ix_i + \frac{1}{3}\sum_{i=1}^{n}(r_id_i + x_ic_i) + \frac{1}{6}\left(\sum_{i=1}^{n}c_id_i\right) - \alpha^2\left(\sum_{i=1}^{n}c_id_i\right) + \alpha\left(\sum_{i=1}^{n}(r_id_i + x_ic_i) - 2\sum_{i=1}^{n}c_id_i\right)\right.$$
$$\left. - \left(\sum_{i=1}^{n}r_ix_i + \sum_{i=1}^{n}(r_id_i + x_ic_i) + \sum_{i=1}^{n}c_id_i\right)\right]^2 d\alpha$$
$$= \int_0^1 2\alpha\left[\left(\alpha - \frac{2}{3}\right)\left(\sum_{i=1}^{n}(r_id_i + x_ic_i)\right) + \left(-\alpha^2 + 2\alpha - \frac{5}{6}\right)\left(\sum_{i=1}^{n}c_id_i\right)\right]^2 d\alpha$$
$$= \frac{1}{18}\left[\left(\sum_{i=1}^{n}(r_id_i + x_ic_i)\right) + \frac{4}{5}(c_1d_1 + c_2d_2)\right]^2 + \frac{1}{300}\left(\sum_{i=1}^{n}c_id_i\right)^2$$

By analysing the lower and upper possibilistic variances associated with a portfolio with n securities, the variances associated with the portfolio described in Equations (29) and (30) can be divided into two parts. The first part is defined as the variance of the securities investment, and the second part is the variance of vagueness of incomplete information in a depressed economy with short investments. From Equation (29), we can obtain the upper possibility standard deviation for securities investments and the deviation of incomplete information in the investment environment as shown in Equations (31) and (32).

$$\sigma_1^* = \frac{1}{\sqrt{18}}\left[\sum_{i=1}^{n}(r_i d_i + x_i c_i) + \frac{4}{5}\sum_{i=1}^{n}(c_i d_i)\right], \quad (31)$$

$$\sigma_2^* = \frac{1}{\sqrt{300}}\sum_{i=1}^{n}(c_i d_i). \quad (32)$$

By contrast, from Equation (30), the standard deviations of the lower possibilistic variance are obtained as:

$$\sigma_1^* = \frac{1}{\sqrt{18}}\left[\sum_{i=1}^{n}(r_i d_i + x_i c_i) - \frac{4}{5}\sum_{i=1}^{n}(c_i d_i)\right], \quad (33)$$

$$\sigma_2^* = \frac{1}{\sqrt{300}}\sum_{i=1}^{n}(c_i d_i). \quad (34)$$

Therefore, the expected standard deviation for securities investments can be obtained as in Equation (35), and the expected standard deviation of incomplete information in the investment environment as in Equation (36).

$$\sigma_1 = \frac{\sigma_1^* + \sigma_1^*}{2} = \frac{1}{\sqrt{18}}\left[\sum_{i=1}^{n}(r_i d_i + x_i c_i)\right] \quad (35)$$

$$\sigma_2 = \frac{\sigma_2^* + \sigma_2^*}{2} = \frac{1}{\sqrt{300}}\sum_{i=1}^{n}(c_i d_i) \quad (36)$$

Analogous to Markowitz's mean–variance methodology, the possibilistic mean value represents the invested return of the portfolio, which is the objective function to be maximised, while the possibilistic standard deviation represents the risk of the portfolio to be constrained by the upper bound of the desired values of the investor. Starting from this point of view, the possibilistic mean–standard deviation model of portfolio selection with fuzzy return rate and fuzzy proportion is

$$\begin{aligned} &Max \sum_{i=1}^{n} r_i x_i + \frac{1}{6}\left(\sum_{i=1}^{n} c_i d_i\right) \\ &s.t. \frac{\sqrt{2}}{6}\left[\sum_{i=1}^{n}(r_i d_i + x_i c_i)\right] \le q_1' \\ &\frac{1}{10\sqrt{3}}\left(\sum_{i=1}^{n} c_i d_i\right) \le q_2' \quad \sum_{i=1}^{n} \tilde{x}_i \cong 1 \\ &0 \le d_i \le g_i,\ f_i \le x_i \le 1, \forall i = 1, 2, ..., n \end{aligned} \quad (37)$$

where q_1' denotes the upper bound of the expected standard deviation for securities investment, and q_2' denotes the upper bound of the expected standard deviation of incomplete information of a portfolio. Next, f_i and 1 are the lower and upper bounds of the central value x_i, and g_i is the upper bound of the spread value d_i of the fuzzy number $\tilde{x}_i$. In addition, the model in Equation (37) requires the sum of the fuzzy proportions of $\sum_{i=1}^{n} \tilde{x}_i$ in the portfolio to be near 1. As the spread values d_i of $\tilde{x}_i$ are required to be lower than the value of g_i, $\forall i = 1,2,\ldots,n$, then, the sum of the fuzzy proportions being near 1 implies that the sum of the central values $\sum_{i=1}^{n} x_i$ is essentially smaller than or equal to 1, as shown below:

$$\sum_{i=1}^{n} x_i \underset{\sim}{\le} 1 \quad (38)$$

If Equation (38) is used to revise the model in Equation (37), then the feasible region of the model in Equation (37) is enlarged (Zimmermann 1987) by relaxing the left-hand side constraint of Equation (38) to be possibly larger than the value of the right-hand side. Because the feasible region is fuzzified, the objective function should be fuzzified to rewrite the model in Equation (37) as in Equation (39).

$$\begin{aligned} &\widetilde{Max} \sum_{i=1}^{n} r_i x_i + \frac{1}{6}\left(\sum_{i=1}^{n} c_i d_i\right) \\ &s.t. \frac{\sqrt{2}}{6}\left[\sum_{i=1}^{n}(r_i d_i + x_i c_i)\right] \le q_1' \\ &\frac{1}{10\sqrt{3}}\left(\sum_{i=1}^{n} c_i d_i\right) \le q_2' \quad \sum_{i=1}^{n} \tilde{x}_i \underset{\sim}{\le} 1 \\ &0 \le d_i \le g_i,\ f_i \le x_i \le 1, \forall i = 1, 2, ..., n \end{aligned} \quad (39)$$

To obtain the maximum fuzzy objective value, a lower bound value u for the return rate associated with the portfolio is chosen in the model of Equation (39). Thus, the model in Equation (39) is a fuzzy linear programming model, which can be transformed into the model shown in Equation (40):

$$\begin{aligned} &\sum_{i=1}^{n} r_i x_i + \frac{1}{6}\left(\sum_{i=1}^{n} c_i d_i\right) \underset{\sim}{\ge} u \\ &s.t. \left[\sum_{i=1}^{n}(r_i d_i + x_i c_i)\right] \le q_1 \\ &\left(\sum_{i=1}^{n} c_i d_i\right) \le q_2 \quad \sum_{i=1}^{n} x_i \underset{\sim}{\le} 1 \\ &0 \le d_i \le g_i;\ f_i \le x_i \le 1, \forall i = 1, 2, \ldots, n, \end{aligned} \quad (40)$$

where the membership functions of "$\underset{\sim}{\geq}$" for $\sum_{i=1}^{n} r_i x_i + \frac{1}{6}(\sum_{i=1}^{n} c_i d_i)$ and "$\underset{\sim}{\leq}$" for $\sum_{i=1}^{n} x_i$ are defined in Equations (41) and (42), and are shown in Figures 1 and 2, respectively.

$$u_{\underset{\sim}{\geq}}(x) = \begin{cases} 1 & if \sum_{i=1}^{n} r_i x_i + \frac{1}{6}\left(\sum_{i=1}^{n} c_i d_i\right) \geq u \\ 1 - \dfrac{u - [\sum_{i=1}^{n} r_i x_i + \frac{1}{6}\left(\sum_{i=1}^{n} c_i d_i\right)]}{p_1} & if \sum_{i=1}^{n} r_i x_i + \frac{1}{6}\left(\sum_{i=1}^{n} c_i d_i\right) - p_1 \leq x \leq \sum_{i=1}^{n} r_i x_i + \frac{1}{6}\left(\sum_{i=1}^{n} c_i d_i\right) \\ 0 & otherwise \end{cases} \tag{41}$$

$$u_{\underset{\sim}{\leq}}(x) = \begin{cases} 1 & if \sum_{i=1}^{n} x_i \leq 1 \\ 1 - \dfrac{\sum_{i=1}^{n} x_i - 1}{p_2} & if 1 \leq \sum_{i=1}^{n} x_i \leq 1 + p_2 \\ 0 & otherwise \end{cases} \tag{42}$$

In Equations (41) and (42), p_1 denotes the tolerance value that signifies that the expected return $\sum_{i=1}^{n} r_i x_i + \frac{1}{6}(\sum_{i=1}^{n} c_i d_i)$ can be smaller than the desired objective value u, and p_2 denotes the tolerance value that signifies that the sum of the central values of the investment proportion $\sum_{i=1}^{n} x_i$ can be larger than the desired invested proportion value. In order to obtain the minimum fuzziness of the expected return $\sum_{i=1}^{n} r_i x_i + \frac{1}{6}(\sum_{i=1}^{n} c_i d_i)$ in the portfolio, the value u is suggested to be larger than zero. In addition, the value of p_1 is suggested to be at most, equal to the given value of u, because the lower bound of the expected return (u–p_1) in the portfolio should be larger than 0. Similarly, p_2 should be less than 1 because the range of the spread ($1 + p_2$) of the fuzzy proportion associated with the portfolio should be less than 1 when the investor is confident about the future economy and the margin of purchasing more securities than those in previous portfolios. To obtain the optimal solution that satisfies the model in Equation (40) to the maximum level, a new variable λ is introduced to derive the model in Equation (43). Besides, if the chosen tolerance values cannot obtain the optimal solution of the model in Equation (43), then a larger u, or q_1 and q_2, and tolerance values p_1 and p_2 should be chosen. Finally, the objective value λ represents the satisfactory level under the given tolerance values of $\sum_{i=1}^{n} r_i x_i + \frac{1}{6}(\sum_{i=1}^{n} c_i d_i)$ and $\sum_{i=1}^{n} x_i$, where a larger objective value of λ does not imply a better solution but instead it implies the possibility that the optimal solution can be obtained under the given tolerance values.

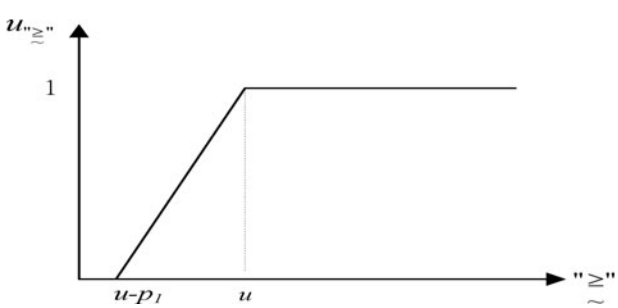

Figure 1. The membership function for the fuzzy constraint of $\sum_{i=1}^{n} r_i x_i + \frac{1}{6}(\sum_{i=1}^{n} c_i d_i)$.

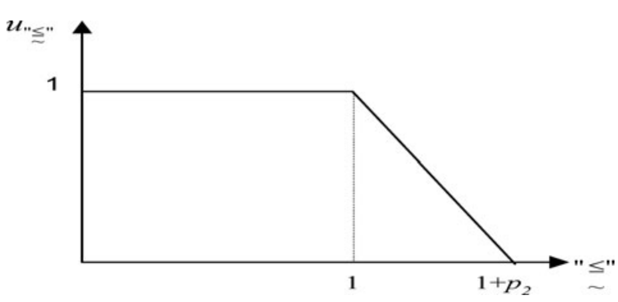

Figure 2. The membership function for the fuzzy constraint of $\sum_{i=1}^{n} x_i$.

$$\begin{aligned} &\text{Max } \lambda \\ &\text{s.t.} \sum_{i=1}^{n} r_i x_i + \tfrac{1}{6}\left(\sum_{i=1}^{n} c_i d_i\right) \geq u - (1-\lambda)p_1 \\ &\left[\sum_{i=1}^{n} (r_i d_i + x_i c_i)\right] \leq q_1 \\ &\left(\sum_{i=1}^{n} c_i d_i\right) \leq q_2 \\ &\sum_{i=1}^{n} x_i \leq 1 + (1-\lambda)p_2 \\ &0 \leq d_i \leq g_i; f_i \leq x_i \leq 1; 0 \leq \lambda \leq 1 \end{aligned} \tag{43}$$

4. Numerical example

To illustrate how our proposed model solves the fuzzy portfolio problem with incomplete information, a portfolio selection example is chosen and revised from Zhang (2007). In this example, five selected securities were chosen as a portfolio. Based on historical data, the financial reports of the corporations and future information, the return rate of each selected stock in the portfolio can be estimated with the following possibility distributions: $\tilde{r}_1 = (0.073, 0.054)$, $\tilde{r}_2 = (0.15, 0.075)$, $\tilde{r}_3 = (0.138, 0.096)$, $\tilde{r}_4 = (0.168, 0.126)$ and $\tilde{r}_5 = (0.208, 0.168)$, where the first and second items are the central values and spread values of the fuzzy return $\tilde{r}_i$, for $i = 1, 2,\ldots, 5$. The lower and upper bounds of the centre value x_i in the fuzzy proportion $\tilde{x}_i$, for $i = 1, 2,\ldots, 5$, is given by 0 and 1, respectively. Similarly, the upper bound of its spread value d_i is given by 0.1. Using the model in Equation (43), the possibilistic portfolios under some different tolerated risk levels p_1 and p_2 are constructed as

$$
\begin{aligned}
&\text{Max } \lambda \\
&s.t.\, 0.073x_1 + 0.15x_2 + 0.138x_3 + 0.168x_4 + 0.208x_5 \\
&+ \frac{1}{6}(0.054d_1 + 0.075d_2 + 0.096d_3 + 0.126d_4 + 0.168d_5 \\
&\geq u - (1-\lambda)p_1.\, 0.073d_1 + 0.15d_2 + 0.138d_3 + 0.168d_4 \\
&+0.208d_5 + 0.054x_1 + 0.075x_2 + 0.096x_3 + 0.126x_4 \\
&+0.168x_5 \leq q_1 0.054d_1 + 0.075d_2 + 0.096d_3 + 0.126d_4 \\
&+0.168d_5 \leq q_2 x_1 + x_2 + x_3 + x_4 + x_5 \leq 1 + (1-\lambda)p_2 \\
&0 \leq d_1 \leq 0.1; 0 \leq d_2 \leq 0.1; 0 \leq d_3 \leq 0.1; 0 \leq d_4 \\
&\leq 0.1; 0 \leq d_5 \leq 0.1\; 0.1 \leq x_1 \leq 1; 0.1 \leq x_2 \leq 1; \\
&0.1 \leq x_3 \leq 1; 0.1 \leq x_4 \leq 1; 0.1 \leq x_5 \leq 1\; 0 \leq \lambda \leq 1 \quad (44)
\end{aligned}
$$

We solve the model of Equation (44) by setting the input values of p_1, p_2, q_1, q_2 and the lower bound value u of the expected fuzzy return in the portfolio. The results of the possibilistic portfolio for each security are shown in Table 1. Table 1 indicates that the investor is sure to choose the selective securities from $\tilde{x}_1$ to $\tilde{x}_5$ as the portfolio. However, because investors have incomplete information about this portfolio, they do not know how to allocate their capital to these selected securities. For example, the input parameters are given as $p_1 = 0.05, p_2 = 0.5, q_1 = 0.25$ and $q_2 = 0.1$. When the expected return is required to be possibly larger than 0.27, the fuzzy proportion for each security can be obtained as $\tilde{x}_1 = (0.1, 0.0928)$, $\tilde{x}_2 = (0.1, 0)$, $\tilde{x}_3 = (0.1, 0)$, $\tilde{x}_4 = (0.12, 0.1)$ and $\tilde{x}_5 = (1, 0.1)$. The sum of the fuzzy proportions for the securities is $\sum_{i=1}^{n} \tilde{x}_i = (1.42, 0.2928)$, with a lower bound of 1.1272 and an upper bound of 1.7128, and where the investor can use his own capital for investment, but also needs to borrow more capital for margin purchasing to enlarge his portfolio because the expected return is very large. In addition, the fuzzy proportion of each selected security tends to margin purchasing when u exceeds 0.2, but the investor has insufficient information to make crisp portfolio selections, and thus, the invested proportions are fuzzy numbers. When the expected returns are required to be possibly larger than 0.15, the invested proportion of x_5 is greater than the other securities, because security x_5 has a better invested return rate in this portfolio. When the expected returns are required to be smaller than 0.1, the total proportions are less than 1 in these portfolios because the expected return is too small to make a positive investment and thus induce the investor to save some capital for future investments. In contrast, if the expected return value is less than 0.06, the invested proportion for each security is at its minimum value of 0.1. Therefore, under the specific risks, the investor can choose an efficient portfolio according to the incomplete information during the period of depression.

From Table 2 to Table 5, we make some sensitivity analyses to compare the effects of the tolerance values p_1 p_2, q_1 and q_2 to the invested proportions. In Table 2, by increasing the value of p_1 we can find that a larger tolerance value p_1 means that a lower expected return (u–p_1) is contributed; then a lower invested proportion for a specific security (i.e. x_5) can be solved. Therefore, we suggest that if an investor can stand the tolerance range of the expected return to be smaller than 3%, then a reasonable p_1 value is suggested as 0.03 for the portfolio selection. In Table 3, we can find that the feasible region for the sum of the fuzzy proportions is enlarged by increasing the value of p_2; the larger the value of p_2, the larger the invested proportions we can obtain. In practice, we suggest that if the economic change is better, then a larger p_2 value can be chosen; otherwise, a smaller p_2 value is suggested during the period of depression. In addition, we have defined the q_1 value to be the risk of security investment, thus, a larger q_1 value implies larger

Table 1. The possibilistic efficient portfolios.

	u	x_1	x_2	x_3	x_4	x_5	d_1	d_2	d_3	d_4	d_5
$p_1 = 0.05$,	0.27	0.1	0.1	0.1	0.12	1	0.0928	0	0	0.1	0.1
$p_2 = 0.5$,	0.25	0.1	0.1	0.1	0.1	0.9107	0.1	0.0213	0.1	0.1	0.1
$q_1 = 0.25$,	0.2	0.1	0.1	0.1	0.1	0.6656	0.1	0.1	0.1	0.1	0.1
$q_2 = 0.1$	0.18	0.1	0.1	0.1	0.1	0.6	0	0	0	0	0.0821
	0.15	0.3052	0.1	0.1	0.1	0.3948	0	0	0	0	0
	0.13	0.4533	0.1	0.1	0.1	0.2467	0	0	0	0	0
	0.1	0.1	0.1	0.1	0.1	0.2264	0	0	0	0	0
	0.08	0.1	0.1	0.1	0.1	0.1303	0	0	0	0	0
	0.06	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.04	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0

Table 2. Sensitivity analysis for parameter p_1.

	u	x_1	x_2	x_3	x_4	x_5	d_1	d_2	d_3	d_4	d_5
$p_1 = 0.01$,	0.27	0.1	0.1	0.1	0.12	1	0.0928	0	0	0.1	0.1
$p_2 = 0.5$,	0.25	0.1	0.1	0.1	0.1	0.9107	0.1	0.0213	0.1	0.1	0.1
$q_1 = 0.25$,	0.2	0.1	0.1	0.1	0.1	0.6656	0.1	0.1	0.1	0.1	0.1
$q_2 = 0.1$	0.18	0.147	0.1	0.1	0.1	0.553	0.1	0.1	0.1	0.1	0.1
	0.15	0.1	0.5776	0.1	0.1	0.1224	0	0	0	0	0
	0.13	0.4553	0.1	0.1	0.1	0.2467	0	0	0	0	0
	0.1	0.4603	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.08	0.1	0.1	0.1	0.1	0.1303	0	0	0	0	0
	0.06	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.04	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
$p_1 = 0.03$,	0.27	0.1	0.1	0.1	0.12	1	0.0928	0	0	0.1	0.1
$p_2 = 0.5$,	0.25	0.1	0.1	0.1	0.1	0.9107	0.1	0.0213	0.1	0.1	0.1
$q_1 = 0.25$,	0.2	0.1	0.1	0.1	0.1	0.6656	0.1	0.1	0.1	0.1	0.1
$q_2 = 0.1$	0.18	0.1	0.1	0.1	0.1	0.6	0	0	0	0	0.0821
	0.15	0.3052	0.1	0.1	0.1	0.3948	0	0	0	0	0
	0.13	0.4553	0.1	0.1	0.1	0.2467	0	0	0	0	0
	0.1	0.1	0.1	0.1	0.1	0.2264	0	0	0	0	0
	0.08	0.1	0.1	0.1	0.1	0.1303	0	0	0	0	0
	0.06	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.04	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
$p_1 = 0.1$,	0.27	0.1	0.1	0.1	0.1	0.8405	0.1	0.1	0.1	0.1	0.1
$p_2 = 0.5$,	0.25	0.1	0.1	0.1	0.1	0.8405	0.1	0.1	0.1	0.1	0.1
$q_1 = 0.25$,	0.2	0.1	0.1	0.1	0.1	0.6656	0.1	0.1	0.1	0.1	0.1
$q_2 = 0.1$	0.18	0.1	0.1	0.1	0.1	0.6	0	0	0	0	0.0821
	0.15	0.1	0.1	0.1	0.1	0.4668	0	0	0	0	0
	0.13	0.1	0.1	0.1	0.1	0.3707	0	0	0	0	0
	0.1	0.1	0.1	0.1	0.1	0.2264	0	0	0	0	0
	0.08	0.1	0.1	0.1	0.1	0.1303	0	0	0	0	0
	0.06	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.04	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
$p_1 = 0.15$,	0.27	0.1	0.1	0.1	0.1	0.6	0.1	0.1	0.1	0.1	0.1
$p_2 = 0.5$,	0.25	0.1	0.1	0.1	0.1	0.6	0.1	0.1	0.1	0.1	0.1
$q_1 = 0.25$,	0.2	0.1	0.1	0.1	0.1	0.6	0.1	0.1	0.1	0.1	0.1
$q_2 = 0.1$	0.18	0.1	0.1	0.1	0.1	0.6	0	0	0	0	0.0821
	0.15	0.1	0.1	0.1	0.1	0.4668	0	0	0	0	0
	0.13	0.1	0.1	0.1	0.1	0.3707	0	0	0	0	0
	0.1	0.1	0.1	0.1	0.1	0.2264	0	0	0	0	0
	0.08	0.1	0.1	0.1	0.1	0.1303	0	0	0	0	0
	0.06	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.04	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0

investment risk. In Table 4, we can find that if we set a larger q_1 value, then the profitable security x_5 is increasing in the selected portfolio. An investor can choose the value q_1 according to the condition of economics. We can also find that when the invested risk $q_1 = 0.1$, the portfolio solution is infeasible at expected returns of 0.25 and 0.27 because we cannot obtain a portfolio at a higher expected return with the smallest investment risk. Finally, the value q_2 has been defined as the risk of vagueness of incomplete information in a depressed economy. In Table 4 we can find that the sum of the spreads for the fuzzy proportions fitting with larger q_2 values are larger than or equal to the values of smaller q_2s. We suggest that if an investor has incomplete information about the condition of economics, then a larger q_2 value is chosen, otherwise, a smaller q_2 is suggested.

Summarily, by comparing Table 2 to Table 5, we find out that the proposed portfolio selection model can provide a possible solution to an investor under incomplete information during the period of depression. Furthermore, whether the input values p_1, p_2, q_1 and q_2 are changed or not, a given expected return u smaller than 0.1 always derives the same invested proportion in these portfolios. Therefore, for an investor to obtain fair profit, we can

Table 3. Sensitivity analysis for parameter p_2.

	u	x_1	x_2	x_3	x_4	x_5	d_1	d_2	d_3	d_4	d_5
$p_1 = 0.05$,	0.27	Infeasible solution									
$p_2 = 0.1$,	0.25	0.1	0.1	0.1	0.1	0.6656	0.1	0.1	0.1	0.1	0.1
$q_1 = 0.25$,	0.2	0.1	0.1	0.1	0.1	0.6	0.1	0.1	0.1	0.1	0.1
$q_2 = 0.1$	0.18	0.1	0.1	0.1	0.1	0.6	0	0	0	0	0.0821
	0.15	0.3052	0.1	0.1	0.1	0.3948	0	0	0	0	0
	0.13	0.4533	0.1	0.1	0.1	0.2467	0	0	0	0	0
	0.1	0.1	0.1	0.1	0.1	0.2264	0	0	0	0	0
	0.08	0.1	0.1	0.1	0.1	0.1303	0	0	0	0	0
	0.06	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.04	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
$p_1 = 0.05$,	0.27	0.1	0.1	0.1	0.1	0.9	0.1	0.0333	0.1	0.1	0.1
$p_2 = 0.3$,	0.25	0.1	0.1	0.1	0.1	0.9	0.1	0.0333	0.1	0.1	0.1
$q_1 = 0.25$,	0.2	0.1	0.1	0.1	0.1	0.6656	0.1	0.1	0.1	0.1	0.1
$q_2 = 0.1$	0.18	0.1	0.1	0.1	0.1	0.6	0	0	0	0	0.0821
	0.15	0.3052	0.1	0.1	0.1	0.3984	0	0	0	0	0
	0.13	0.4533	0.1	0.1	0.1	0.2467	0	0	0	0	0
	0.1	0.1	0.1	0.1	0.1	0.2264	0	0	0	0	0
	0.08	0.1	0.1	0.1	0.1	0.1303	0	0	0	0	0
	0.06	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.04	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
$p_1 = 0.05$,	0.27	0.1	0.1	0.1	0.12	1	0.0928	0	0	0.1	0.1
$p_2 = 0.7$,	0.25	0.1	0.1	0.1	0.1	0.9107	0.1	0.0213	0.1	0.1	0.1
$q_1 = 0.25$,	0.2	0.1	0.1	0.1	0.1	0.6656	0.1	0.1	0.1	0.1	0.1
$q_2 = 0.1$	0.18	0.1	0.1	0.1	0.1	0.6	0	0	0	0	0.0821
	0.15	0.3052	0.1	0.1	0.1	0.3984	0	0	0	0	0
	0.13	0.4533	0.1	0.1	0.1	0.2467	0	0	0	0	0
	0.1	0.1	0.1	0.1	0.1	0.2264	0	0	0	0	0
	0.08	0.1	0.1	0.1	0.1	0.1303	0	0	0	0	0
	0.06	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.04	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
$p_1 = 0.05$,	0.27	0.1	0.1	0.1	0.12	1	0.0928	0	0	0.1	0.1
$p_2 = 1$,	0.25	0.1	0.1	0.1	0.1	0.9107	0.1	0.0213	0.1	0.1	0.1
$q_1 = 0.25$,	0.2	0.1	0.1	0.1	0.1	0.6656	0.1	0.1	0.1	0.1	0.1
$q_2 = 0.1$	0.18	0.1	0.1	0.1	0.1	0.6	0	0	0	0	0.0821
	0.15	0.3052	0.1	0.1	0.1	0.3984	0	0	0	0	0
	0.13	0.4533	0.1	0.1	0.1	0.2467	0	0	0	0	0
	0.1	0.1	0.1	0.1	0.1	0.2264	0	0	0	0	0
	0.08	0.1	0.1	0.1	0.1	0.1303	0	0	0	0	0
	0.06	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.04	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0

suggest them to choose the portfolio at the expected return of 0.1.

5. Conclusions

In this paper, we proposed a fuzzy portfolio model with fuzzy return and fuzzy proportion under incomplete information in a period of depression. The results indicated considerable returns for the portfolios derived by the proposed model under the specific risks. If a larger invested return is required, the derived proportion for each security in the portfolio is a fuzzy number; otherwise, the derived proportion is a crisp number. In addition, under varying investment risk or vagueness risk, the investor can derive shortage investment or excess investment using the proposed fuzzy portfolio model. Most importantly, the sensitivity analysis performed on the illustrative example shows that the proposed portfolio model can be easily applied to derive the invested proportion for an investor. If the fuzzy portfolio model with fuzzy return and fuzzy proportion meets expectations, then this approach will be an important tool for investments. However, this work only examines input tol-

Table 4. Sensitivity analysis for parameter q_1.

	u	x_1	x_2	x_3	x_4	x_5	d_1	d_2	d_3	d_4	d_5
$p_1 = 0.05$,	0.27	Infeasible solution									
$p_2 = 0.5$,	0.25										
$q_1 = 0.1$,	0.2	0.1	0.7413	0.1	0.1	0.1	0	0	0	0	0
$q_2 = 0.1$	0.18	0.1	0.7413	0.1	0.1	0.1	0	0	0	0	0
	0.15	0.1831	0.3841	0.1	0.1	0.2327	0	0	0	0	0
	0.13	0.4533	0.1	0.1	0.1	0.2467	0	0	0	0	0
	0.1	0.1	0.1	0.1	0.1	0.2264	0	0	0	0	0
	0.08	0.1	0.1	0.1	0.1	0.1303	0	0	0	0	0
	0.06	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.04	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
$p_1 = 0.05$,	0.27	0.1	0.8516	0.1	0.1	0.3484	0	0	0	0	0
$p_2 = 0.5$,	0.25	0.1	0.8516	0.1	0.1	0.3484	0	0	0	0	0
$q_1 = 0.15$,	0.2	0.1	0.1847	0.1	0.1	0.6461	0.1	0	0	0	0
$q_2 = 0.1$	0.18	0.1	0.1	0.1	0.1	0.6022	0	0	0	0	0.066
	0.15	0.3052	0.1	0.1	0.1	0.3948	0	0	0	0	0
	0.13	0.4533	0.1	0.1	0.1	0.2467	0	0	0	0	0
	0.1	0.1	0.1	0.1	0.1	0.2264	0	0	0	0	0
	0.08	0.1	0.1	0.1	0.1	0.1303	0	0	0	0	0
	0.06	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.04	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
$p_1 = 0.05$,	0.27	0.1	0.314	0.1	0.1	0.886	0	0	0	0	0
$p_2 = 0.5$,	0.25	0.1	0.1	0.1	0.1	0.9435	0	0	0	0	0.0308
$q_1 = 0.2$,	0.2	0.1	0.1	0.1	0.1	0.6757	0.1	0	0.047	0.1	0.1
$q_2 = 0.1$	0.18	0.1	0.1	0.1	0.1	0.6	0	0	0	0	0.0821
	0.15	0.3052	0.1	0.1	0.1	0.3948	0	0	0	0	0
	0.13	0.4533	0.1	0.1	0.1	0.2467	0	0	0	0	0
	0.1	0.1	0.1	0.1	0.1	0.2264	0	0	0	0	0
	0.08	0.1	0.1	0.1	0.1	0.1303	0	0	0	0	0
	0.06	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.04	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
$p_1 = 0.05$,	0.27	0.1	0.1	0.1	0.1027	1	0.1	0.1	0.1	0.1	0.1
$p_2 = 0.5$,	0.25	0.1	0.1	0.1	0.1	0.906	0.1	0.1	0.1	0.1	0.1
$q_1 = 0.3$,	0.2	0.1	0.1	0.1	0.1	0.6656	0.1	0.1	0.1	0.1	0.1
$q_2 = 0.1$	0.18	0.1	0.1	0.1	0.1	0.6	0	0	0	0	0.0821
	0.15	0.3052	0.1	0.1	0.1	0.3948	0	0	0	0	0
	0.13	0.4533	0.1	0.1	0.1	0.2467	0	0	0	0	0
	0.1	0.1	0.1	0.1	0.1	0.2264	0	0	0	0	0
	0.08	0.1	0.1	0.1	0.1	0.1303	0	0	0	0	0
	0.06	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.04	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0

erance values, the lower bound return, the input investment risk, and the vagueness risk to determine the selection of securities for a portfolio. Therefore, future research should consider the best input values for the parameters and determine the relationships among these parameters. In addition, a multi-stage transaction cost for the portfolio selection can be considered during the different stages of business cycles.

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Table 5. Sensitivity analysis for parameter q_2.

	u	x_1	x_2	x_3	x_4	x_5	d_1	d_2	d_3	d_4	d_5
$p_1 = 0.05$,	0.27	0.1	0.1	0.1	0.1442	1	0.1	0.0613	0	0	0
$p_2 = 0.5$,	0.25	0.1	0.1	0.1	0.1	0.9396	0.1	0.0613	0	0	0
$q_1 = 0.25$,	0.2	0.1	0.1	0.1	0.1	0.6992	0	0	0	0	0.0595
$q_2 = 0.01$	0.18	0.1	0.1	0.1	0.1	0.603	0	0	0	0	0.0595
	0.15	0.3052	0.1	0.1	0.1	0.3948	0	0	0	0	0
	0.13	0.4533	0.1	0.1	0.1	0.2467	0	0	0	0	0
	0.1	0.1	0.1	0.1	0.1	0.2264	0	0	0	0	0
	0.08	0.1	0.1	0.1	0.1	0.1303	0	0	0	0	0
	0.06	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.04	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
$p_1 = 0.05$,	0.27	0.1	0.1	0.1	0.12	1	0.0928	0	0	0.1	0.1
$p_2 = 0.5$,	0.25	0.1	0.1	0.1	0.1	0.9107	0.1	0.0213	0.1	0.1	0.1
$q_1 = 0.25$,	0.2	0.1	0.1	0.1	0.1	0.6671	0.1	0.0747	0.1	0.1	0.1
$q_2 = 0.05$	0.18	0.1	0.1	0.1	0.1	0.6	0	0	0	0	0.0821
	0.15	0.3052	0.1	0.1	0.1	0.3948	0	0	0	0	0
	0.13	0.4533	0.1	0.1	0.1	0.2467	0	0	0	0	0
	0.1	0.1	0.1	0.1	0.1	0.2264	0	0	0	0	0
	0.08	0.1	0.1	0.1	0.1	0.1303	0	0	0	0	0
	0.06	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.04	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
$p_1 = 0.05$,	0.27	0.1	0.1	0.1	0.12	1	0.0928	0	0	0.1	0.1
$p_2 = 0.5$,	0.25	0.1	0.1	0.1	0.1	0.9107	0.1	0.0213	0.1	0.1	0.1
$q_1 = 0.25$,	0.2	0.1	0.1	0.1	0.1	0.6656	0.1	0.1	0.1	0.1	0.1
$q_2 = 0.08$	0.18	0.1	0.1	0.1	0.1	0.6	0	0	0	0	0.0821
	0.15	0.3052	0.1	0.1	0.1	0.3948	0	0	0	0	0
	0.13	0.4533	0.1	0.1	0.1	0.2467	0	0	0	0	0
	0.1	0.1	0.1	0.1	0.1	0.2264	0	0	0	0	0
	0.08	0.1	0.1	0.1	0.1	0.1303	0	0	0	0	0
	0.06	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.04	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
$p_1 = 0.05$,	0.27	0.1	0.1	0.1	0.12	1	0.0928	0	0	0.1	0.1
$p_2 = 0.5$,	0.25	0.1	0.1	0.1	0.1	0.9107	0.1	0.0213	0.1	0.1	0.1
$q_1 = 0.25$,	0.2	0.1	0.1	0.1	0.1	0.6656	0.1	0.1	0.1	0.1	0.1
$q_2 = 0.15$	0.18	0.1	0.1	0.1	0.1	0.6	0	0	0	0	0.0821
	0.15	0.3052	0.1	0.1	0.1	0.3948	0	0	0	0	0
	0.13	0.4533	0.1	0.1	0.1	0.2467	0	0	0	0	0
	0.1	0.1	0.1	0.1	0.1	0.2264	0	0	0	0	0
	0.08	0.1	0.1	0.1	0.1	0.1303	0	0	0	0	0
	0.06	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
	0.04	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0

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