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Inventory lot-size policies for deteriorating items with expiration dates and advance payments

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ABSTRACT

For deteriorating items with seasonal demand, a supplier usually requests that the buyer (retailer) prepays a fraction of the acquisition cost as a deposit. The expiration date of a deteriorating item is an important factor in a buyer's purchase decision. Despite its importance, relatively little attention has been paid to the effects of the expiration date; the versions of economic order quantity models that are available consider fixed deterioration rates. This paper considers a more realistic situation where the deterioration rate of a product gradually increases as the expiration date approaches. In this paper, the optimal cycle time and the cycle fraction of no shortages are the decision variables that minimize the total cost. The total annual relevant cost is shown to be strictly pseudo-convex for each of the decision variables, which simplifies the search for the global solution to a local minimum. This paper provides an improvement on earlier work, as it provides an optimal rather than a near-optimal solution. Several numerical examples are provided to illustrate the behaviour of the model and to highlight some managerial insights.

1. Introduction

It is common practice by suppliers to ask their buyers (retailers) to prepay deposits for purchased seasonal and deteriorating items. By doing so, a retailer may obtain a price discount or an on-time delivery in return. In a recent study, Taleizadeh [1] proposed an economic order quantity (EOQ) model for a deteriorating product, in which: (1) the deterioration rate is constant and not subject to an expiration date, (2) the supplier requests a fraction of the acquisition cost to be prepaid in equal-sized multiple installments, and the remaining balance at the time of delivery, and (3) shortages are allowed with complete backlogging. Concurrently, Taleizadeh [2] extended his work in [1] to consider partial backlogging, where he obtained a near-optimal solution by using a truncated Taylor series expansion. In his case, the deterioration rate approaches, but never reaches, zero.

The expiration date of a product impacts a customer's purchase decision. Unlike earlier work, this paper addresses these shortcomings by assuming that the deterioration rate gradually increases as the expiration date of the product approaches,

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Table 1
Comparisons among various inventory models.

Reference	Payment	Trade credit	Shortage	Backlogging	Deterioration
[10]	Delay	Up	No	No	Constant
[35]	Delay	Up	No	No	Time-varying
[23]	Immediate	No	No	No	Time-varying
[22]	Immediate	No	No	No	Constant
[5]	Delay	Up	No	No	No
[12]	Delay	Up/Down	No	No	No
[11]	Delay	Up	Yes	Complete	Constant
[13]	Delay	Up/Down	No	No	Constant
[14]	Delay	Up/Down	No	No	Constant
[15]	Delay	Up	Yes	Complete	No
[28]	Delay	Up	No	No	Constant
[1]	Advance	Up	Yes	Complete	Constant
[2]	Advance	Up	Yes	Partial	Constant
[32]	Immediate	Up	No/Yes	Partial	Constant
[8]	Delay	Up	No	No	No
[9]	Delay	Up/Down	No	No	No
[30]	Delay	Down	No	No	Time-varying
[31]	Delay	Up/Down	No	No	Time-varying
[17]	Advance	No	No	No	No
The paper	Advance	No	No/Yes	Partial	Time-varying

by which point it is fully (100%) deteriorated. As a result, the retailer's annual cost is minimized for the cycle time and the fraction of no shortages as decision variables. This results in a cost function which is strictly pseudo-convex for decision variables. This approach simplifies the search for the global and local minimum solutions.

It is well-known that Harris [3] established the classical EOQ model on the assumptions that the buyer must pay for the acquisition cost as soon as the purchase items are received, and the purchase items are non-perishable and can be sold indefinitely. Vast inventory models had been developed since then. Recently, several review papers on the lot size models were studied by Andriolo et al. [4], Bushuev et al. [5], and Glock et al. [6]. In practice, there are three different strategies for the seller to collect the buyer's acquisition cost: (*i*) to get the entire payment as soon as the order quantity is delivered, such as in Harris [3], (*ii*) to grant a permissible delay in payment in order to increase sales, such as in Goyal [7] and Teng [8,9], and (*iii*) to request a partial prepayment prior to delivery in exchange of on-line delivery or price discount, such as in Taleizadeh [1,2].

In current highly competitive markets, the seller frequently offers the buyer a permissible delay in payments (i.e., credit period) to stimulate sales and consequently reduce inventory. During the credit period, interest is earned on the accumulative revenue, and hence reduces the cost. However, if the buyer cannot pay off the acquisition cost after the credit period then he/she is charged the interest on the unpaid balance. In a seminal work, Goyal [5] developed the optimal EOQ when the seller offers the buyer a permissible delay in payments. Several relevant inventory models for deteriorating items with types of deterioration rates and payment options are summarized in Table 1. For a detailed review on trade credit financing, please see Seifert et al. [16].

On the other hand, for highly seasonal products or flammable deteriorating items, the seller usually requests buyers to prepay a fraction of the acquisition cost as a good-faith deposit. In return, the buyer gets an on-time delivery guarantee or a price discount. Zhang [17] studied the optimal advanced payment scheme with fixed prepayment costs. Taleizadeh et al. [8] constructed an EOQ model with partial backordering and multiple prepayments. Taleizadeh [1,2] extended the EOQ model for deteriorating items with multiple prepayments, and obtained an explicit closed-form solution when the deterioration rate is near zero. However, none of the researchers in the area of advance payments took the product expiration date into consideration based on our best knowledge.

In fact, many products (e.g., bakeries, fruits, meat, milk, vegetables, fashion- merchandises and high-tech products) are perishable and deteriorate continuously due to several reasons, such as evaporation, spoilage, and obsolescence. For detailed reviews in the area of deteriorating items, we refer readers to the works of Raafat [18], Goyal and Giri [19] and Bakker et al. [20]. Additionally, for a complete survey with regard to deterioration and lifetime constraints in production and supply chain planning we refer to readers to the paper of Pahl and Voß [21]. In a pioneering piece of work, Ghare and Schrader [22] proposed an EOQ model with a constant exponentially deterioration rate. Covert and Philip [23] generalized the constant exponential deterioration rate to a two-parameter Weibull distribution. Dave and Patel [24] then developed an EOQ model for deteriorating items with linearly increasing demand and no shortages. For generality, Hariga [25] studied the EOQ models for deteriorating items with time-varying demand. Teng et al. [26] expanded EOQ models with shortages to any fluctuating demand pattern. Teng et al. [27] explored the model to allow for partial backlogging and lost sales. Skouri et al. [28] developed the model for deteriorating items with ramp-type demand and permissible delay in payments. Wu et al. [29] derived optimal replenishment policies for non-instantaneous deteriorating items. Wang et al. [30] and Wu et al. Wu et al. [31] captured the relevant fact that the deterioration rate for a deteriorating product increases with time and reaches

100% by the time it reaches its expiration date, and then derived the optimal credit period and cycle time in a supply chain in which trade credit increases not only the sales revenue but also the default risk and the opportunity cost. Currently, there are several interesting and relevant papers related to inventory models with partial backlogging such as Taleizadeh and Nematollahi [32], Teng et al. [33], Wee et al. [34], and others. The brief comparisons among various models are given in Table 1.

In this paper, we incorporate the concept of 100% deterioration rate at the product expiration date to build an EOQ model in which the supplier requests a fraction of procurement cost be prepaid in multiple equal installments, and the remaining balance is paid at the point of delivery. The retailer's objective is to decide on the optimal replenishment cycle time and the fraction of no shortages such that the total relevant cost is minimized. The rest of the paper is organized as follows. In Section 2, the notations and assumptions that are used throughout the entire paper are defined. The mathematical models with and without shortages are developed in Section 3. Next, theoretical results and an algorithm are established in Section 4. Numerical examples and a sensitivity analysis are provided in Section 5. Finally, the conclusions and the future research directions are given in Section 6.

2. Notation and assumptions

The following notation and assumptions are introduced to define the EOQ model for deteriorating items with multiple prepayments.

2.1. Notation

The following parameters and variables are used to develop the problem.

α	the fraction of the acquisition cost to be prepaid before the time of delivery, $0 \le \alpha \le 1$.
A	the acquisition cost per cycle in dollars, $A > 0$.
λ	the fraction of shortages to be backordered, $0 \le \lambda \le 1$.
b	the backordering cost per unit per year, $b > 0$.
B	the backordering cost per cycle in dollars.
c	the purchase cost per unit in dollars, $c > 0$.
C	the capital cost per cycle in dollars before receiving the order quantity.
D	the market annual demand rate in units.
h	the holding cost including capital cost per unit per year in dollars, $h > 0$.
H	the holding cost per cycle in dollars after receiving the order quantity.
m	the expiration date or the maximum lifetime in years, $0 < m < 5$.
n	the number of equal prepayments before receiving the order quantity.
O	the ordering cost in dollars per order, $O > 0$.
p	the length of time in years during which the prepayments are paid, $p > 0$.
r	the interest rate of capital cost per dollar per year, $0 \le r \le 1$.
s	the cost of lost sales in dollars per unit, $s > c$.
S	the cost of lost sales per cycle in dollars.
$\theta(t)$	the deterioration rate at time t, $0 \le \theta(t) \le 1$.
E	the partially backordered quantity.
$I(t)$	the inventory level in units at time t.
K	the fraction of no shortages (or inventory), $0 \le K \le 1$ (a decision variable).
Q	the order quantity in units.
T	the length of inventory cycle time in years, $T \le m$ (a decision variable).
TC	the total annual relevant cost in dollars.

For convenience, the asterisk symbol on a variable denotes the optimal solution of the variable; for instance, $T*$ is the optimal solution of T.

2.2. Assumptions

Next, the following assumptions are made to establish the mathematical inventory model.

1. All deteriorating items continuously deteriorate with time, and cannot be sold when time exceeds the expiration date m. For simplicity, we assume the same as in Wang et al. [30] and Chen and Teng [35], that the deterioration rate is

$$\theta(t) = \frac{1}{1+m-t}, \ 0 \le t \le T \le m. \tag{1}$$

   Notice that it is clear that the replenishment cycle time T is less than or equal to the product expiration date m.
2. There is no replacement or repair of deteriorated items during the replenishment cycle $[0, T]$.

3. For highly seasonal products or deteriorating items, the seller usually demands α fractions of acquisition cost A to be prepaid (i.e., αA) before the time of delivery. Then the remaining balance $(1-\alpha)A$ is paid at the point of delivery. Notice that if $\alpha = 0$ then the seller does not request prepayment. On the other hand, if $\alpha = 1$ then the seller requests the buyer to prepay the entire acquisition cost.
4. The buyer agrees to pre-pay αA by n equal installments in p years prior to the time of delivery, and pay the rest of $(1-\alpha)A$ at the time of delivery.
5. The replenishment rate is infinite.
6. The lead time is zero.

3. Mathematical model

Given the above notation and assumptions, the proposed EOQ models for deteriorating items with equally multiple installments of α fractions of acquisition cost have two situations: one is without shortages and the other is with shortages and partial backordering. We discuss the case of no shortages first, then follow this with the case with partial backordering.

3.1. The model without shortages

In general, customers will not take a rain check on fresh products (e.g., fruits and vegetables), doughnuts and breads, meat and milk, etc. Therefore, the case without shortages is necessary to discuss.

As shown in Fig. 1, the retailer pays the supplier α fractions of the acquisition cost A by n equal installments in p years prior to the time of delivery. The supplier then delivers the order quantity Q units to the retailer at time 0, and receives the remaining unpaid balance $(1-\alpha)A$ immediately. Thereafter, the retailer's inventory level is gradually depleted to zero by the end of the replenishment cycle T, due to the combination of demand and deterioration. Hence, the inventory level at time t is governed by the following differential equation:

$$\frac{dI(t)}{dt} = -D - \frac{1}{1+m-t} I(t),\ 0 \le t \le T, \tag{2}$$

with boundary condition $I(T) = 0$. From Wang et al. [34], we know that the solution of the above differential equation is:

$$I(t) = D(1+m-t)\ln\left(\frac{1+m-t}{1+m-T}\right),\ 0 \le t \le T. \tag{3}$$

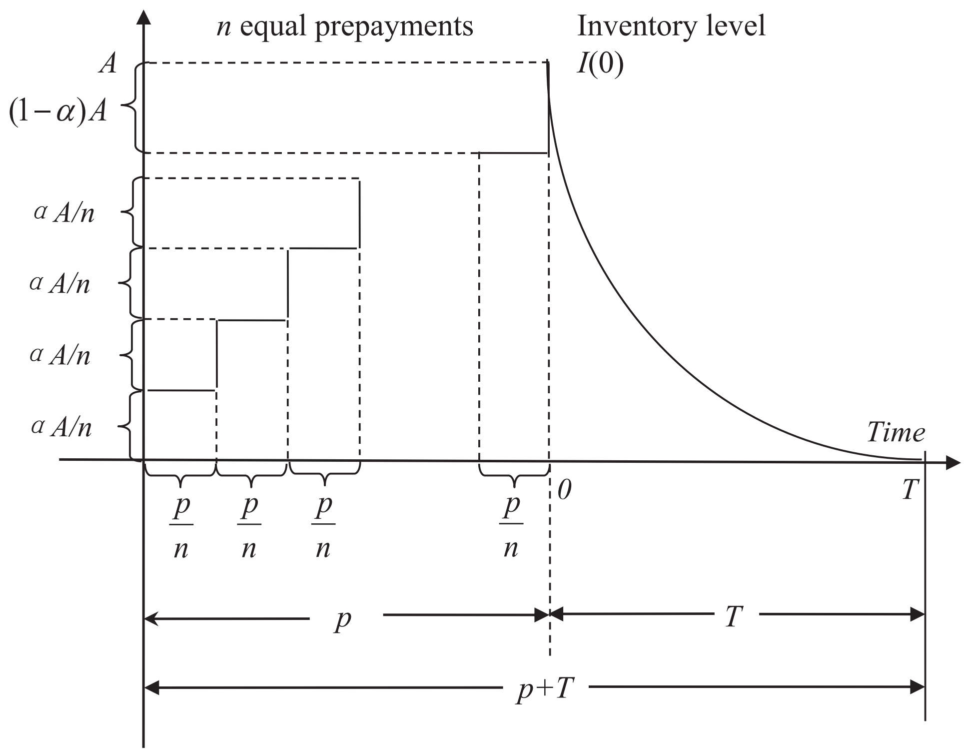

Fig. 1. Graphical representation of the inventory system without shortages

Consequently, the order quantity per replenishment cycle time is:

$$Q = I(0) = D(1+m)\ln\left(\frac{1+m}{1+m-T}\right). \tag{4}$$

The acquisition cost per replenishment cycle is:

$$A = cQ = cD(1+m)\ln\left(\frac{1+m}{1+m-T}\right). \tag{5}$$

The holding cost per cycle including capital cost after receiving Q units at time 0 is:

$$H = h\int_0^T I(t)\,dt = hD\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right]. \tag{6}$$

From Fig. 1 or Taleizadeh [1,2], we know the capital cost per cycle is:

$$C = r\left[\frac{\alpha A}{n}\left(\frac{p}{n}\right)(1+2+\cdots+n)\right] = \frac{n+1}{2n}\,\alpha prA = \frac{n+1}{2n}\,\alpha cDpr(1+m)\ln\left(\frac{1+m}{1+m-T}\right). \tag{7}$$

The total relevant cost per cycle is comprised of the sum of the ordering cost O, the acquisition cost A, the capital cost before receiving the order quantity C and the holding cost after receiving the order quantity H. As a result, the total relevant cost per cycle is calculated as follows:

$$\begin{aligned} O+A+C+H = O + cD(1+m)\ln\left(\frac{1+m}{1+m-T}\right)\left[1+\frac{\alpha pr}{2n}(n+1)\right] \\ + hD\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right]. \end{aligned} \tag{8}$$

As a result, the total annual relevant cost is a function of the replenishment cycle time T as follows:

$$\begin{aligned} TC(T) = \frac{1}{T}\left\{O + cD(1+m)\ln\left(\frac{1+m}{1+m-T}\right)\left[1+\frac{\alpha pr}{2n}(n+1)\right]\right. \\ \left. + hD\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right]\right\}. \end{aligned} \tag{9}$$

The retailer's objective is to find the optimal cycle time $T*$ such that the total annual relevant cost $TC(T)$ in (9) is minimized. Notice that we are unable to obtain the closed-form optimal solution in (9) by applying the Lambert W function (e.g., see Gambini et al. [36] and Warburton [37]). Next, we discuss the case in which shortages are permitted but partially backordered.

3.2. The model with shortages

As shown in Fig. 2, the retailer pre-pays the supplier αA by n equal installments in p years prior to the time of delivery, and pays the remaining balance $(1-\alpha)A$ as soon as they receive the order quantity of Q units at time 0. The inventory level is then gradually depleted to zero at time KT due to a combination of demand and deterioration. Thereafter, shortages are partially backordered at the rate of λ during the time interval $[KT, T]$.

Similar to the results in the case of without shortages, the inventory level at time t is governed by the following differential equation:

$$\frac{dI(t)}{dt} = -D - \frac{1}{1+m-t}\,I(t),\ 0 \le t \le KT, \tag{10}$$

with boundary condition $I(KT) = 0$. The solution of the above differential equation is:

$$I(t) = D(1+m-t)\ln\left(\frac{1+m-t}{1+m-KT}\right),\ 0 \le t \le KT. \tag{11}$$

It is clear that the partially backordered quantity is $E = \lambda D(1-K)T$, and the lost sales quantity is $(1-\lambda)D(1-K)T$. Hence, the order quantity per replenishment cycle time in this case is:

$$Q = D(1+m)\ln\left(\frac{1+m}{1+m-KT}\right) + \lambda D(1-K)T. \tag{12}$$

Then, the acquisition cost per replenishment cycle is:

$$A = cD\left[(1+m)\ln\left(\frac{1+m}{1+m-KT}\right) + \lambda(1-K)T\right]. \tag{13}$$

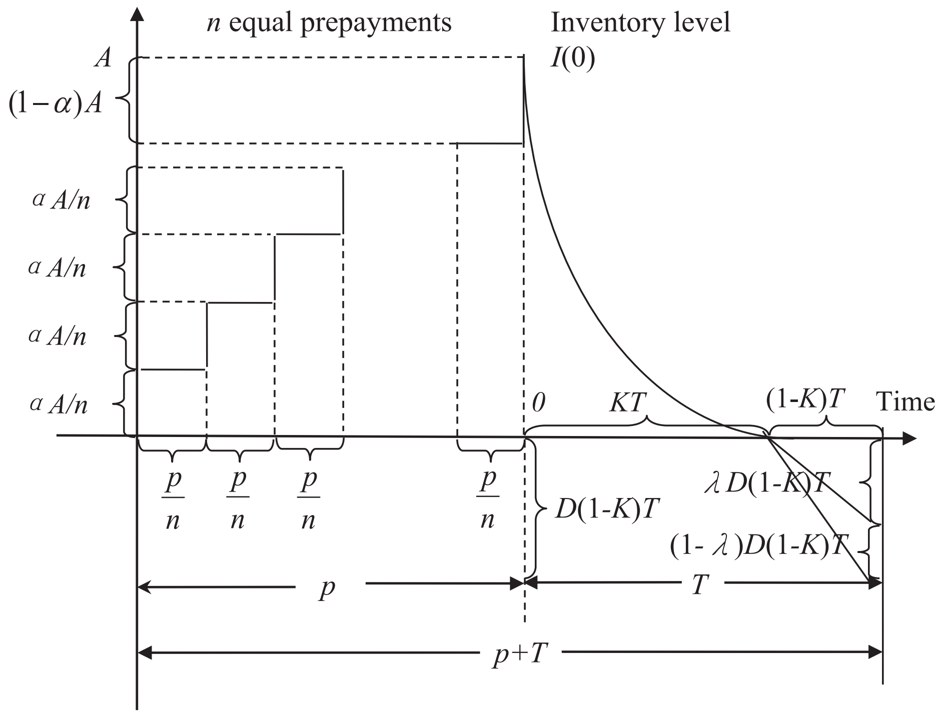

Fig. 2. Graphical representation for the inventory system with shortages.

The holding cost per cycle from time 0 to time KT is given by:

$$H = hD\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-KT}\right) + \frac{(KT)^2}{4} - \frac{(1+m)KT}{2}\right]. \tag{14}$$

From Fig. 2, we know the capital cost per cycle is:

$$C = \alpha cDpr\left(\frac{n+1}{2n}\right)\left[(1+m)\ln\left(\frac{1+m}{1+m-KT}\right) + \lambda(1-K)T\right]. \tag{15}$$

It is clear from Fig. 2 that the partially backordered cost per cycle is:

$$B = \frac{1}{2}b\lambda D(1-K)^2T^2. \tag{16}$$

The cost of lost sales per cycle is $S = s(1-\lambda)D(1-K)T$. The total relevant cost per cycle in this case is the sum of the ordering cost, the acquisition cost, the capital cost before receiving the order quantity, the holding cost after receiving the order quantity, the partially backordered cost and the cost of lost sales. As a result, the total relevant cost per cycle is:

$$\begin{aligned} O+A+H+C+B+S = {} & O + cD\left[(1+m)\ln\left(\frac{1+m}{1+m-KT}\right) + \lambda(1-K)T\right]\left[1 + \frac{\alpha pr}{2n}(n+1)\right] \\ & + hD\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-KT}\right) + \frac{(KT)^2}{4} - \frac{(1+m)KT}{2}\right] \\ & + \frac{1}{2}b\lambda D(1-K)^2T^2 + s(1-\lambda)D(1-K)T. \end{aligned} \tag{17}$$

Consequently, the total annual relevant cost is a function of the replenishment cycle time T, and the fraction of no shortages K is as follows:

$$\begin{aligned} TC(K,T) = {} & \frac{1}{T}\left\{O + D(1+m)\ln\left(\frac{1+m}{1+m-KT}\right)\left[c\left(1+\alpha pr\frac{n+1}{2n}\right) + \frac{h}{2}(1+m)\right]\right\} \\ & + (1-K)D\left[c\lambda\left(1+\alpha pr\frac{n+1}{2n}\right) + \frac{b\lambda}{2}(1-K)T + s(1-\lambda)\right] + hD\left[\frac{K^2T}{4} - \frac{(1+m)K}{2}\right]. \end{aligned} \tag{18}$$

The retailer's objective here is to find the optimal cycle time $T*$ and the optimal fraction of no shortages $K*$such that the total annual relevant cost $TC(K, T)$ in (18) is minimized. Notice that if $K=1$, then the case with shortages is simplified to the case of without shortages. In the next section, we derive the necessary and sufficient conditions for the optimal solution.

4. The theoretical results

Let us discuss the case without shortages first, and then the case in which shortages are partially backordered with some possible lost sales.

4.1. The model without shortages

According to Theorems 3.2.9, and 3.2.10 in Cambini and Martein [38], the fractional function

$$q(x) = \frac{f(x)}{g(x)}, \tag{19}$$

is (strictly) pseudo-convex, if $f(x)$ is non-negative, differentiable and (strictly) convex, and $g(x)$ is positive, differentiable and concave. By applying the above theoretical results, we can demonstrate that the total annual relevant cost $TC(T)$ is a strictly pseudo-convex function of T, which simplifies the unique optimal solution to a local minimum. For convenience, let us define

$$L_1 = \frac{(1+m)D}{m^2}\left[c\left(1+\alpha pr\frac{n+1}{2n}\right)+\frac{h(1+m)}{2}\right][m-\ln(1+m)] - \frac{O}{m^2} + \frac{hD}{4}. \tag{20}$$

Then we have the following result.

Theorem 1.

(a) $TC(T)$ in (9) is a strictly pseudo-convex function of T, and hence there exists a unique minimum solution $T*$.
(b) If $L_1 \geq 0$, then there exists a unique $T* \in (0, m]$ such that $TC(T)$ in (9) is minimized.
(c) If $L_1 < 0$, then $TC(T)$ in (9) is minimized at $T^* = m$.

Proof. See Appendix A.

To find the optimal cycle time $T*$, taking the first-order derivative of $TC(T)$ in (9) with respect to T, setting the result to zero, and re-arranging terms, we get the necessary and sufficient condition for $T*$as:

$$(1+m)D\left[c\left(1+\alpha pr\frac{n+1}{2n}\right)+\frac{h(1+m)}{2}\right]\left[\frac{T}{1+m-T}-\ln\left(\frac{1+m}{1+m-T}\right)\right] - O + \frac{hD}{4}T^2 = 0. \tag{21}$$

Next, we discuss the case in which shortages are partially backordered.

4.2. The model with shortages

By using the analogous argument as in Section 4.1, one can easily obtain the following theoretical results. From (18), we can easily obtain the following corollary.

Corollary 1. *$TC(K, T)$ is increasing with α, b, c, h, r, O, and s; while decreasing with n.*

Proof. It immediately follows from (9) and (18).

Likewise, we define

$$L_2 = \frac{(1+m)D}{m^2}\left[c\left(1+\alpha pr\frac{n+1}{2n}\right)+\frac{h(1+m)}{2}\right]\left[\frac{Km}{1+m-Km}-\ln\left(\frac{1+m}{1+m-Km}\right)\right] - \frac{O}{m^2} + \frac{hDK^2}{4} + \frac{b\lambda D(1-K)^2}{2}. \tag{22}$$

Then we have the following result.

Theorem 2. For any given K, we get

(a) $TC(K, T)$ in (18) is a strictly pseudo-convex function in T, and hence there exists a unique minimum solution $T*$.
(b) If $L_2 \geq 0$, then there exists a unique $T* \in (0, m]$ such that $TC(T)$ in (18) is minimized.
(c) If $L_2 < 0$, then $TC(T)$ in (18) is minimized at $T^* = m$.

Proof. The proof is omitted because it is similar to that of Theorem 1.

To find the optimal cycle time $T*$, taking the first-order derivative of $TC(K, T)$ in (18) with respect to T, setting the result to zero, and re-arranging terms, we get the necessary condition for $T*$:

$$(1+m)\left[c\left(1+\alpha pr\frac{n+1}{2n}\right)+\frac{h(1+m)}{2}\right]\left[\frac{KT}{1+m-KT}-\ln\left(\frac{1+m}{1+m-KT}\right)\right] - \frac{O}{D} + \frac{h(KT)^2}{4} + \frac{b\lambda(1-K)^2T^2}{2} = 0\,. \tag{23}$$

Similarly, for convenience, we define

$$L_3 = Dc(1-\lambda)\left(1+\alpha pr\frac{n+1}{2n}\right) - D[b\lambda T + s(1-\lambda)], \tag{24}$$

and

$$L_4 = \left[\frac{D(1+m)}{1+m-T}\right]\left[c\left(1+\alpha pr\frac{n+1}{2n}\right)+\frac{h}{2}(1+m)\right] - D\left[c\lambda\left(1+\alpha pr\frac{n+1}{2n}\right)+s(1-\lambda)\right] - \frac{hD}{2}(1+m-T). \tag{25}$$

We know from Appendix B that $L_3 < L_4$. Consequently, one can mathematically prove the following result.

Theorem 3. For any given T, we have:

(a) $TC(K, T)$ in (18) is a strictly convex function in K, and hence exists a unique minimum solution $K*$.
(b) If $L_4 \le 0$, then $TC(K)$ in (18) is minimized at $K^* = 1$.
(c) If $L_3 < 0$ and $L_4 > 0$, then there exists a unique $K* \in (0,1)$ such that $TC(K)$ in (18) is minimized.
(d) If $L_3 \ge 0$, then $TC(K)$ in (18) is minimized at $K^* = 0$.

Proof. See Appendix B.

To find the optimal fraction of no shortages $K*$, taking the first-order derivative of $TC(K, T)$ in (18) with respect to K, setting the result to zero, and simplifying terms, we derive the following necessary condition:

$$\left(\frac{1+m}{1+m-KT}\right)\left[c\left(1+\alpha pr\frac{n+1}{2n}\right)+\frac{h}{2}(1+m)\right] - \left[c\lambda\left(1+\alpha pr\frac{n+1}{2n}\right)+b\lambda(1-K)T+s(1-\lambda)\right] + \frac{h}{2}(KT-1-m) = 0. \tag{26}$$

Summarizing the results of Theorems 2 and 3, we propose the following algorithm to find the optimal solution for the model with shortages.

Algorithm to find the optimal solution

Step 1. Input all values of parameters.
Step 2. Solve (23) and (26) simultaneously, and get the solution (T, K).
Step 3. If $T \le m$ and $0 \le K \le 1$, then set the optimal $T^* = T$ and $K^* = K$, and stop.
Step 4. If $T \le m$ and $K > 1$, then set the optimal $T^* = T$ and $K^* = 1$, and stop.
Step 5. If $T \le m$ and $K < 0$, then set the optimal $T^* = T$ and $K^* = 0$, and stop.
Step 6. If $T > m$ and $0 \le K \le 1$, then set $T^* = m$, compute L_3 and L_4, obtain $K*$ based on the results in Theorem 3, and then stop.
Step 7. If $T > m$ and $K > 1$, then set $T^* = m$ and $K^* = 1$, and then stop.
Step 8. If $T > m$ and $K < 0$, then set $T^* = m$ and $K^* = 0$, and then stop.

In the next section, we provide several numerical examples to illustrate theoretical results and gain some managerial insights.

5. Numerical examples

In this section, to illustrate theoretical results as well as to gain some managerial insights, we use MATHEMATICA 10.2 to run several numerical examples. We then study the sensitivity analysis on the optimal solution with respect to some important parameters in appropriate units.

Example 1. The model without shortages

Let $D=1000$ units/year, $O=\$100$/order, $h=\$10$/unit/year, $c=\$30$/unit, $n=3$, $p=0.17$ years (or 62 days), $r=0.10$/dollar/year, $\alpha=0.4$, and $m=0.25$ years (or 91 days).

By using MATHEMATICA 10.2, we have the following unique optimal solution:
$Q*=76.2323$ units, $T*=0.0740$ years (or 27 days), and $TC*=\$32{,}793.86$.

Example 2. Sensitivity analysis without shortages.

Using the same data as used in Example 1, we study the sensitivity analysis of the optimal solution with respect to relevant and important parameters such as α and m. The computational results are shown in Table 2.

Table 2 shows that the optimal order quantity $Q*$ and cycle time $T*$ are insensitive to the change in the fraction of advanced payment α. However, the optimal total annual cost $TC*$ is slightly increasing when α is increasing. Finally, an increase in the maximum lifetime m causes increases in the optimal $Q*$ and $T*$, but a decrease in the optimal $TC*$.

Table 2
Sensitivity analysis when shortages are prohibited.

Parameter	Order quantity $Q*$	Cycle time $T*$	Total cost $TC*$
$\alpha=0.20$	76.2936	0.0740	\$32,723.80
$\alpha=0.40$	76.2323	0.0740	\$32,793.86
$\alpha=0.60$	76.1712	0.0739	\$32,864.00
$m=0.25$	76.2323	0.0740	\$32,793.86
$m=0.35$	78.3516	0.0761	\$32,720.75
$m=0.45$	80.3246	0.0781	\$32,656.18

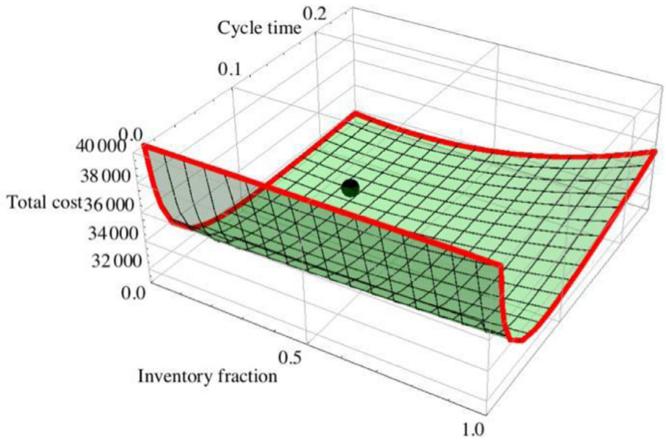

Fig. 3. Optimal solution with shortages and $\lambda = 0.50$.

Table 3
Sensitivity analysis when shortages are allowed.

Parameter	$E*$	$K*$	$Q*$	$T*$	$TC*$
$\alpha=0.40$	63.0369	0.2131	97.6620	0.1602	\$31,328.74
$\alpha=0.60$	63.4049	0.2082	97.2106	0.1602	\$31,370.10
$\alpha=0.80$	63.7641	0.2033	96.7475	0.1601	\$31,411.28
$m=0.25$	63.0369	0.2131	97.6620	0.1602	\$31,328.74
$m=0.35$	62.6828	0.2224	99.0221	0.1612	\$31,321.65
$m=0.45$	62.3501	0.2310	100.3102	0.1622	\$31,315.00
$\lambda = 0.50$	63.0369	0.2131	97.6620	0.1602	\$31,328.74
$\lambda = 0.75$	72.8895	0.2946	114.1430	0.1378	\$31,559.79
$\lambda = 1.00$	79.7039	0.3623	125.8271	0.1250	\$31,730.08
$s=15.00$	70.7107	0.0000	70.7107	0.1414	\$23,982.21
$s=35.00$	63.0369	0.2131	97.6620	0.1602	\$31,328.74
$s=60.00$	0.0000	1.0000	76.2323	0.0740	\$32,793.80

Example 3. The model with shortages.

The data are the same as those used in Example 1. However, we assume that $b=\$20$/unit/year, and $s=\$30$/unit if shortages are backlogged.

If shortages are completely backordered (i.e., $\lambda=1$), we obtain the following unique optimal solution:
$E*=79.7039$ units, $K*=0.3623$, $Q*=125.8271$ units, $T*=0.1250$ years (or 46 days), and $TC*=\$31,730.08$.
If shortages are partially backordered (e.g., $\lambda=0.5$), we obtain the unique optimal solution as shown in Fig. 3:
$E*=63.0369$ units, $K*=0.2131$, $Q*=97.6620$ units, $T*=0.1602$ years (or 59 days), and $TC*=\$31,328.74$.
Fig. 3 reveals that $TC(K, T)$ is a strictly pseudo-convex function in both K and T. Notice that the black point in Fig. 3 is the location of the optimal solution.

Example 4. Sensitivity analysis with shortages.

Using the same data as those used in Example 3, we present a sensitivity analysis of the optimal solution with respect to relevant and important parameters such as α, m, λ and s. The computational results are shown in Table 3.

The sensitivity analysis in Table 3 reveals that: (i) the longer the expiration date m, the longer the cycle time $T*$(as well as the order quantity $Q*$), while the lower the fraction of inventory $K*$ as well as the total cost $TC*$, (ii) an increase in the fraction of advance payment α causes increases in $E*$ and $TC*$ but decreases in $K*$, $Q*$ and $T*$, (iii) an increase in the backlogging rate λ causes increases in $E*$, $K*$, $Q*$ and $TC*$ while a decrease in $T*$, and (iv) a higher value of the cost of lost sales s implies a lower value of $E*$; meanwhile a higher value of $K*$ or $TC*$.

6. Conclusion and future research

In contrast to those previous EOQ models with advance payments such as in Taleizadeh [1,2], without taking expiration dates into consideration, we have enhanced their shortcoming by incorporating the important fact that the deterioration rate gradually approaches 100% as the expiration date approaches. As a result, we have developed an EOQ model for the retailer with partial backordering and lost sales when the supplier requests a partial prepayment before delivery, and the product gradually deteriorates to 100% as its expiration date approaches. Then, we have mathematically demonstrated that the total annual relevant cost is strictly pseudo-convex on decision variables: the replenishment cycle time and the fraction of no shortages. Consequently, the search for the global solution has been simplified to find the unique local minimum. In contrast to a near-optimal solution as in Taleizadeh [1,2], we have established the optimal solution for arbitrary deterioration rates in this study. Following the generalized assumption that the deterioration rate is time-varying instead of constant, we have obtained some intuitive understandings from computational results. For instance, the numerical results reveal that the longer the expiration date, the larger the cycle time (as well as the order quantity), while the lower the total relevant cost. Hence, the expiration date does affect the order quantity.

The proposed model can be extended in several ways. For a non-classical approach to inventory management, the entropy cost as shown in Jeber et al. [39] could be used as a simple way to measure the hidden cost that arises from disorder in production-inventory systems. Furthermore, the supplier may grant permissible delay in payment to the remaining unpaid balance, instead of collecting it at the point of delivery. In addition, we can generalize the constant demand pattern to a demand function of time, price, advertising and product quality. Also, we could expand the model from a single player to a two-player model for both the retailer and the supplier. Finally, we might consider the number of prepayments before delivery as another decision variable.

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Appendix A. Proof of Theorem 1

By applying (9), we define

$$f_1(T) = O + cD(1+m)\ln\left(\frac{1+m}{1+m-T}\right)\left[1+\frac{\alpha pr}{2n}(n+1)\right] + hD\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}\right], \tag{A1}$$

and

$$g_1(T) = T > 0. \tag{A2}$$

Hence, $TC(T) = f_1(T)/g_1(T)$. Taking the first- and second-order derivatives of $f_1(T)$ with respect to T, we obtain:

$$f_1{}'(T) = \frac{cD(1+m)}{1+m-T}\left(1+\alpha pr\frac{n+1}{2n}\right) + \frac{hD}{2}\left[\frac{(1+m)^2}{1+m-T}+T-(1+m)\right], \tag{A3}$$

and

$$f_1''(T) = \frac{cD(1+m)}{(1+m-T)^2}\left(1+\alpha pr\frac{n+1}{2}\right) + \frac{hD}{2}\left[\left(\frac{1+m}{1+m-T}\right)^2+1\right] > 0\,, \tag{A4}$$

respectively. Therefore, $TC(T)$ is a strictly pseudo-convex function of T, and hence there exists a unique global minimum $T*$. This completes the proof of Part (a) of Theorem 1.

By taking the first-order derivative of $TC(T)$ in (9) with respect to T, and rearranging terms, we get

$$TC'(T) = \frac{(1+m)D}{T^2}\left[c\left(1+\alpha pr\frac{n+1}{2n}\right)+\frac{h(1+m)}{2}\right]\left[\frac{T}{1+m-T}-\ln\left(\frac{1+m}{1+m-T}\right)\right] - \frac{O}{T^2}+\frac{hD}{4}. \tag{A5}$$

Using L'Hospital's Rule, we obtain:

$$\lim_{T\to 0}\left[\frac{1}{(1+m-T)T}-\frac{1}{T^2}\ln\left(\frac{1+m}{1+m-T}\right)\right] = \lim_{T\to 0}\left[\frac{T-(1+m-T)\ln\left(\frac{1+m}{1+m-T}\right)}{(1+m-T)T^2}\right]$$
$$= \lim_{T\to 0}\left[\frac{\ln\left(\frac{1+m}{1+m-T}\right)}{2T+2mT-3T^2}\right] = \lim_{T\to 0}\left[\frac{1}{(2+2m-6T)(1+m-T)}\right] = \frac{1}{2(1+m)^2}. \tag{A6}$$

Combining (A5) and (A6), we have

$$\lim_{T\to 0} TC'(T) = -\infty. \tag{A7}$$

We know from (A5) that

$$TC'(m) = \frac{(1+m)D}{m^2}\left[c\left(1+\alpha pr\frac{n+1}{2n}\right)+\frac{h(1+m)}{2}\right][m-\ln(1+m)] - \frac{O}{m^2}+\frac{hD}{4} = L_1. \tag{A8}$$

If $L_1 \geq 0$, by applying the Mean Value Theorem to (A7) and (A8), there exists a unique $T* \in (0, m]$ such that $TC(T)$ in (9) is minimized. This completes the proof of Part (b) of Theorem 1.

Finally, If $L_1 < 0$, then $TC'(T) < 0$, for all $T \in (0, m]$. Hence, $TC(T)$ in (9) is decreasing in T, and thus is minimized at $T^* = m$. This completes the proof of Part (c) of Theorem 1.

Appendix B. Proof of Theorem 3

For any given T, by applying (18) we define

$$f_3(K) = \frac{O}{T}+\frac{D(1+m)}{T}\ln\left(\frac{1+m}{1+m-KT}\right)\left[c\left(1+\alpha pr\frac{n+1}{2n}\right)+\frac{h}{2}(1+m)\right]$$
$$+(1-K)D\left[c\lambda\left(1+\alpha pr\frac{n+1}{2n}\right)+\frac{b\lambda}{2}(1-K)T+s(1-\lambda)\right]+hD\left[\frac{K^2T}{4}-\frac{(1+m)K}{2}\right]. \tag{B1}$$

Taking the first-order and second-order derivatives of $f_3(K)$ with respect to K, we derive:

$$f_3'(K) = \left[\frac{D(1+m)}{1+m-KT}\right]\left[c\left(1+\alpha pr\frac{n+1}{2n}\right)+\frac{h}{2}(1+m)\right]$$
$$-D\left[c\lambda\left(1+\alpha pr\frac{n+1}{2n}\right)+b\lambda(1-K)T+s(1-\lambda)\right]+\frac{hD}{2}(KT-1-m)\,, \tag{B2}$$

and

$$f_3''(K) = \frac{DT(1+m)}{(1+m-KT)^2}\left[c\left(1+\alpha pr\frac{n+1}{2n}\right)+\frac{h}{2}(1+m)\right]+\frac{DhT}{2}+b\lambda DT > 0, \tag{B3}$$

respectively. Consequently, for any give T, $TC(K, T) = f_3(K)$ is a strictly convex function in K.

If $f_3'(0) = L_3 < 0$ and $f_3'(1) = L_4 \leq 0$, then $f_3'(K) < 0$ for all $K \in [0, 1]$ and $f_3(K)$is decreasing in K. Hence, $TC(K)$ in (18) is minimized at $K^* = 1$.

If $f_3'(0) = L_3 < 0$ and $f_3'(1) = L_4 > 0$, then we know from the mean value theorem that there exists a unique $K* \in (0, 1)$ such that $TC(K)$ in (18) is minimized.

Finally, if $L_3 \geq 0$, then $f_3'(K) > 0$ for all $K \in (0, 1]$ and $f_3(K)$is increasing in K. Thus, $TC(K)$ in (18) is minimized at $K^* = 0$. This completes the proof of Theorem 3.

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