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On Normal Approximation of χ^2 Distribution

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Abstract

According to the central limit theorem, if $X_1, X_2, \ldots X_\upsilon$ is a random sample drawn from $\chi^2(1)$, then, when $\upsilon \to \infty$, the distribution function of the sample mean $\overline{X} = \frac{\sum_{i=1}^{\upsilon} X_i}{\upsilon}$ would asymptotically approximate to $N\left(1, \frac{2}{\upsilon}\right)$, or the distribution function of $\frac{\overline{X} - 1}{\sqrt{\frac{2}{\upsilon}}}$ would approximate to the standard normal distribution $N(0, 1)$. Also, the distribution function of $\sum_{i=1}^{\upsilon} X_i$ would asymptotically approximate to the normal distribution $N(\upsilon, 2\upsilon)$. Many statistics textbooks or applied statistics research accept the use of a sample size of $\upsilon \geq 30$ for the assumption of $N(\upsilon, 2\upsilon)$ approximating to $\overline{X_\upsilon}$. Therefore, in the present study, computer simulation was adopted to test the required sample size υ for the normal distribution to approximate to the χ^2 distribution. This information is useful for the applications of the central limit theorem.

***Key Words*:** Computer Simulation, The Central Limit Theorem, χ^2 Distribution, Normal Distribution

1. Introduction

The normal distribution has a symmetric, unimodal, bell-shaped curve, and it is frequently applied to describe social, natural, and industrial phenomena as well as findings from academic research. For example, data from meteorological experiments, precipitation studies, and component manufacturing measurements are often analyzed and interpreted using the normal distribution. In addition, the normal distribution is also suitable for explaining errors in scientific measurements. In a sense, the normal distribution is the most important type of probability distributions in statistics. In real life, there are assorted of probability distributions, such as unimodal vs. multimodal distributions, symmetrical vs. asymmetrical distributions, high vs. low skewness distributions, and a non-skewed, nonmodal, and no-tail uniform distribution. Some of these distributions have a pattern similar to that of the normal distribution, while others may have a pattern quite distinctive from that of the normal distribution. Take the χ^2 distribution, such as the gamma distribution, derived from the normal distribution as an example, its kurtosis and skewness change according to the degrees of freedom, and this distribution is commonly used for making statistical inferences and in various statistical applications.

Let X be a continuous random variable, and the probability density function be

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-u)^2}{2\sigma^2}}, \ -\infty < x < \infty, \ \sigma > 0 \tag{1}$$

In this case, the $f(x)$ is called a normal distributed or a normal probability density function [1,2]. It can be

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found from the normal distribution that the mean u and the variance σ^2 affect the normal distribution curve; the mean u determines the position of the normal distribution, while the variance σ^2 determines the degrees of dispersion of the normal distribution. Therefore, a normal distribution would vary according to the mean u and the variance σ^2 these two important parameters. In general, the mean u and the variance σ^2 of a normal distribution alter the shape of the curve, and to make the curve of a normal distribution more consistent, $u = 0$ and $\sigma^2 = 1$ are adopted to form the standard normal distribution [3]. The standard normal distribution reference table is convenient and useful in data analysis.

Let the density function of the continuous random variable X be

$$f(x) = \frac{1}{\Gamma(\upsilon/2)2^{\frac{\upsilon}{2}}} x^{\upsilon/2-1} e^{-x/2}, \quad x \geq 0 \tag{2}$$

The continuous random variable X in this case has a χ^2 distribution with degrees of freedom of υ [4], and denoted by $X \sim \chi^2(\upsilon)$. If there are variables $X_1 \sim \chi^2(\upsilon_1)$ and $X_2 \sim \chi^2(\upsilon_2)$ that are mutually independent, then $X_1 + X_2 \sim \chi^2(\upsilon_1 + \upsilon_2)$. In other words, the sum of the two variables X_1 and X_2 has a χ^2 distribution with degrees of freedom of $(\upsilon_1 + \upsilon_2)$. Therefore, if $X_i \sim \chi^2(1)$, then $X = X_1 + X_2 + \ldots X_\upsilon \sim \chi^2(\upsilon)$. A χ^2 distribution has a mean $E(X) = \upsilon$ and avariance $V(X) = 2\upsilon$; in other words, for χ^2 random variables with degrees of freedom υ, the mean happens to be the degrees of freedom υ, and the variance is twice the degrees of freedom. Degrees of freedom υ reflects the skewness of the probability density function of the χ^2 distributions. See Figure 1. The smaller the degrees of freedom υ is, the more right-skewed the probability density function curve of the χ^2 distribution is, where as the larger the degrees of freedom υ is, the more symmetric the probability density function curve of the χ^2 distribution is.

According to the central limit theorem, if X_1, X_2, $\ldots X_\upsilon$ is a random sample drawn from $\chi^2(1)$, then, when $\upsilon \to \infty$, the distribution function of the sample mean $\overline{X} = \frac{\sum_{i=1}^{\upsilon} X_i}{\upsilon}$ would asymptotically approximate to $N(1, 2/\upsilon)$, or the distribution function of $\frac{\overline{X} - 1}{\sqrt{\frac{2}{\upsilon}}}$ would approximate to the standard normal distribution $N(0, 1)$. Also, the distribution function of $\sum_{i=1}^{\upsilon} X_i$ would asymptotically approximate to the normal distribution $N(\upsilon, 2\upsilon)$, or the distribution function of $\frac{\sum_{i=1}^{\upsilon} X_i - \upsilon}{\sqrt{2\upsilon}}$ would approximate to the standard normal distribution $N(0, 1)$. Furthermore, since $X_1, X_2, \ldots X_\upsilon$ is a random sample drawn from $\chi^2(1)$, we have $\sum_{i=1}^{\upsilon} x_i \sim \chi^2(\upsilon)$.

Many statistics textbooks or applied statistics research [5–13] accept the use of a sample size of $\upsilon \geq 30$ for the assumption of $N(\upsilon, 2\upsilon)$ approximating to $\overline{X_\upsilon}$. Nonetheless, Chang et al. showed in using central limit theorem for weibull and gamma distribution [14, 15]. The sample size of $n \geq 30$ was not larger enough. So the random sampling distribution of sample means could not be approximated to the normal distribution. Chang et al. confirmed that when employing the central limit theorem, the sample size should vary depending on the probability distribution type. Therefore, in the present study, computer simulation was adopted to test the required sample size υ for the normal distribution to approximate to the χ^2 distribution. This infor-

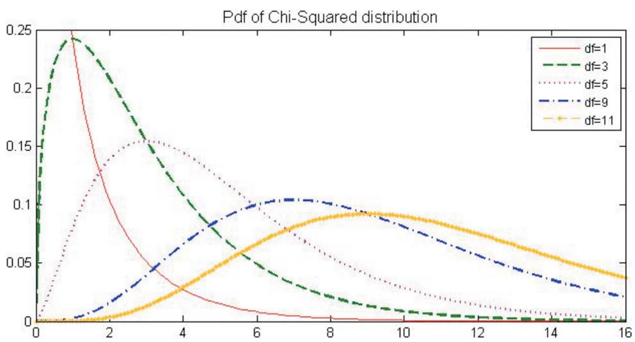

Figure 1. The probability density function curve of the χ^2 distribution with a degrees of freedom df = 1, 3, 5, 9, 11.

mation is useful for the applications of the central limit theorem.

2. The Sample Size of χ^2 Distribution Based on Central Limit Theorem

2.1 Statistical Test and Computer Simulation

The study used the built-in NORM.S.INV function of Excel statistical software for simulating sampling from a χ^2 distribution random sample. With degrees of freedom of υ, a sample of $n = 200$ was obtained, and that gave a sample mean of $\bar{X}_1$. The procedure was repeated 200 times to give a new sample set containing 200 sample means $\bar{X}_1, \bar{X}_2, ..., \bar{X}_{200}$. Next, the W-test of Shapiro and Wilk was employed (with a significance level less than 0.05) to test if the 200 sample means were normally distributed. For each degrees of freedom where $\upsilon = 2, \ldots, 400$, will produce 200 test results, i.e., accepting or rejecting normal distribution, were generated. If a test result rejects the normal distribution assumption, then it is treated as a "success". The above test was performed 200 times for each degrees of freedom υ, and 200 Bernoulli trial results were obtained. The number of "success" (m) was recorded and there was a success ratio of $m' = \frac{m}{200}$. This information indicated whether the normal distribution should be accepted or rejected. In the end, a total of 641,592,000,000 [= 200 × (2 + 3 + 4 + … + 400) × 200 × 200] random numbers were generated, and 79,800 (= 399 × 200) normality tests were performed.

The Shapiro-Wilk W-test proposed by [16] was used for normality testing, and the definition is presented below:

$$W = \left\{ \sum_{i=1}^{h} a_{in} (x_{n-i+1} - x_{(i)}) \right\}^2 \Big/ \sum_{i=1}^{n} (x_i - \bar{x})^2, x_{(1)} \leq .... \leq x_{(n)} \quad (3)$$

When n is an even number, $h = \frac{1}{2} n$, and if n is an odd number, then $h = \frac{1}{2}(n-1)$. Shaoiro and Wilk also provided a cross-reference table for the parameter a_{in}. Compared to other normality tests, the Shapiro-Wilk W-test is more sensitive; it works for small sample sizes ($n < 20$) or if there are outliers [17]. Pearson et al. [18] also mentioned that among various normality tests, the Shapiro-Wilk W-test remains very sensitive even with skewness, and they also considered the Shapiro-Wilk W-test the most robust normality test. Therefore, the study used the Shapiro-Wilk W-test statistic to be the normality test statistic for testing the sampling distribution of sample means. Using the above simulation method and statistical tests, the authors performed computer simulation using degrees of freedom $\upsilon = 2, 3, 4, 5, \ldots 350$. The results are presented in Table 1.

2.2 Simulation Results

Table 1 shows that with a significance level less than 0.05, the ratio of the number of times rejecting the normal assumption (m') was 0.1 for the χ^2 distribution with degrees of freedom of 2, and moreover, it was found that as the degrees of freedom υ increased, m', the ratio of the number of times rejecting the normality assumption became smaller. This finding demonstrated that the approximation of the normal distribution to the χ^2 distribution is more acceptable when the degrees of freedom is greater than 2. When the degrees of freedom $\upsilon = 30$, the ratio of the number of times rejecting the normality assumption was 0.055. Many general statistics textbooks or applied theses accept that when $\upsilon \geq 30$, the normal distribution can replace the χ^2 distribution. Theoretically, when the degrees of freedom $\upsilon > 30$, the ratio of the number of times rejecting the normality assumption should be less than 0.055. Nevertheless, from Table 1, it can be found that for $\upsilon > 30$, it was still frequent to get a ratios of the number of times rejecting the normality assumption (m') greater than 0.055. Apparently treating the χ^2 distribution as the normal distribution when $\upsilon \geq 30$ is too lenient. Next, the W-test result of each degrees of freedom of the χ^2 distribution was plotted into a line graph. See Figure 2. The x-axis is the degrees of freedom, while the y-axis is the number of times rejecting the normality assumption. It can be found from Figure 2 that as the degrees of freedom of the χ^2 distribution increased from 11 to 400, the ratio of the number of times rejecting the normality assumption (m′) decreased slowly, and most of the m' obtained were between 0.04 and 0.08.

Table 1. W test results of χ^2 distribution as υ varies (υ = 2, 3, 4, …, 400; ratio m' in the table are reject frequency of repeating 200 W tests)

υ	m'	υ	m'	υ	m'	υ	m'	υ	m'
2	0.10	52	0.060	102	0.060	152	0.055	202	0.060
3	0.095	53	0.070	103	0.065	153	0.050	203	0.050
4	0.095	54	0.060	104	0.055	154	0.055	204	0.055
5	0.090	55	0.075	105	0.050	155	0.060	205	0.055
6	0.090	56	0.055	106	0.055	156	0.075	206	0.065
7	0.085	57	0.065	107	0.070	157	0.065	207	0.045
8	0.080	58	0.070	108	0.060	158	0.060	208	0.070
9	0.075	59	0.065	109	0.055	159	0.050	209	0.060
10	0.080	60	0.050	110	0.055	160	0.055	210	0.050
11	0.085	61	0.055	111	0.050	161	0.050	211	0.060
12	0.080	62	0.070	112	0.045	162	0.050	212	0.065
13	0.080	63	0.050	113	0.070	163	0.050	213	0.055
14	0.075	64	0.070	114	0.065	164	0.055	214	0.050
15	0.070	65	0.050	115	0.065	165	0.060	215	0.050
16	0.075	66	0.070	116	0.065	166	0.060	216	0.055
17	0.065	67	0.055	117	0.070	167	0.060	217	0.045
18	0.075	68	0.050	118	0.050	168	0.050	218	0.050
19	0.060	69	0.060	119	0.065	169	0.050	219	0.065
20	0.070	70	0.060	120	0.065	170	0.060	220	0.050
21	0.070	71	0.050	121	0.055	171	0.055	221	0.065
22	0.060	72	0.060	122	0.065	172	0.060	222	0.050
23	0.065	73	0.060	123	0.055	173	0.050	223	0.055
24	0.070	74	0.050	124	0.050	174	0.070	224	0.045
25	0.065	75	0.075	125	0.045	175	0.060	225	0.060
26	0.055	76	0.070	126	0.060	176	0.055	226	0.060
27	0.070	77	0.065	127	0.055	177	0.055	227	0.050
28	0.070	78	0.060	128	0.065	178	0.045	228	0.065
29	0.070	79	0.060	129	0.050	179	0.050	229	0.045
30	0.055	80	0.060	130	0.055	180	0.055	230	0.050
31	0.070	81	0.075	131	0.050	181	0.045	231	0.040
32	0.075	82	0.070	132	0.050	182	0.060	232	0.055
33	0.080	83	0.060	133	0.050	183	0.045	233	0.045
34	0.060	84	0.070	134	0.060	184	0.050	234	0.045
35	0.060	85	0.060	135	0.050	185	0.045	235	0.040
36	0.065	86	0.060	136	0.070	186	0.050	236	0.060
37	0.055	87	0.055	137	0.050	187	0.060	237	0.060
38	0.045	88	0.060	138	0.055	188	0.045	238	0.050
39	0.050	89	0.070	139	0.065	189	0.050	239	0.045
40	0.050	90	0.075	140	0.055	190	0.050	240	0.050
41	0.075	91	0.075	141	0.050	191	0.055	241	0.060
42	0.060	92	0.060	142	0.060	192	0.055	242	0.040
43	0.065	93	0.055	143	0.075	193	0.050	243	0.040
44	0.070	94	0.055	144	0.065	194	0.050	244	0.055
45	0.045	95	0.055	145	0.060	195	0.065	245	0.050
46	0.070	96	0.055	146	0.060	196	0.045	246	0.050
47	0.070	97	0.060	147	0.060	197	0.070	247	0.055
48	0.070	98	0.055	148	0.060	198	0.050	248	0.050
49	0.065	99	0.060	149	0.060	199	0.060	249	0.055
50	0.060	100	0.075	150	0.060	200	0.050	250	0.045
51	0.050	101	0.070	151	0.065	201	0.040	251	0.045

Table 1. Continued

υ	m'	υ	m'	υ	m'	υ	m'	υ	m'
252	0.050	282	0.050	312	0.050	342	0.040	372	0.040
253	0.050	283	0.045	313	0.045	343	0.035	373	0.040
254	0.040	284	0.045	314	0.030	344	0.040	374	0.045
255	0.060	285	0.045	315	0.040	345	0.040	375	0.025
256	0.045	286	0.045	316	0.050	346	0.030	376	0.030
257	0.050	287	0.040	317	0.050	347	0.040	377	0.040
258	0.050	288	0.045	318	0.045	348	0.045	378	0.040
259	0.050	289	0.045	319	0.050	349	0.045	379	0.045
260	0.050	290	0.050	320	0.045	350	0.045	380	0.040
261	0.045	291	0.045	321	0.045	351	0.040	381	0.030
262	0.050	292	0.045	322	0.050	352	0.055	382	0.035
263	0.040	293	0.040	323	0.040	353	0.050	383	0.035
264	0.040	294	0.045	324	0.050	354	0.050	384	0.035
265	0.035	295	0.045	325	0.055	355	0.050	385	0.025
266	0.040	296	0.045	326	0.040	356	0.040	386	0.020
267	0.035	297	0.045	327	0.040	357	0.035	387	0.030
268	0.050	298	0.030	328	0.050	358	0.040	388	0.040
269	0.050	299	0.040	329	0.045	359	0.040	389	0.040
270	0.050	300	0.050	330	0.050	360	0.045	390	0.040
271	0.040	301	0.045	331	0.045	361	0.050	391	0.045
272	0.055	302	0.045	332	0.045	362	0.060	392	0.045
273	0.050	303	0.045	333	0.020	363	0.050	393	0.040
274	0.045	304	0.055	334	0.040	364	0.025	394	0.050
275	0.045	305	0.050	335	0.040	365	0.040	395	0.035
276	0.050	306	0.035	336	0.040	366	0.045	396	0.030
277	0.045	307	0.050	337	0.045	367	0.055	397	0.040
278	0.045	308	0.040	338	0.045	368	0.050	398	0.030
279	0.045	309	0.040	339	0.025	369	0.045	399	0.035
280	0.040	310	0.045	340	0.045	370	0.045	400	0.035
281	0.045	311	0.045	341	0.040	371	0.040		

3. Speed of Cutoff Value of Normal Distribution Approximating to Standard Normal Cutoff Value of χ^2 Distribution

3.1 Normal Approximation to χ^2 Distribution

According to the central limit theorem, when the degrees of freedom approaches infinity ($\upsilon \to \infty$), the normal distribution can approximate the χ^2 distribution [19], i.e., $N(\upsilon, 2\upsilon)$ approximates to $\chi^2(\upsilon)$. In other words, if $X \sim \chi^2(\upsilon)$, $E(X) = \upsilon$, and $Var(X) = 2\upsilon$, then $E(\overline{X}_\upsilon) = \upsilon$ and $Var(\overline{X}_\upsilon) = \frac{2\upsilon}{\upsilon}$. If $E(X_i) = 1$ and $Var(X_i) = 2$, then $E(\overline{X}_\upsilon) = 1$ and $Var(\overline{X}_\upsilon) = \frac{2}{\upsilon}$. Furthermore, we also have

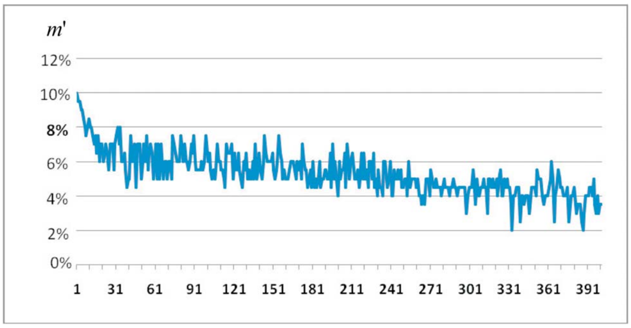

Figure 2. Relation between the degrees of freedom υ and number of times rejecting normality test m' under χ^2 distribution.

$$\overline{X}_\upsilon = \frac{X_1 + X_2 + \ldots\ldots + X_\upsilon}{\upsilon} \sim N\left(1, \frac{2}{\upsilon}\right) \tag{4}$$

and

$$Z=\frac{\overline{X}_{\upsilon}-1}{\sqrt{2/\upsilon}}\sim N(0,1) \quad (5)$$

Substituting (4) into (5), one obtains

$$\frac{(X_1+X_2+.........X_{\upsilon}/\upsilon)-1}{\sqrt{2/\upsilon}}=Z$$

where $X_1+X_2+\ldots+X_{\upsilon}\sim\chi^2(\upsilon)$. In addition to that, by the relation $\frac{\chi^2(\upsilon)-\upsilon}{\sqrt{2\upsilon}}=Z$, it implies

$$\chi^2(\upsilon)=Z\sqrt{2\upsilon}+\upsilon \quad (6)$$

Since

$$X\sim\chi^2(\upsilon) \text{ and } P(X>x)=\alpha \quad (7)$$

this gives

$$x=\chi^2_{\alpha}(\upsilon) \quad (8)$$

Combining (6) to (8), one has

$$P\left[X>\chi^2_{a}(\upsilon)\right]=\alpha\Rightarrow P\left[Z\sqrt{2\upsilon}+\upsilon>\chi^2_{\alpha}(\upsilon)\right]=\alpha \quad (9)$$

Rearranging (9),

$$P\left(Z>\frac{\chi^2_{\alpha}(\upsilon)-\upsilon}{\sqrt{2\upsilon}}\right)\Rightarrow\frac{\chi^2_{\alpha}(\upsilon)-\upsilon}{\sqrt{2\upsilon}}\approx z_{\alpha}$$

we obtain

$$\chi^2_{\alpha}(\upsilon)\approx z_{\alpha}\sqrt{2\upsilon}+\upsilon \quad (10)$$

3.2 Simulation Results

The study then used the built-in $\chi^2(\upsilon)$ of the χ^2 distribution with degrees of freedom of υ provided by the Excel statistics software, and the cutoff value $\chi^2_{\alpha}(\upsilon)$ is defined as $P(\chi^2>\chi^2_{\alpha}(\upsilon))=\alpha$ minus the mean degrees of freedom υ of $\chi^2(\upsilon)$ and then divided by the standard deviation $\sqrt{2\upsilon}$. In other words, it is to have $\frac{\chi^2_{\alpha}(\upsilon)-\upsilon}{\sqrt{2\upsilon}}$ undergo standard normalization. It can be found in Table 2

Table 2. $\tilde{\chi}^2_{a}(\upsilon)=\frac{\chi^2_{a}(\upsilon)-\upsilon}{\sqrt{2\upsilon}}$

υ	$\tilde{\chi}^2_{0.1}(\upsilon)$	$\tilde{\chi}^2_{0.05}(\upsilon)$	$\tilde{\chi}^2_{0.025}(\upsilon)$	$\tilde{\chi}^2_{0.01}(\upsilon)$	$\tilde{\chi}^2_{0.005}(\upsilon)$
1	1.206	2.009	2.845	3.984	4.864
2	1.303	1.996	2.689	3.605	4.298
3	1.327	1.966	2.592	3.407	4.016
4	1.336	1.940	2.526	3.280	3.840
5	1.340	1.920	2.477	3.190	3.716
6	1.341	1.903	2.439	3.121	3.622
7	1.341	1.889	2.409	3.067	3.549
8	1.340	1.877	2.384	3.023	3.489
9	1.340	1.867	2.362	2.985	3.439
10	1.339	1.858	2.344	2.954	3.396
11	1.338	1.850	2.328	2.926	3.359
12	1.337	1.842	2.314	2.902	3.327
13	1.336	1.836	2.302	2.881	3.299
14	1.335	1.830	2.290	2.861	3.273
15	1.334	1.825	2.280	2.844	3.250
16	1.333	1.820	2.271	2.828	3.229
17	1.332	1.816	2.262	2.814	3.210
18	1.332	1.812	2.254	2.801	3.193
19	1.331	1.808	2.247	2.789	3.177
20	1.330	1.804	2.240	2.777	3.162
21	1.329	1.801	2.234	2.767	3.148
22	1.329	1.798	2.228	2.757	3.135
23	1.328	1.795	2.223	2.748	3.123
24	1.327	1.792	2.218	2.739	3.112
25	1.327	1.789	2.213	2.731	3.101
26	1.326	1.787	2.208	2.724	3.091
27	1.326	1.784	2.204	2.717	3.082
28	1.325	1.782	2.200	2.710	3.073
29	1.325	1.780	2.196	2.703	3.064
30	1.324	1.778	2.192	2.697	3.056
31	1.324	1.776	2.188	2.691	3.048
32	1.323	1.774	2.185	2.686	3.041
33	1.323	1.773	2.182	2.680	3.034
34	1.322	1.771	2.179	2.675	3.027
35	1.322	1.769	2.176	2.670	3.021
36	1.321	1.768	2.173	2.666	3.015
37	1.321	1.766	2.170	2.661	3.009
38	1.321	1.765	2.167	2.657	3.003
39	1.320	1.763	2.165	2.653	2.998
40	1.320	1.762	2.162	2.649	2.993
41	1.319	1.761	2.160	2.645	2.987
42	1.319	1.759	2.158	2.641	2.983
43	1.319	1.758	2.156	2.638	2.978
44	1.318	1.757	2.153	2.634	2.973
45	1.318	1.756	2.151	2.631	2.969
46	1.318	1.755	2.149	2.627	2.965
47	1.318	1.754	2.147	2.624	2.961
48	1.317	1.752	2.146	2.621	2.957

Table 2. Continued

υ	$\tilde{\chi}^2_{0.1}(\upsilon)$	$\tilde{\chi}^2_{0.05}(\upsilon)$	$\tilde{\chi}^2_{0.025}(\upsilon)$	$\tilde{\chi}^2_{0.01}(\upsilon)$	$\tilde{\chi}^2_{0.005}(\upsilon)$
49	1.317	1.751	2.144	2.618	2.953
50	1.317	1.750	2.142	2.615	2.949
51	1.316	1.750	2.140	2.613	2.945
52	1.316	1.749	2.139	2.610	2.942
53	1.316	1.748	2.137	2.607	2.938
54	1.316	1.747	2.135	2.605	2.935
55	1.315	1.746	2.134	2.602	2.932
56	1.315	1.745	2.132	2.600	2.929
57	1.315	1.744	2.131	2.597	2.926
58	1.315	1.743	2.130	2.595	2.923
59	1.314	1.743	2.128	2.593	2.920
60	1.314	1.742	2.127	2.591	2.917
61	1.314	1.741	2.125	2.589	2.914
62	1.314	1.740	2.124	2.586	2.911
63	1.314	1.740	2.123	2.584	2.909
64	1.313	1.739	2.122	2.582	2.906
65	1.313	1.738	2.120	2.580	2.904
66	1.313	1.738	2.119	2.579	2.901
67	1.313	1.737	2.118	2.577	2.899
68	1.313	1.736	2.117	2.575	2.896
69	1.312	1.736	2.116	2.573	2.894
70	1.312	1.735	2.115	2.571	2.892
71	1.312	1.735	2.114	2.570	2.889
72	1.312	1.734	2.113	2.568	2.887
73	1.312	1.733	2.112	2.566	2.885
74	1.312	1.733	2.111	2.565	2.883
75	1.311	1.732	2.110	2.563	2.881
76	1.311	1.732	2.109	2.562	2.879
77	1.311	1.731	2.108	2.560	2.877
78	1.311	1.731	2.107	2.559	2.875
79	1.311	1.730	2.106	2.557	2.873
80	1.311	1.730	2.105	2.556	2.871
81	1.310	1.729	2.104	2.554	2.870
82	1.310	1.729	2.103	2.553	2.868
83	1.310	1.728	2.103	2.552	2.866
84	1.310	1.728	2.102	2.550	2.864
85	1.310	1.727	2.101	2.549	2.863
86	1.310	1.727	2.100	2.548	2.861
87	1.310	1.726	2.099	2.547	2.859
88	1.309	1.726	2.099	2.545	2.858
89	1.309	1.726	2.098	2.544	2.856
90	1.309	1.725	2.097	2.543	2.855
91	1.309	1.725	2.096	2.542	2.853
92	1.309	1.724	2.096	2.541	2.852
93	1.309	1.724	2.095	2.539	2.850
94	1.309	1.724	2.094	2.538	2.849
95	1.309	1.723	2.094	2.537	2.847
96	1.308	1.723	2.093	2.536	2.846
97	1.308	1.722	2.092	2.535	2.844
98	1.308	1.722	2.092	2.534	2.843

Table 2. Continued

υ	$\tilde{\chi}^2_{0.1}(\upsilon)$	$\tilde{\chi}^2_{0.05}(\upsilon)$	$\tilde{\chi}^2_{0.025}(\upsilon)$	$\tilde{\chi}^2_{0.01}(\upsilon)$	$\tilde{\chi}^2_{0.005}(\upsilon)$
99	1.308	1.722	2.091	2.533	2.842
100	1.308	1.721	2.090	2.532	2.840
110	1.307	1.718	2.084	2.522	2.828
120	1.306	1.715	2.079	2.514	2.817
130	1.305	1.712	2.075	2.507	2.808
140	1.304	1.710	2.071	2.500	2.800
150	1.304	1.708	2.067	2.495	2.792
165	1.303	1.705	2.062	2.487	2.782
170	1.303	1.704	2.061	2.484	2.779
180	1.302	1.703	2.058	2.480	2.773
190	1.302	1.701	2.055	2.476	2.768
200	1.301	1.700	2.053	2.472	2.763
210	1.301	1.698	2.051	2.469	2.759
220	1.300	1.697	2.049	2.466	2.755
230	1.300	1.696	2.047	2.462	2.751
240	1.300	1.695	2.045	2.460	2.747
250	1.299	1.694	2.043	2.457	2.743
260	1.299	1.693	2.042	2.454	2.740
270	1.299	1.692	2.040	2.452	2.737
280	1.298	1.691	2.039	2.450	2.734
290	1.298	1.691	2.037	2.448	2.732
300	1.298	1.690	2.036	2.446	2.729
310	1.298	1.689	2.035	2.444	2.726
320	1.297	1.689	2.034	2.442	2.724
330	1.297	1.688	2.033	2.440	2.722
340	1.297	1.687	2.032	2.438	2.720
350	1.297	1.687	2.031	2.437	2.718
360	1.296	1.686	2.030	2.435	2.716
370	1.296	1.686	2.029	2.434	2.714
380	1.296	1.685	2.028	2.432	2.712
390	1.296	1.685	2.027	2.431	2.710
400	1.296	1.684	2.026	2.430	2.708
410	1.296	1.684	2.025	2.429	2.707
420	1.295	1.683	2.024	2.427	2.705
430	1.295	1.683	2.024	2.426	2.704
440	1.295	1.682	2.023	2.425	2.702
450	1.295	1.682	2.022	2.424	2.701
460	1.295	1.681	2.022	2.423	2.700
470	1.295	1.681	2.021	2.422	2.698
480	1.295	1.681	2.020	2.421	2.697
490	1.294	1.680	2.020	2.420	2.696
500	1.294	1.680	2.019	2.419	2.694
680	1.293	1.675	2.011	2.406	2.678
700	1.292	1.675	2.010	2.405	2.676
800	1.292	1.673	2.007	2.400	2.670
890	1.291	1.671	2.004	2.396	2.665
1000	1.291	1.670	2.002	2.392	2.660
1100	1.290	1.669	2.000	2.389	2.656
1110	1.290	1.669	2.000	2.389	2.655

that with $\alpha \in \{0.1, 0.05, 0.025, 0.01, 0.005\}$ and $\upsilon = 1, 2, 3, 4, 5 \ldots 1110$, the smaller the degrees of freedom, the more asymmetric the distribution pattern, and similarly, the greater the degrees of freedom, the more symmetric the pattern. Moreover, the greater the α, the smaller the standard normal value of the χ^2 distribution. As the degrees of freedom increased, the standard normal value of the χ^2 distribution decreased progressively. Nevertheless, only when $\alpha = 0.1$ did the degrees of freedom start to decrease progressively. The error $D_a(\upsilon) = \left|\tilde{\chi}^2_\alpha(\upsilon) - z_a\right|$ of z_a and $\tilde{\chi}^2_\alpha(\upsilon) = \frac{\chi^2_a(\upsilon) - \upsilon}{\sqrt{2\upsilon}}$ were shown in Table 3. According to the above, when $\upsilon \to \infty$, $\chi^2(\upsilon)$ can be approximated by the normal distribution $N(\upsilon, 2\upsilon)$. General statistics textbooks and applied theses use $\upsilon \geq 30$ for the normal distribution to replace the χ^2 distribution, but it was clearly shown in Table 3 that the error was very big.

As shown in Table 3, when the degrees of freedom was 30, the smaller the α, the bigger the error. For $\chi^2_{0.1}(30)$, the error was 0.042. With $\chi^2_{0.05}(30)$, the error was 0.133. With $\chi^2_{0.025}(30)$, the error was 0.232. With $\chi^2_{0.01}(30)$, the error was 0.371. With $\chi^2_{0.005}(30)$, the error was 0.480. Therefore, treating the χ^2 distribution as normally distributed when $\upsilon \geq 30$ based on the central limit theorem is too lenient. It can be found from Excel that the maximum degrees of freedom is 1110, but even with degrees of freedom of 1110, there was still an error of 0.008 for $\chi^2_{0.1}(1110)$, and 0.024 for $\chi^2_{0.05}(1110)$, 0.04 for $\chi^2_{0.025}(1110)$, 0.063 for $\chi^2_{0.01}(1110)$, and 0.079 for $\chi^2_{0.005}(1110)$.

3.3 Discussion

It can be found from in Table 3, with $\alpha \in \{0.1, 0.05, 0.025, 0.01, 0.005\}$, the investigators determined the least degrees of freedom for errors of 0.08, 0.07, 0.06, 0.05 and 0.04. It can be found in Figure 3 that the greater the α, the smaller the error, and the smaller the α, the greater the error. Therefore, the largest error of $\chi^2_{0.1}$ was found to be close to 0.06, the smallest error of $\chi^2_{0.1}$ was found to be 0.06, and the smallest error of $\chi^2_{0.005}$ was even as high as 0.08.

To better observe the degrees of freedom required for the normal approximation to the χ^2 distribution, in-

Table 3. $D_a(\upsilon) = \left|\tilde{\chi}^2_\alpha(\upsilon) - z_a\right|$

υ	$D_{0.1}(\upsilon)$	$D_{0.05}(\upsilon)$	$D_{0.025}(\upsilon)$	$D_{0.01}(\upsilon)$	$D_{0.005}(\upsilon)$
1	0.076	0.364	0.885	1.658	2.288
2	0.021	0.0351	0.729	1.279	1.722
3	0.045	0.321	0.632	1.081	1.440
4	0.054	0.295	0.566	0.954	1.264
5	0.058	0.275	0.517	0.864	1.140
6	**0.059**	0.258	0.479	0.795	1.046
7	0.059	0.244	0.449	0.741	0.973
8	0.058	0.232	0.424	0.697	0.913
9	0.058	0.222	0.402	0.659	0.863
10	0.057	0.213	0.384	0.628	0.820
11	0.056	0.205	0.368	0.600	0.783
12	0.055	0.197	0.354	0.576	0.751
13	0.054	0.191	0.342	0.555	0.723
14	0.053	0.185	0.330	0.535	0.697
15	0.052	0.180	0.320	0.518	0.674
16	0.051	0.175	0.311	0.502	0.653
17	**0.050**	0.171	0.302	0.488	0.634
18	0.050	0.167	0.294	0.475	0.617
19	0.049	0.163	0.287	0.463	0.601
20	0.048	0.159	0.280	0.451	0.586
21	0.047	0.156	0.274	0.441	0.572
22	0.047	0.153	0.268	0.431	0.559
23	0.046	0.150	0.263	0.422	0.547
24	0.045	0.147	0.258	0.413	0.536
25	0.045	0.144	0.253	0.405	0.525
26	0.044	0.142	0.248	0.398	0.515
27	0.044	0.139	0.244	0.391	0.506
28	0.043	0.137	0.240	0.384	0.497
29	0.043	0.135	0.236	0.377	0.488
30	**0.042**	**0.133**	**0.232**	**0.371**	**0.480**
31	0.042	0.131	0.228	0.365	0.472
32	0.041	0.129	0.225	0.360	0.465
33	0.041	0.128	0.222	0.354	0.458
34	**0.040**	0.126	0.219	0.349	0.451
35	0.040	0.124	0.216	0.344	0.445
36	0.039	0.123	0.213	0.340	0.439
37	0.039	0.121	0.210	0.335	0.433
38	0.039	0.120	0.207	0.331	0.427
39	0.038	0.118	0.205	0.327	0.422
40	0.038	0.117	0.202	0.323	0.417
41	0.037	0.116	0.200	0.319	0.411
42	0.037	0.114	0.198	0.315	0.407
43	0.037	0.113	0.196	0.312	0.402
44	0.036	0.112	0.193	0.308	0.397
45	0.036	0.111	0.191	0.305	0.393
46	0.036	0.110	0.189	0.301	0.389
47	0.036	0.109	0.187	0.298	0.385
48	0.035	0.107	0.186	0.295	0.381
49	0.035	0.106	0.184	0.292	0.377

Table 3. Continued

υ	$D_{0.1}(\upsilon)$	$D_{0.05}(\upsilon)$	$D_{0.025}(\upsilon)$	$D_{0.01}(\upsilon)$	$D_{0.005}(\upsilon)$
50	0.035	0.105	0.182	0.289	0.373
51	0.034	0.105	0.180	0.287	0.369
52	0.034	0.104	0.179	0.284	0.366
53	0.034	0.103	0.177	0.281	0.362
54	0.034	0.102	0.175	0.279	0.359
55	0.033	0.101	0.174	0.276	0.356
56	0.033	0.100	0.172	0.274	0.353
57	0.033	0.099	0.171	0.271	0.350
58	0.033	0.098	0.170	0.269	0.347
59	0.032	0.098	0.168	0.267	0.344
60	0.032	0.097	0.167	0.265	0.341
61	0.032	0.096	0.165	0.263	0.338
62	0.032	0.095	0.164	0.260	0.335
63	0.032	0.095	0.163	0.258	0.333
64	0.031	0.094	0.162	0.256	0.330
65	0.031	0.093	0.160	0.254	0.328
66	0.031	0.093	0.159	0.253	0.325
67	0.031	0.092	0.158	0.251	0.323
68	0.031	0.091	0.157	0.249	0.320
69	0.030	0.091	0.156	0.247	0.318
70	0.030	0.090	0.155	0.245	0.316
71	0.030	0.090	0.154	0.244	0.313
72	0.030	0.089	0.153	0.242	0.311
73	0.030	0.088	0.152	0.240	0.309
74	0.030	0.088	0.151	0.239	0.307
75	0.029	0.087	0.150	0.237	0.305
76	0.029	0.087	0.149	0.236	0.303
77	0.029	0.086	0.148	0.234	0.301
78	0.029	0.086	0.147	0.233	0.299
79	0.029	0.085	0.146	0.231	0.297
80	0.029	0.085	0.145	0.230	0.295
81	0.028	0.084	0.144	0.228	0.294
82	0.028	0.084	0.143	0.227	0.292
83	0.028	0.083	0.143	0.226	0.290
84	0.028	0.083	0.142	0.224	0.288
85	0.028	0.082	0.141	0.223	0.287
86	0.028	0.082	0.140	0.222	0.285
87	0.028	0.081	0.139	0.221	0.283
88	0.027	0.081	0.139	0.219	0.282
89	0.027	0.081	0.138	0.218	0.280
90	0.027	**0.080**	0.137	0.217	0.279
91	0.027	0.080	0.136	0.216	0.277
92	0.027	0.079	0.136	0.215	0.276
93	0.027	0.079	0.135	0.213	0.274
94	0.027	0.079	0.134	0.212	0.273
95	0.027	0.078	0.134	0.211	0.271
96	0.026	0.078	0.133	0.210	0.270
97	0.026	0.077	0.132	0.209	0.268
98	0.026	0.077	0.132	0.208	0.267
99	0.026	0.077	0.131	0.207	0.266

Table 3. Continued

υ	$D_{0.1}(\upsilon)$	$D_{0.05}(\upsilon)$	$D_{0.025}(\upsilon)$	$D_{0.01}(\upsilon)$	$D_{0.005}(\upsilon)$
100	0.026	0.076	0.130	0.206	0.264
110	0.025	0.073	0.124	0.196	0.252
120	0.024	**0.070**	0.119	0.188	0.241
130	0.023	0.067	0.115	0.181	0.232
140	0.022	0.065	0.111	0.174	0.224
150	0.022	0.063	0.107	0.169	0.216
165	0.021	**0.060**	0.102	0.161	0.206
170	0.021	0.059	0.101	0.158	0.203
180	0.020	0.058	0.098	0.154	0.197
190	0.020	0.056	0.095	0.150	0.192
200	0.019	0.055	0.093	0.146	0.187
210	0.019	0.053	0.091	0.143	0.183
220	0.018	0.052	0.089	0.140	0.179
230	0.018	0.051	0.087	0.136	0.175
240	0.018	**0.050**	0.085	0.134	0.171
250	0.017	0.049	0.083	0.131	0.167
260	0.017	0.048	0.082	0.128	0.164
270	0.017	0.047	**0.080**	0.126	0.161
280	0.016	0.046	0.079	0.124	0.158
290	0.016	0.046	0.077	0.122	0.156
300	0.016	0.045	0.076	0.120	0.153
310	0.016	0.044	0.075	0.118	0.150
320	0.015	0.044	0.074	0.116	0.148
330	0.015	0.043	0.073	0.114	0.146
340	0.015	0.042	0.072	0.112	0.144
350	0.015	0.042	0.071	0.111	0.142
360	0.014	0.041	**0.070**	0.109	0.140
370	0.014	0.041	0.069	0.108	0.138
380	0.014	0.040	0.068	0.106	0.136
390	0.014	0.040	0.067	0.105	0.134
400	0.014	0.039	0.066	0.104	0.132
410	0.014	0.039	0.065	0.103	0.131
420	0.013	0.038	0.064	0.101	0.129
430	0.013	0.038	0.064	0.100	0.128
440	0.013	0.037	0.063	0.099	0.126
450	0.013	0.037	0.062	0.098	0.125
460	0.013	0.036	0.062	0.097	0.124
470	0.013	0.036	0.061	0.096	0.122
480	0.013	0.036	**0.060**	0.095	0.121
490	0.012	0.035	0.060	0.094	0.120
500	0.012	0.035	0.059	0.093	0.118
680	0.011	0.030	0.051	**0.080**	0.102
700	0.010	0.030	**0.050**	0.079	0.100
800	0.010	0.028	0.047	0.074	0.094
890	0.009	0.026	0.044	**0.070**	0.089
1000	0.009	0.025	0.042	0.066	0.084
1100	0.008	0.024	**0.040**	**0.063**	**0.080**
1110	0.008	0.024	0.040	0.063	0.079

formation in Figure 3 was expressed in Table 4. It can be found that it is more appropriate to apply the central limit theory with a bigger α because the greater the α, the smaller the degrees of freedom required for the normal approximation. The least degrees of freedom required for an error less than 0.08 is $\upsilon = 90$ for $\chi^2_{0.05}$, $\upsilon = 270$ for $\chi^2_{0.025}$, $\upsilon = 680$ for $\chi^2_{0.01}$, and $\upsilon = 1100$ for $\chi^2_{0.005}$. The least degrees of freedom required for an error less than 0.07 is $\upsilon = 120$ for $\chi^2_{0.05}$, $\upsilon = 360$ for $\chi^2_{0.025}$, and $\upsilon = 890$ for $\chi^2_{0.01}$. The least degrees of freedom required for an error less than 0.06 is $\upsilon = 6$ for $\chi^2_{0.1}$, $\upsilon = 165$ for $\chi^2_{0.05}$, $\upsilon = 480$ for $\chi^2_{0.025}$, and $\upsilon = 1100$ for $\chi^2_{0.01}$. The least degrees of freedom required for an error less than 0.05 is $\upsilon = 17$ for $\chi^2_{0.1}$, $\upsilon = 240$ for $\chi^2_{0.05}$, and $\upsilon = 700$ for $\chi^2_{0.025}$. The least degrees of freedom required for an error less than 0.04 is $\upsilon = 34$ for $\chi^2_{0.1}$, $\upsilon = 380$ for $\chi^2_{0.05}$, and $\upsilon = 1100$ for $\chi^2_{0.025}$.

In order to test the accuracy of the least degrees of freedom required for the normal approximation to the χ^2 distribution in Table 4, the investigators examined the relation among the standard normal value of the χ^2 distribution, and the error $D_a(\upsilon)$ and degrees of freedom υ of

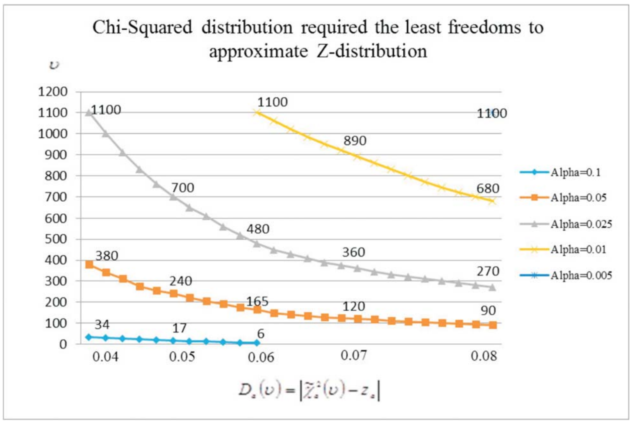

Figure 3. The least degrees of freedom required for χ^2 distribution for errors ranged between 0.04 and 0.08.

Table 4. The least degrees of freedom required for normal approximation to χ^2 distribution

$D_a(\upsilon)$ \ υ α	0.1	0.05	0.025	0.01	0.005
0.08		90	270	680	1100
0.07		120	360	890	
0.06		165	480	1100	
0.05	17	240	700		
0.04	34	380	1100		

z_a. The results in Table 3 showed a tendency between $D_a(\upsilon)$ and the degrees of freedom υ. The study then used the inverse regression model $D_\alpha(\upsilon) = b_0 + b_1 / \upsilon + \varepsilon$ to demonstrate this tendency. With $\alpha \in \{0.1, 0.05, 0.025, 0.01, 0.005\}$, the error $D_a(\upsilon)$ was set to be 0.08, 0.07, 0.06, 0.05 and 0.04, and the coefficient of determination and the p-value of the regression model were listed in Table 5. It can be found from Table 5 that there was a significant association between $D_a(\upsilon)$ and the degrees of freedom υ. Moreover, the coefficient of determination R^2 was between 0.882 and 1.000. Overall, the explanatory power of the least degrees of freedom required by the normal approximation to the χ^2 distribution was excellent.

4. Conclusions

The second section examined whether the normal approximation can be applied to the χ^2 distribution with degrees of freedom of υ. The investigators tested if the sample mean of a random sample of a size of υ randomly drawn from $\chi^2(1)$ can be approximated to the normal distribution. An approximated value m' of the probability of type I error was obtained from computer simulation. It was found that m' decreased slowly (mostly between 0.04 and 0.08) with an increase in the degrees of freedom of the χ^2 distribution. It was also observed that the speed of the normal approximation to the χ^2 distribution was fast when the degrees of freedom was greater than 2. Although the speed of approximation was high, it was nev-

Table 5. Regression model: coefficient of determination R^2 and P-value

$D_a(\upsilon)$ \ R^2 (P) α	0.1	0.05	0.025	0.01	0.005
0.08		0.977 (0.000)	0.989 (0.000)	0.999 (0.000)	1.000 (0.000)
0.07		0.974 (0.000)	0.996 (0.000)	0.999 (0.001)	
0.06		0.979 (0.000)	0.996 (0.000)	1.000 (0.000)	
0.05	0.882 (0.000)	0.985 (0.000)	0.997 (0.000)		
0.04	0.940 (0.000)	0.996 (0.000)	0.998 (0.000)		

ertheless still not acceptable. In section 3, the standard normal value of the χ^2 distribution and the cutoff value of the standard normal distribution were compared, and it was found that an increase in the degrees of freedom of the χ^2 distribution was associated with a smaller difference between the standard normal value of the χ^2 distribution and the cutoff value of the standard normal distribution. The above result was similar to the result presented in section 2.

It was also observed that the computer simulation results in Table 1 (section 2) and the errors $D_a(\upsilon) = \left|\tilde{\chi}^2_\alpha(\upsilon) - z_a\right|$ in Table 3 (section 3) were consistent. For example, the values m' in Table 1 were mostly between 0.04 and 0.08, and the main errors in Table 3 were also between 0.04 and 0.08. Next, it can be found in Figure 3 that when $\alpha = 0.05$, the least degrees of freedom required for errors of 0.08, 0.07, 0.06, 0.05, and 0.04 were 90, 120, 165, 240, and 380 respectively. Compared with the degrees of freedom of 90, 120, 165, 240, and 380 in Table 1, the ratios of the number of times rejecting the normality assumption were 0.075, 0.065, 0.06, 0.05, and 0.04 respectively, indicating a high consistency between the two, i.e., m' in Table 1 and errors in Table 3, in testing the normal approximation to the χ^2 distribution.

Taken together, the computer simulation results in section 2 showed that most ratios of the number of times rejecting the normal assumption of the degrees of freedom greater than 30 ($\upsilon > 30$) were greater than the ratio of the number of time rejecting the normal assumption of $\upsilon = 30$. When applying the central limit theorem, apparently, the use of $\upsilon \geq 30$, preferred by general statistics textbooks or applied theses and research, for accepting the normal approximation to the random distribution of the sample mean, i.e., having the normal distribution replacing the random distribution, is too lenient. In section 3, the standard normal value of the χ^2 distribution and the error of the standard normal distribution were used to estimate the least degrees of freedom required for the normal approximation to the χ^2 distribution. It was shown that treating the χ^2 distribution as the normal distribution when $\upsilon \geq 30$ may satisfy the criteria for $\alpha \geq 0.1$ in Table 4, but when $\alpha < 0.1$, the use of degrees of freedom greater than 30 for determining if the central limit theorem can be applied or not is inappropriate, and moreover, the degrees of freedom may not.

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