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## Statistical inferences of a two-parameter distribution with the bathtub shape based on progressive censored sample

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This study considers the hypothesis test and interval estimation of the shape parameter of a new two-parameter distribution with the bathtub shape or increasing failure rate function under type-II progressive censoring based on the proposed  $m$  pivotal quantities. The confidence region of two parameters of this distribution is also proposed. We give one numerical example to illustrate the proposed methods. Next, the Monte Carlo simulation is done to assess the behaviour of these pivotal quantities. Lastly, we find the optimal pivotal quantity based on the criteria of highest power, minimum confidence length and the minimum confidence region.

**Keywords:** two-parameter distribution; bathtub shape; increasing failure rate function; shape parameter; type-II progressive censoring; testing hypothesis; confidence interval; confidence region

### 1. Introduction

In the area of lifetime analysis, there has been a lot of research on distributions that have bathtub-shaped failure rate function (FRF). Xie *et al.* [1] proposed a modified Weibull extension with three parameters. When parameter  $\alpha = 1$ , their model is reduced to a two-parameter distribution with the bathtub shape or increasing FRF proposed by Chen [2]. Bebbington *et al.* [3] proposed a new two-parameter ageing distribution, which is a generalization of the Weibull, and studied its properties. This flexible Weibull extension distribution is able to model various ageing classes of lifetime distributions with IFR, IFRA and MBT by choosing appropriate parameter values. Zhang and Xie [4] addressed the extended Weibull distribution in Marshall and Olkin [5] by introducing an additional parameter to an existing model and also investigated some properties of this distribution and studied the versatility of the model in modelling failure data. Dimitrakopoulou *et al.* [6] proposed a lifetime distribution with an upside-down bathtub-shaped hazard function with a minimum number of parameters, at least as flexible as the Weibull distribution. Pham and Lai [7] presented a short summary of some well-known recent generalizations of Weibull-related lifetime models and also discussed the properties of this general class. In this article, we focus on a two-parameter distribution with the bathtub shape or increasing FRF proposed by

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Chen [2]. This lifetime distribution has bathtub-shaped FRF if  $\beta < 1$ ; increasing FRF if  $\beta \geq 1$  and this distribution becomes the exponential power distribution if  $\lambda = 1$ . Wu *et al.* [8] proposed the optimal estimation of the parameters of this lifetime distribution based on the doubly type-II censored sample. The reason why we are focusing our research on this distribution is because other generalizations of the Weibull distribution cannot obtain exact statistical inferences for their parameters based on the maximum likelihood estimation (MLEs) of the parameters. We establish the hypothesis test and interval estimation of the shape parameter of this lifetime distribution based on the progressively type-II censored sample. In addition, we proposed  $m-1$  pivotal quantities to construct the confidence region of two parameters of this given distribution.

Censoring arises when some lifetimes of products are missing or for implementing some purposes of experimental designs. There are several types of censoring schemes and the type-II censoring scheme is the most common one. The progressively type-II censoring scheme is described as follows. First, the experimenter place  $n$  units on test. The first failure is observed and then  $r_1$  of surviving units are randomly selected and removed. When the  $i$ th failure unit is observed,  $r_i$  of surviving units are randomly selected and removed,  $i = 2, \dots, m$ . This experiment terminates when the  $m$ th failure unit is observed and  $r_m = n - r_1 - \dots - r_{m-1} - m$  of surviving units are all removed. Cohen [9] and Cohen and Norgaard [10] studied the statistical inference on the parameter of several failure time distributions under type-II progressive censoring when the censoring schemes  $r_1, \dots, r_m$  are all pre-fixed. In Section 2,  $m$  pivotal quantities are proposed to do the test and establish the confidence interval for the shape parameter of the given distribution based on the type-II progressive censored sample. In these  $m$  pivotal quantities,  $m-1$  pivotal quantities are extended from the idea of Wu *et al.* [8]. There are some numerical characteristics jointly depended on two parameters of the extreme-value distribution. Therefore, it is necessary to obtain a confidence region for the two parameters and the confidence region is proposed in Section 3. In Section 4, we give a numerical example to illustrate the proposed methods. In Section 5, a simulation study is done to compare the performances of all proposed pivotal quantities and find the optimal pivotal quantity based on the criteria of highest power, minimum confidence length and the minimum confidence region. Lastly, the conclusion is discussed in Section 6.

## 2. Hypothesis tests and interval estimations for the shape parameter

Let random variable  $X$  have a new two-parameter distribution with the bathtub shape or increasing FRF with parameter  $\lambda$  and  $\beta$ , where  $\lambda$  is the scale parameter and  $\beta$  is the shape parameter. The probability density function of  $X$  is given by

$$f(x) = \begin{cases} \lambda\beta x^{\beta-1} e^{x^\beta} \exp\{\lambda(1 - e^{x^\beta})\}, & x > 0, \lambda > 0, \beta > 0 \\ 0 & \text{otherwise} \end{cases}. \quad (1)$$

The cumulative distribution function  $F(x)$  is given by  $F(x) = 1 - e^{\lambda(1 - e^{x^\beta})}$ . Let  $X_1 < X_2 < \dots < X_m$  denote a progressively type-II censored sample. With a predetermined number of removals ( $r_1, \dots, r_{m-1}$ ), the likelihood function can be written as [9]

$$L(\mathbf{x}; \lambda, \beta) = c^* \prod_{i=1}^m (\lambda\beta x_i^{\beta-1} e^{x_i^\beta} \exp\{\lambda(1 - e^{x_i^\beta})\})(\exp\{\lambda(1 - e^{x_i^\beta})\})^{r_i}, \quad (2)$$

where  $c^* = n(n - r_1 - 1) \cdots (n - r_1 - \dots - r_{m-1} - m + 1)$ ,  $x_1 < x_2 < \dots < x_m$  and  $r_i$  can be any integer value between 0 and  $n - m - (r_1 + r_2 + \dots + r_{i-1})$ ,  $i = 1, \dots, m-1$ .

It can be shown that after the transformation of  $Y_i = \lambda(e^{X_i^\beta} - 1)$ ,  $Y_1 < Y_2 < \dots < Y_m$  is a progressively type-II censored sample from the standard exponential distribution.

In order to do the test and interval estimation for the shape parameter of type-II progressive censoring, we proposed  $m$  pivotal quantities as follows.

First, define

$$S(\beta) = \frac{\sum_{i=1}^m (1+r_i)Y_i/n}{\prod_{i=1}^m Y_i^{(1+r_i)/n}} = \frac{\sum_{i=1}^m (1+r_i)(\exp(X_i^\beta) - 1)/n}{\prod_{i=1}^m (\exp(X_i^\beta) - 1)^{(1+r_i)/n}}. \quad (3)$$

It is easy to see that the distribution of  $S(\beta)$  is independent of parameters and depending on the censoring structure. Therefore, it can be used as a pivotal quantity for establishing the statistical inferences for the shape parameter  $\beta$ .

In addition, we propose the following  $m - 1$  pivotal quantities (which are extended from the idea of Wu *et al.* [8]) as follows:

$$\begin{aligned} h_j &= \frac{(j)}{(m-j)} \frac{-(n-j-r_1-\cdots-r_j)Y_j + \sum_{i=j+1}^m (r_i+1)Y_i}{\sum_{i=1}^{j-1} (r_i+1)Y_i + (n-j-r_1-\cdots-r_{j-1}+1)Y_j} \\ &= \frac{(j)}{(m-j)} \frac{-(n-j-r_1-\cdots-r_j)(\exp(X_j^\beta) - 1) + \sum_{i=j+1}^m (r_i+1)(\exp(X_i^\beta) - 1)}{\sum_{i=1}^{j-1} (r_i+1)(\exp(X_i^\beta) - 1) + (n-j-r_1-\cdots-r_{j-1}+1)(\exp(X_j^\beta) - 1)}, \\ j &= 1, \dots, m-1. \end{aligned} \quad (4)$$

For a fixed censoring scheme  $(r_1, \dots, r_{m-1})$ , let  $Z_1 = nY_1$ ,  $Z_2 = (n-r_1-1)(Y_2-Y_1)$ ,  $Z_3 = (n-r_1-r_2-2)(Y_3-Y_2)$ , ...,  $Z_m = (n-r_1-\cdots-r_{m-1}-m+1)(Y_m-Y_{m-1})$ .

Balakrishnan and Aggarwala [11] showed that the generalized spacings  $Z_1, \dots, Z_m$  are all independent and identically distributed as standard exponentials. Hence,  $U_j = 2 \sum_{i=1}^j Z_i$  and  $V_j = 2(\sum_{i=1}^m Z_i - U_j)$  are independently chi-square distributed with  $2j$  and  $2(m-j)$  degrees of freedom, respectively. Therefore,

$$h_j = \frac{(j)}{(m-j)} \frac{-(n-j-r_1-\cdots-r_j)Y_j + \sum_{i=j+1}^m (r_i+1)Y_i}{\sum_{i=1}^{j-1} (r_i+1)Y_i + (n-j-r_1-\cdots-r_{j-1}+1)Y_j} = \frac{(j)V_j}{(m-j)U_j}$$

has a  $F$  distribution with  $2(m-j)$  and  $2j$  degrees of freedom, where  $j = 1, \dots, m-1$ .

The following two lemmas are required in order to derive the tests and confidence intervals for the scale parameter of the new two-parameter distribution with the bathtub shape or increasing FRF.

**LEMMA 1** *For a given set of observations  $X_1 < X_2 < \cdots < X_m$ , the function  $S(\beta)$  is a strictly decreasing function of  $\beta$  when  $\beta > 0$ . There is a unique solution for the given equation  $S(\beta) = t$  when  $t > 1$ .*

*Proof* Take the derivative of the  $\ln S(\beta)$  with respect to  $\beta$ , then we have

$$\frac{\partial \ln S(\beta)}{\partial \beta} = \frac{(1/n) \sum_{i=1}^m (1+r_i) \exp(X_i^\beta) X_i^\beta \ln X_i}{(1/n) \sum_{i=1}^m (1+r_i)(\exp(X_i^\beta) - 1)} - \frac{1}{n} \sum_{i=1}^m \frac{(1+r_i) \exp(X_i^\beta) X_i^\beta \ln X_i}{\exp(X_i^\beta) - 1}.$$

Define the p.d.f of a random variable  $w$  as  $f(w) = (1+r_i)/n$ ,  $w = x_1, x_2, \dots, x_m$ .

Consider two strictly increasing functions of  $\beta$  as  $g(w) = \exp(w^\beta) - 1$  and  $h(w) = (\exp(w^\beta) w^\beta \ln w) / \exp(w^\beta) - 1$ , where  $w > 0$ .

Then  $\text{cov}(g(w), h(w)) = E(g(w)h(w)) - E(g(w))E(h(w)) > 0$ . Observed that  $\partial \ln S(\theta) / \partial \theta = E(g(w)h(w))/E(g(w)) - E(h(w)) > 0$ . Then, we obtained that the function  $S(\beta)$  is a strictly increasing function of  $\beta$  when  $\beta > 0$ .

Furthermore, we have  $\lim_{\beta \rightarrow 0} S(\beta) = 1$  and  $\lim_{\beta \rightarrow \infty} S(\beta) = \infty$ . Therefore, there is a unique solution for the given equation  $S(\beta) = t$  when  $t > 1$ . ■

**LEMMA 2** *For a given set of observations  $X_1 < X_2 < \dots < X_m$ , the function  $h_j$  is a strictly increasing function of  $\beta$  when  $\beta > 0$ . There is a unique solution for the given equation  $h_j = t$  when  $t > 0$ .*

*Proof* Since  $h_j$  can be rewritten as

$$\begin{aligned} h_j &= \frac{j}{(m-j)} \cdot \frac{\sum_{i=j+1}^m (r_i + 1)(e^{X_i^\beta} - 1) - (n - r_1 - r_2 - \dots - r_j - j)(e^{X_j^\beta} - 1)}{(n - r_1 - r_2 - \dots - r_{j-1} - j + 1)(e^{X_j^\beta} - 1) + \sum_{i=1}^{j-1} (r_i + 1)(e^{X_i^\beta} - 1)} \\ &= \frac{j}{(m-j)} \cdot \frac{\sum_{i=j+1}^m (r_i + 1)(e^{X_i^\beta} - 1)/(e^{X_j^\beta} - 1) - (n - r_1 - r_2 - \dots - r_j - j)}{(n - r_1 - r_2 - \dots - r_{j-1} - j + 1) + \sum_{i=1}^{j-1} (r_i + 1)(e^{X_i^\beta} - 1)/(e^{X_j^\beta} - 1)}, \end{aligned}$$

where  $j = 1, \dots, m-1$  and the function of  $(e^{X_i^\beta} - 1)$  is a strictly increasing function of  $\beta$ . Therefore,  $h_j$  is a strictly increasing function of  $\beta$  when  $\beta > 0$ . Furthermore, we have  $\lim_{\beta \rightarrow 0} h_j(\beta; X_1, \dots, X_m) = 0$  and  $\lim_{\beta \rightarrow \infty} h_j(\beta; X_1, \dots, X_m) = \infty$ . Thus, there is a unique solution for the given equation  $h_j = t$  when  $t > 0$ ,  $j = 1, \dots, m-1$ . ■

Now if the user would like to test

$$H_0 : \beta = \beta_0 \quad vs. \quad H_1 : \beta \neq \beta_0.$$

When the pivotal quantity  $S(\beta)$  is used, the decision rule is to reject  $H_0$  if  $S(\beta_0) > S^*(\alpha/2)$  or  $S(\beta_0) < S^*(1 - \alpha/2)$ , where  $S^*(\alpha/2)$  is the right-tail  $\alpha/2$  percentile of the distribution of  $S(\beta)$ .

But the exact distribution of the pivotal quantity  $S(\beta)$  is too hard to derive. Therefore, the Monte Carlo method is used to approximate its distribution. By using the Absoft Fortran software package [12], 600,000 simulation runs are done to obtain the upper percentiles of the distribution of  $S(\beta)$  for  $n = 10, 20$  and  $m = n - 2(1)n$  and different censoring scheme (three kinds of censoring (left, middle and right censoring)) are listed in Table 1.

When the pivotal quantities  $h_j$ ,  $j = 1, \dots, m-1$ , are used, the decision rule is to reject  $H_0$  if  $h_j > F_{\alpha/2}(2(m-j), 2j)$  or  $h_j < F_{1-\alpha/2}(2(m-j), 2j)$ , where  $F_{\alpha/2}(2(m-j), 2j)$  is the right-tail  $\alpha/2$  percentile of a  $F$  distribution with  $2(m-j)$  and  $2j$  degrees of freedom, where  $j = 1, \dots, m-1$ .

Using Lemma 1, we can obtain an exact confidence interval of the scale parameter  $\beta$  as  $(S^{-1}(\alpha/2), S^{-1}(1 - \alpha/2))$ , where  $S^{-1}(\alpha/2)$  is the solution of the equation

$$S(\theta) = S^* \left( \frac{\alpha}{2} \right) \tag{5}$$

and  $S^{-1}(1 - \alpha/2)$  is the solution of the equation

$$S(\theta) = S^* \left( 1 - \frac{\alpha}{2} \right). \tag{6}$$

Using Lemma 2, we can obtain an exact confidence interval of the scale parameter  $\beta$  as  $(h_j^{-1}[F_{\alpha/2}(2(m-j), 2j)], h_j^{-1}[F_{1-\alpha/2}(2(m-j), 2j)])$ , where  $h_j^{-1}[F_{\alpha/2}(2(m-j), 2j)]$  is the solution of the equation

$$h_j = F_{\alpha/2}(2(m-j), 2j) \tag{7}$$

and  $h_j^{-1}[F_{1-\alpha/2}(2(m-j), 2j)]$  is the solution of the equation

$$h_j = F_{1-\alpha/2}(2(m-j), 2j). \tag{8}$$

Table 1. Upper percentiles of distribution of  $S(\beta)$ .

$n$	$m$	Censoring scheme	$\alpha$									
			0.995	0.990	0.975	0.950	0.900	0.100	0.050	0.025	0.010	0.005
10	8	$r_1 = 2, r_i = 0, i \geq 2$	1.126605	1.160609	1.222931	1.291707	1.392115	3.929313	4.993912	6.270279	8.380244	10.35606
		$r_4 = r_5 = 1, r_i = 0, i \neq 4, 5$	1.071403	1.090006	1.124668	1.162299	1.216608	2.264144	2.596391	2.963067	3.495284	3.951188
		$r_8 = 2, r_i = 0, i \leq 7$	1.053477	1.068208	1.09584	1.125099	1.16821	1.962244	2.201746	2.453878	2.818741	3.118443
9	9	$r_1 = 1, r_i = 0, i \geq 2$	1.123658	1.153025	1.206438	1.263917	1.3462	3.041773	3.630495	4.298241	5.309685	6.208941
		$r_5 = 1, r_i = 0, i \neq 5$	1.089565	1.110934	1.148803	1.189447	1.247609	2.300262	2.619396	2.957893	3.447394	3.859908
		$r_9 = 1, r_i = 0, i \leq 8$	1.077564	1.096718	1.130691	1.167354	1.219174	2.139294	2.411264	2.701532	3.117581	3.454578
10	10	$r_i = 0, i = 1, \dots, 10$	1.107945	1.132305	1.173869	1.217997	1.280086	2.348337	2.664743	2.990533	3.468513	3.859882
		$r_1 = 2, r_i = 0, i \geq 2$	1.289719	1.331201	1.401124	1.470563	1.564911	3.036317	3.463651	3.914135	4.562398	5.106811
		$r_7 = r_8 = 1, r_i = 0, i \neq 7, 8$	1.205269	1.233144	1.279973	1.326474	1.387275	2.205817	2.405857	2.607777	2.87688	3.092089
20	18	$r_{18} = 2, r_i = 0, i \leq 17$	1.172467	1.196879	1.237413	1.27681	1.329524	2.014769	2.18133	2.345401	2.5663	2.736687
		$r_1 = 1, r_i = 0, i \geq 2$	1.260099	1.29496	1.354272	1.41216	1.488524	2.576418	2.860616	3.148737	3.55572	3.892139
		$r_9 = 1, r_i = 0, i \neq 9$	1.211791	1.239457	1.286556	1.332194	1.392828	2.182982	2.374567	2.566788	2.823189	3.025907
19	19	$r_{19} = 1, r_i = 0, i \leq 18$	1.196696	1.22236	1.265824	1.309073	1.365612	2.099664	2.277742	2.450538	2.683502	2.869262
		$r_i = 0, i = 1, \dots, 20$	1.221735	1.251269	1.29885	1.34543	1.406813	2.195287	2.385089	2.573478	2.830644	3.025426

### 3. Confidence region for two parameters

In order to construct the confidence region of two parameters, the pivotal quantity defined as  $g = (U_j + V_j) = 2 \sum_{i=1}^m Z_i = 2 \sum_{i=1}^m (r_i + 1)Y_i = 2\lambda \sum_{i=1}^m (r_i + 1)Y_i \cdot (e^{X_i^\beta} - 1)$  is necessary. It can be shown that  $h_j$ ,  $j = 1, \dots, m-1$ , and  $g$  are independent and  $h_j$  has a  $F$  distribution with  $2(m-j)$  and  $2j$  degrees of freedom and  $g$  has a chi-squared distribution with  $2m$  degrees of freedom,  $j = 1, \dots, m-1$ . For any given  $\beta$ , it can be shown that  $g$  is a strictly decreasing function of  $\lambda$  when  $\lambda > 0$ . The equation  $g = k$  has a unique solution of  $\lambda$  when  $0 < k < \infty$ .

Based on Lemma 2 and the above property related to the function  $g$ , we can construct the confidence region of two parameters as the following theorem.

**THEOREM 3** *The  $(1-\alpha)$  100% joint confidence region of  $\lambda$  and  $\beta$  is given by*

$$\begin{cases} h_j^{-1}[F_{(1-\sqrt{1-\alpha})/2}(2(m-j), 2j)] < \beta < h_j^{-1}[F_{(1+\sqrt{1-\alpha})/2}(2(m-j), 2j)] \\ \frac{\chi_{(1-\sqrt{1-\alpha})/2}^2(2m)}{2 \sum_{i=1}^m (r_i + 1)(1 - e^{X_i^\beta})} < \lambda < \frac{\chi_{(1+\sqrt{1-\alpha})/2}^2(2m)}{2 \sum_{i=1}^m (r_i + 1)(1 - e^{X_i^\beta})} \end{cases},$$

where  $F_{(1+\sqrt{1-\alpha})/2}(2(m-j), 2j)$  and  $\chi_{(1+\sqrt{1-\alpha})/2}^2(2m)$  are the upper  $(1 + \sqrt{1-\alpha})/2$  percentile for  $F$  distribution with  $2(m-j)$  and  $2j$  degrees of freedom and the upper  $(1 + \sqrt{1-\alpha})/2$  percentile for chi-squared distribution with  $2m$  degrees of freedom, respectively,  $0 < \alpha < 1$ , and  $h_j^{-1}[t]$  is the solution of  $h_j = t$ ,  $j = 1, \dots, m-1$ .

### 4. One numerical example

To illustrate our proposed methods, suppose that the experimenter places 10 units on test. Under the pre-fixed censoring scheme  $(r_1, r_2, \dots, r_8) = (2, 0, 0, 0, 0, 0, 0, 0)$ , the type-II progressive censored sample of size  $m = 8$  from a new two-parameter distribution with the bathtub shape or increasing FRF is generated at  $(\lambda, \beta) = (1.0, 1.0)$ . The sample is given by  $(X_1, \dots, X_8) = (0.056995, 0.110942, 0.2094099, 0.2174900, 0.286167, 0.3641177, 0.3864243, 0.5978188)$ .

If the user wants to test  $H_0 : \beta = 0.6$  vs.  $H_1 : \beta \neq 0.6$  at the level of significance  $\alpha = 0.05$ , the following percentiles are necessary.

$$\begin{array}{ll} S^*(0.025) = 6.270279 & S^*(0.975) = 1.222931 \\ F_{0.025}(14, 2) = 39.426505 & F_{0.975}(14, 2) = 0.20590122 \\ F_{0.025}(12, 4) = 8.751159 & F_{0.975}(12, 4) = 0.24264727 \\ F_{0.025}(10, 6) = 5.461324 & F_{0.975}(10, 6) = 0.24557165 \\ F_{0.025}(8, 8) = 4.433260 & F_{0.975}(8, 8) = 0.22556765 \\ F_{0.025}(6, 10) = 4.072131 & F_{0.975}(6, 10) = 0.18310579 \\ F_{0.025}(4, 12) = 4.121209 & F_{0.975}(4, 12) = 0.11427058 \\ F_{0.025}(2, 14) = 4.856698 & F_{0.975}(2, 14) = 0.02536365 \end{array}$$

The upper percentiles  $S^*(0.025)$  and  $S^*(0.975)$  are obtained from Table 1. Using the above percentiles, the testing conclusions based on  $m$  pivotal quantities are listed in the following table.

Pivot	Reject $H_0$ if	Conclusion
$S(0.6) = 1.28277$	$S(0.6) > 6.270279$ or $S(0.6) < 1.222931$	DNR $H_0$
$h_1 = 0.35304$	$h_1 > 39.426505$ or $h_1 < 0.20590122$	DNR $H_0$
$h_2 = 0.43557$	$h_2 > 8.751159$ or $h_2 < 0.24264727$	DNR $H_0$
$h_3 = 0.34333$	$h_3 > 5.461324$ or $h_3 < 0.24557165$	DNR $H_0$
$h_4 = 0.53901$	$h_4 > 4.433260$ or $h_4 < 0.22556765$	DNR $H_0$
$h_5 = 0.56337$	$h_5 > 4.072131$ or $h_5 < 0.18310579$	DNR $H_0$
$h_6 = 0.58441$	$h_6 > 4.121209$ or $h_6 < 0.11427058$	DNR $H_0$
$h_7 = 1.16596$	$h_7 > 4.856698$ or $h_7 < 0.02536365$	DNR $H_0$

Next, we want to construct the 95% confidence interval for  $\beta$ . Using the percentiles  $S^*(0.025) = 6.270279$ ,  $S^*(0.975) = 1.222931$  and Equations 5 and 6, the confidence interval of  $\beta$  based on the pivotal quantity  $S(\beta)$  is  $(0.38244, 2.80339)$  with length 1.97613. Using the percentiles  $F_{0.025}(14, 2) = 39.426505$ ,  $F_{0.975}(14, 2) = 0.20590122$  and Equations 7 and 8, the 95% confidence interval of  $\beta$  based on the pivotal quantity  $h_1$  is  $(0.35236, 3.07247)$  with the confidence length 2.34279. As to the 95% joint confidence region for  $\lambda$  and  $\beta$  based on the set of pivotal quantities  $h_1$  and  $g$ , we need the percentiles  $F_{0.0127}(14, 2) = 78.41471$ ,  $F_{0.9873}(14, 2) = 0.16482$ ,  $\chi^2_{0.0127}(16) = 31.20696$  and  $\chi^2_{0.9873}(16) = 6.06839$ . By Theorem 3, the joint confidence region is given by the following inequality:

$$\Rightarrow \left\{ \frac{6.06839}{2 \sum_{i=1}^8 (r_i + 1) \cdot (e^{x_i^\beta} - 1)} < \lambda < \frac{31.20696}{2 \sum_{i=1}^8 (r_i + 1) \cdot (e^{x_i^\beta} - 1)} \right. .$$

The confidence region for  $\lambda$  and  $\beta$  are displayed in Figure 1.

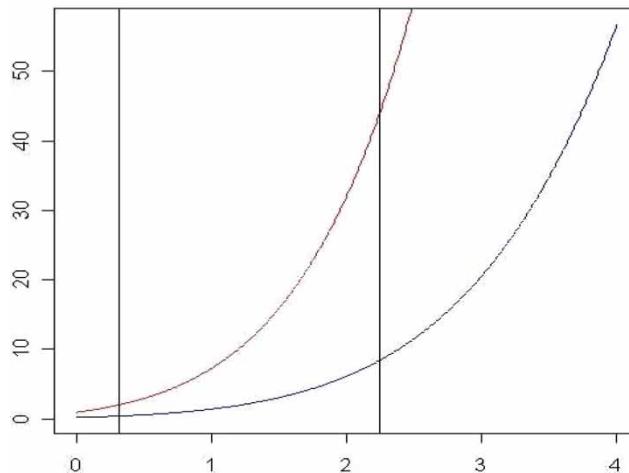


Figure 1. The confidence region for  $\lambda$  and  $\beta$ .

After the integration of  $\int_{0.38244}^{2.38163} ((31.20696 - 6.06839)/2 \sum_{i=1}^8 (r_i + 1)(e^{x_i^\beta} - 1)) d\beta$ , the area of the confidence region is obtained as 13.18542. Similarly, the average length of confidence intervals and the average areas of the confidence regions for all pivotal quantities are listed in the following table.

The 95% confidence length of  $\beta$  and the 95% confidence area of  $\lambda$  and  $\beta$  under the censoring scheme (2,0,0,0,0,0,0) for  $n = 10$  and  $m = 8$ .

Pivots	Average length	Average area
$S(\beta)$	1.97613	
$h_1$	2.34279	13.18531
$h_2$	2.29919	9.16053
$h_3$	2.40045	14.35447
$h_4$	2.05291	8.00384
$h_5$	2.07759	8.63386
$h_6$	2.31724	10.19166
$h_7$	3.76917	5.04497

It can be found that the pivotal quantity  $S(\beta)$  is uniformly better than the other  $m-1$  pivotal quantities in terms of shorter confidence length.

## 5. Simulation study

In order to compare the performance of all methods, the 10,000 simulation runs by Monte Carlo method are done at  $n = 10, 20, m = (n-2), \dots, n$  and under three kinds of censoring (left, middle and right censoring). Based on the pivotal quantity  $S(\beta)$  and other  $m-1$  pivotal quantities  $h_j, j = 1, \dots, m-1$ , the average powers for the true values of  $\beta = 0.1, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.6, 1.8, 2.0, 3.0, 4.0$  and  $\lambda = 1$  when testing  $H_0 : \beta = 0.6$  vs.  $H_1 : \beta \neq 0.6$  at the level of significance  $\alpha = 0.05$  for  $n = 10$  were listed in Tables 2–4. To save space, the tables for the average powers for  $n = 20$  are available upon request at authors' site. It can be seen that all methods can reach the nominal significance level and the pivotal quantity  $S(\beta)$  has higher power than the other  $m-1$  methods. The power comparison figures of the top three better methods are given in Figures 2 and 3. The average lengths of 95% confidence intervals constructed by  $m$  methods and the average areas of 95% confidence regions constructed by  $m-1$  methods under the simulation structure of  $\lambda = 1$  and  $\beta = 0.6$  were listed in Table 5 for all given combination of  $m, n$  and different censoring schemes. All methods can reach the nominal confidence coefficient of 95%. The optimal pivotal quantity with minimum average confidence length for confidence interval or minimum confidence area for confidence region are marked by asterisk (\*) sign, respectively, for any given  $n, m$  and different censoring schemes. It can be seen that the pivotal quantity  $S(\beta)$  outperforms the other  $m-1$  methods in terms of shorter confidence length.

The pivotal quantity  $h_{m-1}$  yields the worst performance either for testing or for constructing confidence intervals. Therefore, the pivotal quantity  $S(\beta)$  is recommended for use and the pivotal quantity  $h_{m-1}$  should be avoided. As to the confidence region, the set of pivotal quantities  $h_1$  and  $g$  is optimal in terms of smaller confidence area for all cases. Furthermore, both the confidence length and confidence area are decreasing when the efficient sample ratio  $m/n$  approaches 1.

Table 2. Power comparisons for  $H_0 : \beta = 0.6$  vs.  $H_1 : \beta \neq 0.6$  at  $n = 10, m = 8$  and  $\alpha = 0.05$ .

$m$	Censoring plan	Pivot	$\beta$									
			0.1	0.4	0.6	0.8	1.0	1.2	1.6	1.8	2	
8	$r_1 = 2, r_i = 0, i \geq 2$	$S(\beta)$	0.9977	0.2638	0.0539	0.0976	0.2287	0.3807	0.6898	0.7881	0.8514	0.9980
		$h_1$	0.9918	0.1730	0.0456	0.0914	0.1814	0.2978	0.5546	0.6418	0.7192	0.9826
		$h_2$	0.9904	0.2218	0.0520	0.0872	0.1894	0.3228	0.5812	0.7122	0.7770	0.9956
		$h_3$	0.9880	0.2280	0.0536	0.0806	0.1820	0.3020	0.5690	0.6758	0.7614	0.9956
		$h_4$	0.9788	0.2336	0.0518	0.0694	0.1524	0.2498	0.5008	0.6118	0.7132	0.9926
		$h_5$	0.9530	0.2320	0.0520	0.0578	0.1196	0.1934	0.3884	0.4826	0.5784	0.9742
		$h_6$	0.9194	0.2086	0.0450	0.0492	0.0840	0.1286	0.2306	0.2968	0.3680	0.8486
		$h_7$	0.7992	0.1854	0.0420	0.0368	0.0546	0.0688	0.1014	0.1104	0.1306	0.2990
	$r_4 = r_5 = 1, r_i = 0, i \neq 4, 5$	$S(\beta)$	0.9974	0.3072	0.0541	0.0923	0.2263	0.3972	0.7209	0.8219	0.8949	0.9997
		$h_1$	0.9862	0.1660	0.0494	0.0932	0.1746	0.2768	0.4906	0.5742	0.6596	0.9668
		$h_2$	0.9896	0.2188	0.0482	0.0844	0.1674	0.2960	0.5242	0.6220	0.7126	0.9862
		$h_3$	0.9858	0.2348	0.0536	0.0796	0.1602	0.2682	0.5120	0.6212	0.7026	0.9848
		$h_4$	0.9742	0.2556	0.0486	0.0668	0.1300	0.2354	0.4470	0.5588	0.6542	0.9832
		$h_5$	0.9532	0.2338	0.0488	0.0602	0.1028	0.1842	0.3536	0.4504	0.5458	0.9666
		$h_6$	0.9024	0.2230	0.0466	0.0458	0.0782	0.1212	0.2218	0.2746	0.3368	0.8246
		$h_7$	0.7946	0.1934	0.0470	0.0386	0.0512	0.0656	0.0890	0.1002	0.1228	0.2588
	$r_8 = 2, r_i = 0, i \leq 7$	$S(\beta)$	0.9935	0.2487	0.0498	0.0909	0.1890	0.3230	0.5907	0.7143	0.8015	0.9975
		$h_1$	0.9732	0.1546	0.0488	0.0806	0.1574	0.2538	0.4596	0.5536	0.6330	0.9588
		$h_2$	0.9746	0.1858	0.0472	0.0778	0.1524	0.2504	0.4608	0.5820	0.6502	0.9816
		$h_3$	0.9662	0.1904	0.0500	0.0696	0.1340	0.2160	0.4426	0.5284	0.6204	0.9708
		$h_4$	0.9466	0.1716	0.0516	0.0642	0.1144	0.1962	0.3618	0.4486	0.5386	0.9506
		$h_5$	0.8974	0.1792	0.0490	0.0572	0.0964	0.1326	0.2630	0.3360	0.4028	0.8714
		$h_6$	0.8132	0.1568	0.0478	0.0470	0.0722	0.0916	0.1520	0.2030	0.2438	0.6128
		$h_7$	0.6182	0.1208	0.0494	0.0410	0.0528	0.0550	0.0766	0.0836	0.0922	0.1984

Table 3. Power comparisons for  $H_0 : \beta = 0.6$  vs.  $H_1 : \beta \neq 0.6$  at  $n = 10, m = 9$  and  $\alpha = 0.05$ .

$m$	Censoring plan	Pivot	$\beta$									
			0.1	0.4	0.6	0.8	1.0	1.2	1.6	1.8	2	4
9	$r_1 = 1, r_i = 0, i \geq 2$	$S(\beta)$	0.9994	0.2940	0.0512	0.1061	0.2651	0.4440	0.7458	0.8455	0.9064	0.9996
		$h_1$	0.9936	0.1702	0.0496	0.1000	0.1954	0.3356	0.5610	0.6680	0.7352	0.9842
		$h_2$	0.9956	0.2250	0.0516	0.0990	0.2008	0.3436	0.6170	0.7234	0.7936	0.9950
		$h_3$	0.9930	0.2488	0.0486	0.0860	0.1976	0.3414	0.6160	0.7336	0.8020	0.9986
		$h_4$	0.9874	0.2474	0.0530	0.0832	0.1854	0.2992	0.5810	0.6982	0.7908	0.9954
		$h_5$	0.9794	0.2410	0.0482	0.0744	0.1498	0.2680	0.5142	0.6186	0.7194	0.9920
		$h_6$	0.9586	0.2452	0.0526	0.0614	0.1206	0.2032	0.3776	0.4870	0.5798	0.9744
		$h_7$	0.9190	0.2118	0.0498	0.0564	0.0860	0.1260	0.2346	0.2918	0.3548	0.8338
		$h_8$	0.8298	0.1708	0.0532	0.0384	0.0510	0.0658	0.0948	0.1040	0.1226	0.2864
	$r_5 = 1, r_i = 0, i \neq 5$	$S(\beta)$	0.9985	0.3380	0.0485	0.1043	0.2628	0.4641	0.7797	0.8759	0.9341	1.0000
		$h_1$	0.9904	0.1754	0.0466	0.0890	0.1894	0.3056	0.5398	0.6416	0.7058	0.9764
		$h_2$	0.9938	0.2186	0.0494	0.0994	0.1974	0.3188	0.5924	0.6768	0.7662	0.9916
		$h_3$	0.9904	0.2438	0.0454	0.0862	0.1964	0.3324	0.5926	0.6926	0.7816	0.9930
		$h_4$	0.9850	0.2592	0.0566	0.0772	0.1714	0.2934	0.5428	0.6644	0.7586	0.9916
		$h_5$	0.9780	0.2478	0.0558	0.0646	0.1450	0.2574	0.4882	0.6036	0.6924	0.9896
		$h_6$	0.9552	0.2402	0.0416	0.0652	0.1230	0.1828	0.3788	0.4802	0.5594	0.9708
		$h_7$	0.9176	0.2292	0.0502	0.0454	0.0762	0.1188	0.2366	0.2932	0.3490	0.8310
		$h_8$	0.8120	0.1862	0.0508	0.0370	0.0550	0.0624	0.1044	0.1124	0.1234	0.2804
	$r_9 = 1, r_i = 0, i \leq 8$	$S(\beta)$	0.9977	0.2846	0.0492	0.0974	0.2341	0.4050	0.7186	0.8133	0.8932	0.9998
		$h_1$	0.9886	0.1674	0.0520	0.0862	0.1832	0.2826	0.5234	0.6102	0.6956	0.9774
		$h_2$	0.9880	0.2000	0.0512	0.0828	0.1896	0.3052	0.5704	0.6774	0.7514	0.9914
		$h_3$	0.9862	0.2170	0.0508	0.0884	0.1760	0.2986	0.5356	0.6514	0.7414	0.9948
		$h_4$	0.9750	0.2128	0.0490	0.0754	0.1580	0.2594	0.5026	0.6016	0.7092	0.9906
		$h_5$	0.9614	0.2058	0.0490	0.0674	0.1276	0.2122	0.4198	0.5348	0.6178	0.9804
		$h_6$	0.9240	0.1928	0.0448	0.0552	0.1056	0.1580	0.3168	0.3864	0.4614	0.9304
		$h_7$	0.8636	0.1770	0.0506	0.0460	0.0744	0.1108	0.1772	0.2360	0.2722	0.6992
		$h_8$	0.7176	0.1392	0.0466	0.0394	0.0484	0.0606	0.0794	0.0944	0.1032	0.2342

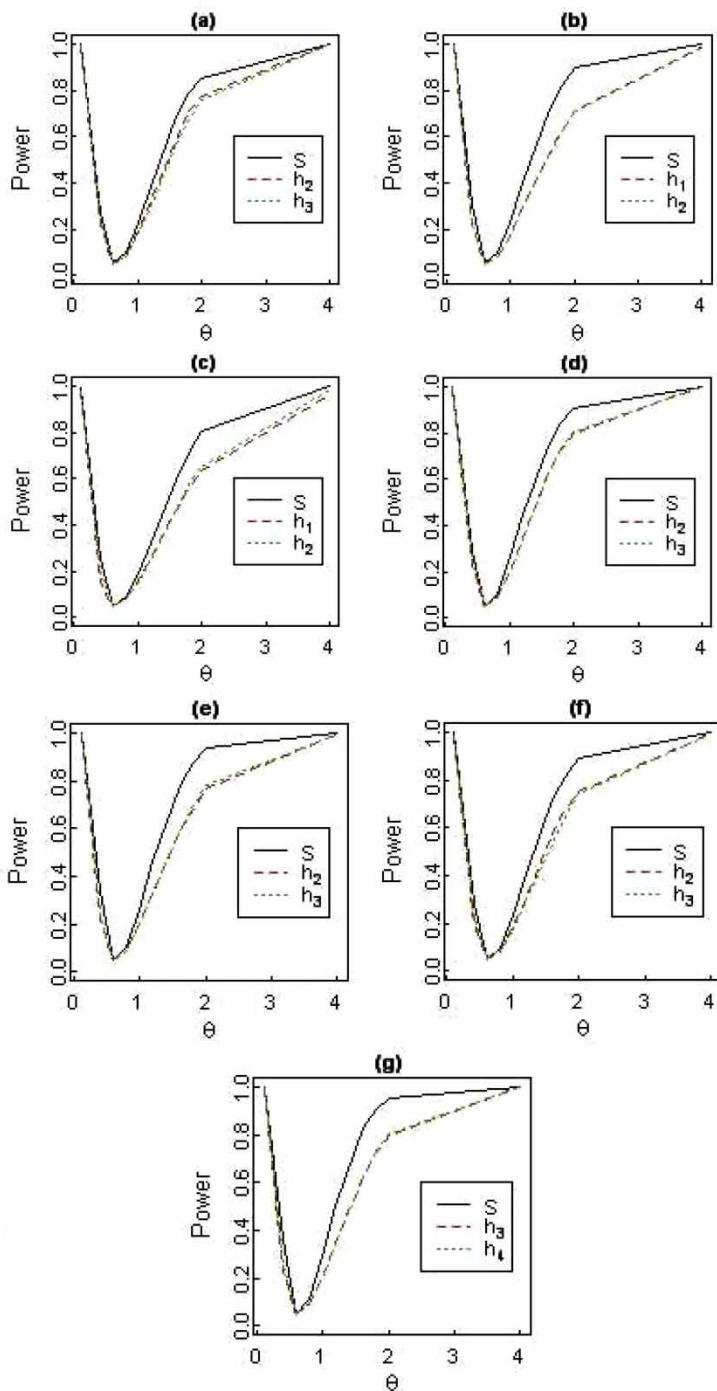


Figure 2. Power comparisons for the top three better methods at  $n = 10$ : (a)  $m = 8$  and  $r_1 = 2, r_i = 0, i \neq 1$ , (b)  $m = 8$  and  $r_4 = r_5 = 1, r_i = 0, i \neq 4, 5$ , (c)  $m = 8$  and  $r_8 = 2, r_i = 0, i \neq 8$ , (d)  $m = 9$  and  $r_1 = 1, r_i = 0, i \neq 1$ , (e)  $m = 9$  and  $r_5 = 1, r_i = 0, i \neq 5$ , (f)  $m = 9$  and  $r_9 = 1, r_i = 0, i \neq 9$  and (g)  $m = 10$ .

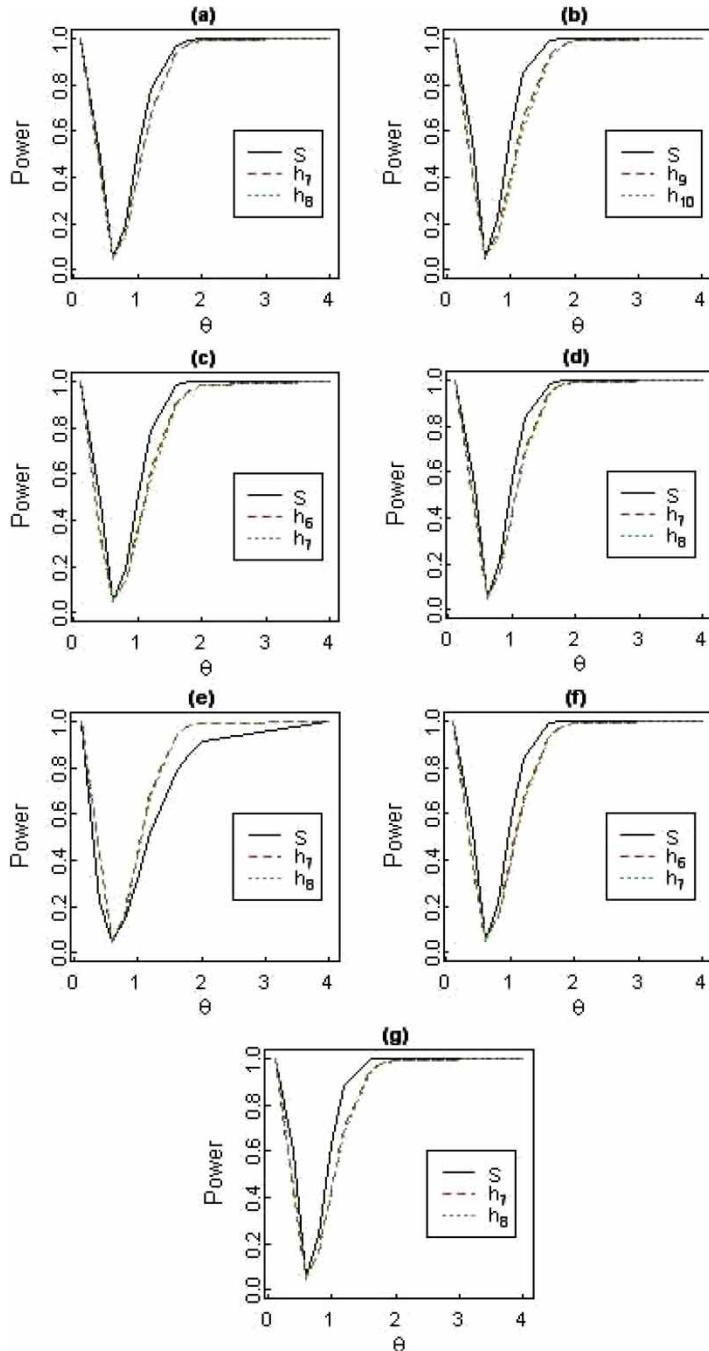


Figure 3. Power comparisons for the top three better methods at  $n = 20$ : (a)  $m = 18$  and  $r_1 = 2$ ,  $r_i = 0$ ,  $i \geq 2$ , (b)  $m = 18$  and  $r_7 = r_8 = 1$ ,  $r_i = 0$ ,  $i \neq 7, 8$ , (c)  $m = 18$  and  $r_{18} = 2$ ,  $r_i = 0$ ,  $i \leq 17$ , (d)  $m = 19$  and  $r_1 = 1$ ,  $r_i = 0$ ,  $i \leq 18$ , (e)  $m = 19$  and  $r_9 = 1$ ,  $r_i = 0$ ,  $i \neq 9$ , (f)  $m = 19$  and  $r_{19} = 1$ ,  $r_i = 0$ ,  $i \leq 18$  and (g)  $m = 20$ .

Table 4. Power comparisons for  $H_0 : \beta = 0.6$  vs.  $H_1 : \beta \neq 0.6$  at  $n = 10, m = 10$  and  $\alpha = 0.05$ .

$m$	Pivot	$\beta$									
		0.1	0.4	0.6	0.8	1.0	1.2	1.6	1.8	2	4
10	$S(\beta)$	0.9988	0.3586	0.0503	0.1136	0.2913	0.4988	0.8298	0.9043	0.9556	0.9999
	$h_1$	0.9950	0.1800	0.0462	0.1008	0.1978	0.3368	0.5828	0.6730	0.7416	0.9820
	$h_2$	0.9972	0.2296	0.0504	0.1012	0.2148	0.3548	0.6496	0.7550	0.8160	0.9976
	$h_3$	0.9956	0.2546	0.0432	0.1026	0.2256	0.3660	0.6582	0.7730	0.8326	0.9966
	$h_4$	0.9952	0.2642	0.0454	0.0860	0.2050	0.3594	0.6332	0.7594	0.8330	0.9988
	$h_5$	0.9890	0.2574	0.0512	0.0826	0.1898	0.3094	0.5976	0.7088	0.8010	0.9972
	$h_6$	0.9784	0.2606	0.0492	0.0744	0.1500	0.2666	0.5188	0.6374	0.7156	0.9956
	$h_7$	0.9648	0.2650	0.0490	0.0630	0.1198	0.1988	0.3870	0.4954	0.5776	0.9776
	$h_8$	0.9322	0.2270	0.0544	0.0542	0.0806	0.1352	0.2364	0.2948	0.3480	0.8248
	$h_9$	0.8302	0.1872	0.0474	0.0394	0.0520	0.0684	0.0974	0.1044	0.1248	0.2802

Table 5. The average length of 95% confidence intervals of  $\beta$  and the average area of 95% confidence area of two parameters under  $(\lambda, \beta) = (1, 0.6)$ .

$m$	Pivot	Left censoring		Middle censoring		Right censoring	
		Length	Area	Length	Area	Length	Area
Censoring plan $n = 10$		$r_1 = 2, r_i = 0, i \neq 1$		$r_4 = r_5 = 1, r_i = 0, i \neq 4; 5$		$r_8 = 2, r_i = 0, i \neq 8$	
8	$S(\beta)$	0.73897*		0.70524*		0.75209*	
	$h_1$	1.13216	3.17123	1.20391	4.19123	1.34655	6.64696
	$h_2$	1.03823	3.09715*	1.07789	3.89668*	1.24086	6.17553*
	$h_3$	1.04548	3.31815	1.08278	4.83205	1.28456	7.55938
	$h_4$	1.10951	3.66188	1.12258	8.49290	1.37448	8.74450
	$h_5$	1.19185	3.91359	1.20235	6.22807	1.51517	12.56590
	$h_6$	1.35055	4.70935	1.35872	12.03924	1.68076	20.94369
	$h_7$	1.63467	5.66392	1.59727	7.39456	1.88842	15.38141
Censoring plan $n = 9$		$r_1 = 1, r_i = 0, i \neq 1$		$r_5 = 1, r_i = 0, i \neq 5$		$r_9 = 1, r_i = 0, i \neq 9$	
9	$S(\beta)$	0.71126*		0.68924*		0.72229*	
	$h_1$	1.12934	2.62205	1.13736	2.68007	1.24564	3.75875
	$h_2$	0.99876	2.46415*	1.02277	2.52650*	1.10606	3.17476*
	$h_3$	0.99178	2.52565	0.99085	2.66485	1.10481	3.35447
	$h_4$	1.00367	2.80234	1.03061	2.92427	1.15440	14.65892
	$h_5$	1.07680	4.08639	1.08328	3.83779	1.22661	4.21509
	$h_6$	1.17053	3.45853	1.15690	3.56776	1.35099	5.48506
	$h_7$	1.32748	3.60746	1.33596	4.36517	1.53849	5.39825
10	$S(\beta)$	$r_i = 0, i \geq 1, 2, \dots, 10$					
	$S(\beta)$	0.68026*					
	$h_1$	1.10117	2.14160				
	$h_2$	0.96342	1.85100*				
	$h_3$	0.93020	1.92392				
	$h_4$	0.94571	1.95095				
	$h_5$	0.98578	2.30535				
	$h_6$	1.03772	2.39754				
18	$S(\beta)$	$r_1 = 3, r_i = 0, i \neq 1$		$r_7 = r_8 = 1, r_i = 0, i \neq 7, 8$		$r_{18} = 2, r_i = 0, i \neq 18$	
	$h_1$	0.60075*		0.55626*		0.59764*	
	$h_2$	0.87247	1.01865	0.88473	1.03705	0.96266	1.25911
	$h_3$	0.72762	0.83195	0.73189	0.84943	0.80233	1.03549
	$h_9$	0.66901	0.77496	0.68320	0.80879	0.74878	0.94473

(Continued)

Table 5. Continued

$m$	Pivot	Left censoring		Middle censoring		Right censoring	
		Length	Area	Length	Area	Length	Area
Censoring plan $n = 20$		$r_1 = 3, r_i = 0, i \neq 1$		$r_7 = r_8 = 1, r_i = 0, i \neq 7, 8$		$r_{18} = 2, r_i = 0, i \neq 18$	
	$h_4$	0.65076	0.75663	0.65692	0.77890	0.72632	0.92125
	$h_5$	0.64461	0.75848	0.64261	0.76207*	0.72108	0.92145
	$h_6$	0.63913	0.75926	0.64511	0.78779	0.71792	0.91582*
	$h_7$	0.64035	0.75199*	0.64422	0.77917	0.73399	0.96486
	$h_8$	0.64847	0.77942	0.64582	0.78406	0.74458	1.00374
	$h_9$	0.65922	0.79699	0.65901	0.82023	0.76884	1.06907
	$h_{10}$	0.68100	0.84405	0.66378	0.82264	1.06907	1.15163
	$h_{11}$	0.69856	0.85645	0.69341	0.86576	0.84185	1.18812
	$h_{12}$	0.73289	0.92516	0.72123	0.91480	0.87926	1.26295
	$h_{13}$	0.77325	0.97780	0.77312	1.01647	0.95849	1.41156
	$h_{14}$	0.84825	1.08057	0.83057	1.08588	1.06188	1.67781
	$h_{15}$	0.93381	1.18342	0.92346	1.23171	1.19700	1.87216
	$h_{16}$	1.10185	1.38006	1.09258	1.43061	1.38519	2.17074
	$h_{17}$	1.45237	1.61347	1.43508	1.63661	1.73538	2.42611
Censoring plan 19	$S(\theta)$	$r_1 = 1, r_i = 0, i \leq 18$		$r_9 = 1, r_i = 0, i \neq 9$		$r_{19} = 1, r_i = 0, i \neq 19$	
		0.56957*		0.54562*		0.57243*	
	$h_1$	0.86844	0.95069	0.87460	0.96962	0.91097	1.08781
	$h_2$	0.71901	0.78691	0.72690	0.81671	0.75384	0.86869
	$h_3$	0.66363	0.74539	0.66687	0.74901	0.70528	0.70528
	$h_4$	0.64440	0.73276	0.64391	0.73121	0.68142	0.80321
	$h_5$	0.63022	0.71102	0.63069	0.72889	0.67108	0.79573
	$h_6$	0.62165	0.70768*	0.62982	0.73488	0.66028	0.77252*
	$h_7$	0.62053	0.70798	0.62385	0.71058*	0.66556	0.78425
	$h_8$	0.62771	0.71414	0.62781	0.72695	0.67628	0.81440
	$h_9$	0.63022	0.71770	0.63157	0.72803	0.69407	0.83967
	$h_{10}$	0.64763	0.74964	0.64614	0.76072	0.70878	0.86124
	$h_{11}$	0.66869	0.78890	0.66505	0.79298	0.73083	0.89177
	$h_{12}$	0.69039	0.84316	0.67981	0.80078	0.76990	0.98492
	$h_{13}$	0.72290	0.85895	0.71798	0.87255	0.81050	1.03076
	$h_{14}$	0.76380	0.91250	0.76236	0.92557	0.86153	1.11764
	$h_{15}$	0.83182	1.01047	0.83054	1.05082	0.96282	1.31804
	$h_{16}$	0.93489	1.13083	0.92276	1.13492	1.07354	1.43047
	$h_{17}$	1.10352	1.33001	1.07994	1.28356	1.28259	1.76616
	$h_{18}$	1.45421	1.49428	1.43375	1.53469	1.61750	1.90659
Censoring plan 20	$S(\theta)$	$r_i = 0, i \geq 1, 2, \dots, 10$					
		0.54365*					
	$h_1$	0.86362	0.90409				
	$h_2$	0.71338	0.76681				
	$h_3$	0.65898	0.71364				
	$h_4$	0.63067	0.68294				
	$h_5$	0.61767	0.67109				
	$h_6$	0.61553	0.67695				
	$h_7$	0.60989	0.66940				
	$h_8$	0.60708	0.66894*				
	$h_9$	0.61530	0.68487				
	$h_{10}$	0.62402	0.69535				
	$h_{11}$	0.63354	0.70102				
	$h_{12}$	0.65114	0.73044				
	$h_{13}$	0.67569	0.77323				
	$h_{14}$	0.71122	0.83039				
	$h_{15}$	0.75994	0.88045				
	$h_{16}$	0.81645	0.97173				
	$h_{17}$	0.91780	1.08505				
	$h_{18}$	1.08822	1.22943				
	$h_{19}$	1.41250	1.45154				

\*The optimal one.

## 6. Conclusions

The topic of type-II progressive censoring has gained more and more attention in recent years. This article proposed the tests and interval estimations of the shape parameter of a new two-parameter distribution with the bathtub shape or increasing FRF under type-II progressive censoring based on  $m$  pivotal quantities. The pivotal quantity  $S(\beta)$  presented in this article is quite attractive since it can be computed easily and it always has the best performance than the other  $m-1$  pivotal quantities to do the tests or constructing the confidence interval for the scale parameter. We also propose the confidence region for the two parameters. The set of pivotal quantities  $(h_1, g)$  is the optimal one in terms of smallest confidence area. In this article, a numerical example is also given to illustrate all the proposed methods.

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