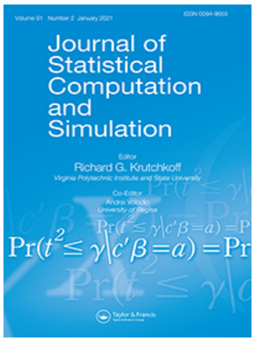

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Statistical inference based on progressively type-II censored data from the Burr X distribution under progressive-stress accelerated life test

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ABSTRACT
In this article, estimation problems for the Burr X distribution under a progressive-stress accelerated life test with progressive type-II censoring are studied. The stress is assumed to be a linearly increasing function of time. The inverse power law and the cumulative exposure model are considered. The classical and Bayesian estimations for the model parameters are obtained by using maximum likelihood method and Markov chain Monte Carlo technique,respectively. The asymptotic confidence intervals are constructed and highest posterior density intervals are also established. A simulation study is conducted to investigate the performance of the proposed point and interval estimations. Finally, a real data set is analysed for illustration.

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Bayes estimate; cumulative exposure model; highest posterior density interval; maximum likelihood method; Markov chain Monte Carlo; progressive censoring

1. Introduction

In recent years, the life testing and reliability experiments of products under normal operating conditions become a tedious job because of the increased average lifetimes of products due to the various technological advancements. Testing such highly reliable products under normal operating condition affects the overall cost per product. To deal with such situation and to get the failure times of products under some life testing experiment, in an affordable period of time, life testing experiments are conducted under higher than normal operating stress. The data obtained under such high-stress conditions are extrapolated to normal operating condition using some appropriate model. The higher stress loadings in accelerated life test (ALT) can be applied in different ways and the three well-known methods are constant-stress, step-stress, and progressive-stress. In constant-stress ALT, the products are tested under a constant-stress level till the test terminates where the test termination can be decided as per requirement such as by fixing the number of failures or by fixing the testing period. There are several researchers investigated constant-stress ALT, for reference one may see Kim and Bai [1], Watkins and John [2], Jaheen et al. [3], Guan et al. [4], and Mohie El-Din et al. [5,6]. In step-stress ALT, the products are tested with increasing stress

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levels step by step by fixing the stress change criteria either by fixing the stress changing time or by fixing the number of failures. The study of step-stress ALT is considered by several authors such as Miller and Nelson [7], Bai et al. [8], Gouno et al. [9], and Balakrishnan et al. [10]. Some recent articles, for example, by Mohie El-Din et al. [11,12], Zhang and Shi [13], Mohie El-Din et al. [14], Abdelmonem and Jaheen [15], and Guan and Tang [16] also considered step-stress ALT for their studies. Furthermore, in progressive-stress ALT, the products are placed in an environment where the stress increases continuously with time. If the stress increases linearly, we call such an experiment as a ramp-stress experiment. Authors such as Yin and Sheng [17], Bai et al. [18], Wang and Fei [19], Abdel-Hamid and AL-Hussaini [20,21], AL-Hussaini et al. [22], Abdel-Hamid and Abushal [23], and Mohie El-Din et al. [24] considered progressive-stress ALT for their studies.

It is observed that it is not possible or sometime not desirable to obtain the failure times of all the test units placed on a life testing experiment because of the associated costs such as high cost of per test units or limitations on experimental time, etc. Thus, such situations are handled by removal of test units before the actual failure occurs and are termed as censoring schemes. Since the removal of these test units can be done in various possible ways, these are further known as various types of censoring scheme. The two widely used censoring schemes are the type-I and type-II censoring schemes. The discussions of these two censoring schemes can be found in many articles. The main drawback of these schemes is that the removal of test units is allowed only after the termination of the experiment. Thus, a different censoring scheme known as progressive censoring was introduced in the literature which allows the removal of the test units during the life experiment. Progressive type-II censoring can be described as follows. Suppose that n test units are placed on a life experiment and m ($m \leq n$) is prefixed. At the occurrence of the first failure, we withdraw R_1 surviving units randomly. At the occurrence of the second failure, we withdraw R_2 surviving units randomly, and so on. Finally, at the occurrence of the mth failure, remaining $n - m - (R_1 + \cdots + R_{m-1})$ surviving units are all withdrawn from the experiment.

In this paper, we consider the progressive-stress ALT with progressive type-II censoring. The lifetime of the test unit is assumed to follow the Burr X distribution. Our aim is to discuss the estimation problems of model parameters based on classical and Bayesian frameworks. The rest of this paper is organized as follows. In Section 2, the considered model is described in brief by describing various assumptions for the progressive-stress ALT. In Section 3, the maximum likelihood estimates (MLEs) are derived. In Section 4, the Bayes estimates are obtained. The asymptotic confidence interval and highest posterior density (HPD) interval are constructed in Section 5. For illustration of the discussed methods, a real data set is analysed in Section 6. In Section 7, a Monte Carlo simulation is conducted to investigate the performance of the proposed methods. At last, in Section 8, some concluding remarks are made.

2. Model description

Among the various family of Burr distributions introduced by Burr [25], Burr X and Burr XII are the two which got the most attention for modelling data in various fields of study. Burr X distribution which is also known as generalized Rayleigh distribution has been considered widely for modelling strength and lifetime data by many authors under complete as well as censored data. It shows many common properties to the distributions such as Weibull, generalized exponential, and gamma. The cumulative distribution function (CDF) of a two-parameter Burr X distribution is given as

$$F(t) = (1 - \mathrm{e}^{-(\gamma t)^2})^{\alpha}, \quad t > 0,$$

and the corresponding probability density function (PDF) is given by

$$f(t) = 2\alpha\gamma^2 t\, \mathrm{e}^{-(\gamma t)^2}(1-\mathrm{e}^{-(\gamma t)^2})^{\alpha-1}, \quad t > 0,$$

where $\alpha > 0$ is the shape parameter and $\gamma > 0$ is the scale parameter. Some of the properties of Burr X distribution are studied by Raqab and Kundu [26]. They observed that the density function of Burr X distribution is decreasing for $\alpha \leq 1/2$, and for $\alpha > 1/2$, it is a right-skewed unimodal function. Similarly, the failure rate function of Burr X distribution is increasing for $\alpha \leq 1/2$, and is bathtub shaped for $\alpha > 1/2$.

Consider a progressive-stress ALT. We have a total of n units available for the life test. Let $S_1(t) < \cdots < S_d(t)$ be stress levels which are functions of time t. Suppose that, at stress level $S_i(t)$, n_i units are placed on a progressively type-II censored life test with censoring scheme $(R_{i1}, \ldots, R_{im_i})$, $i = 1, 2, \ldots, d$, where $\sum_{i=1}^{d} n_i = n$. That is, as the first failure $t_{i1:m_i:n_i}$ occurs, R_{i1} live units are randomly selected and removed. As the second failure $t_{i2:m_i:n_i}$ occurs, R_{i2} of the surviving units are removed, and so on. This experiment terminates at the time when the m_ith failure $t_{im_i:m_i:n_i}$ is observed and the remaining $n_i - m_i - (R_{i1} + \cdots + R_{im_i})$ surviving units are all removed. Thus, the observed progressively censored data under the progressive-stress $S_i(t)$ are $t_{i1:m_i:n_i} < t_{i2:m_i:n_i} < \cdots < t_{im_i:m_i:n_i}$, $i = 1, 2, \ldots, d$.

The following assumptions are made for the progressive-stress ALT framework:

(1) For any stress setting, the lifetime distribution of the test unit has a Burr X distribution.
(2) The progressive-stress $S(t)$ is a linearly increasing function of time t, i.e. $S(t) = vt$, $v > 0$.
(3) The relationship between life characteristic γ and the stress level $S(t)$ is described by inverse power law, i.e. $\gamma(t) = 1/(a[S(t)]^b)$, where $a > 0$ and $b > 0$ are parameters to be estimated.
(4) The linear cumulative exposure model is considered to deal with the effect of stress change from one stress level to another, for more details, see Nelson [27].

From the assumption of the linear cumulative exposure model, the CDF of a test unit under progressive-stress $S_i(t)$ can be written as

$$G_i(t) = F_i(\Delta(t)), \quad i = 1, 2, \ldots, d,$$

where $\Delta(t) = \int_0^t 1/\gamma_i(u)\, \mathrm{d}u = av_i^b t^{b+1}/(b+1)$ and $F_i(\cdot)$ is the CDF of Burr X distribution under progressive-stress $S_i(t)$ with scale parameter taken as 1. Then, we have

$$G_i(t) = \left[1 - \exp\left\{-\left(\frac{av_i^b t^{b+1}}{b+1}\right)^2\right\}\right]^{\alpha}, \quad t > 0, \tag{1}$$

and the corresponding PDF is given by

$$g_i(t) = \frac{2\alpha\left(av_i^b\right)^2 t^{2b+1}}{b+1} \times \exp\left\{-\left(\frac{av_i^b t^{b+1}}{b+1}\right)^2\right\}\left[1-\exp\left\{-\left(\frac{av_i^b t^{b+1}}{b+1}\right)^2\right\}\right]^{\alpha-1}, \quad t>0. \tag{2}$$

3. Maximum likelihood estimation

In this section, the MLEs of parameters a, b, and α are obtained based on progressively type-II censored data under progressive-stress ALT. Let $t_{ij:m_i:n_i}$ be the observed failure times obtained from a progressively type-II censored test with censoring scheme R_{ij} under progressive-stress level $S_i(t)$, $i = 1, \dots, d$ and $j = 1, \dots, m_i$. For simplicity of notation, we use t_{ij} instead of $t_{ij:m_i:n_i}$ throughout the paper. From Balakrishnan and Aggarwala [28], the likelihood function can be written as

$$L(a,b,\alpha) \propto \prod_{i=1}^{d}\prod_{j=1}^{m_i} g_i(t_{ij})[1-G_i(t_{ij})]^{R_{ij}}.$$

Then, using (1) and (2), we can obtain the likelihood function for our model as

$$L(a,b,\alpha) \propto \prod_{i=1}^{d}\prod_{j=1}^{m_i} \frac{2\alpha(av_i)^2 t_{ij}^{2b+1}}{b+1}\, e^{-(av_i^b t_{ij}^{b+1}/(b+1))^2}(1-e^{-(av_i{}^b t_{ij}^{b+1}/(b+1))^2})^{\alpha-1} \times (1-(1-e^{-(av_i^b t_{ij}^{b+1}/(b+1))^2})^{\alpha})^{R_{ij}}.$$

Then, the log-likelihood function can be written as

$$l(a,b,\alpha) \propto (\log\alpha + 2\log a)\sum_{i=1}^{d} m_i + 2d\sum_{i=1}^{d} m_i \log v_i + (2b+1)\sum_{i=1}^{d}\sum_{j=1}^{m_i}\log t_{ij} - \sum_{i=1}^{d}\sum_{j=1}^{m_i}\eta(t_{ij}) + (\alpha-1)\sum_{i=1}^{d}\sum_{j=1}^{m_i}\log(1-\eta(t_{ij})) + \sum_{i=1}^{d}\sum_{j=1}^{m_i} R_{ij}\log(1-(1-\eta(t_{ij}))^{\alpha}),$$

where $\eta(t_{ij}) = (av_i^b t_{ij}^{b+1}/(b+1))^2$. The likelihood equations for the parameters a, b and α are, respectively, given by

$$\frac{\partial l}{\partial a} = \frac{2}{a}\sum_{i=1}^{d} m_i - \frac{2a}{(b+1)^2}\sum_{i=1}^{d}\sum_{j=1}^{m_i}(v_i^b t_{ij}^{b+1})^2 + \frac{2a(\alpha-1)}{(b+1)^2}\sum_{i=1}^{d}\sum_{j=1}^{m_i}\frac{(v_i^b t_{ij}^{b+1})^2\, e^{-\eta(t_{ij})}}{1-e^{-\eta(t_{ij})}}$$

$$- \frac{2a\alpha}{(b+1)^2} \sum_{i=1}^{d} \sum_{j=1}^{m_i} R_{ij} \frac{(v_i^b t_{ij}^{b+1})^2 \,\mathrm{e}^{-\eta(t_{ij})} (1 - \mathrm{e}^{-\eta(t_{ij})})^{\alpha-1}}{\left(1 - (1 - \mathrm{e}^{-\eta(t_{ij})})^{\alpha}\right)},$$

$$\frac{\partial l}{\partial b} = -\frac{1}{b+1} \sum_{i=1}^{d} m_i + 2 \sum_{i=1}^{d} m_i \log v_i + 2 \sum_{i=1}^{d} \sum_{j=1}^{m_i} \log t_{ij}$$

$$- 2 \sum_{i=1}^{d} \sum_{j=1}^{m_i} \eta(t_{ij}) \left(\log v_i + \log t_{ij} - \frac{1}{b+1} \right)$$

$$+ 2(\alpha - 1) \sum_{i=1}^{d} \sum_{j=1}^{m_i} \frac{\eta(t_{ij}) \,\mathrm{e}^{-\eta(t_{ij})}}{1 - \mathrm{e}^{-\eta(t_{ij})}} \left(\log v_i + \log t_{ij} - \frac{1}{b+1} \right)$$

$$- 2\alpha \sum_{i=1}^{d} \sum_{j=1}^{m_i} \frac{\eta(t_{ij}) \,\mathrm{e}^{-\eta(t_{ij})} (1 - \mathrm{e}^{-\eta(t_{ij})})^{\alpha-1}}{(1 - (1 - \mathrm{e}^{-\eta(t_{ij})})^{\alpha})} \left(\log v_i + \log t_{ij} - \frac{1}{b+1} \right),$$

and

$$\frac{\partial l}{\partial \alpha} = \frac{1}{\alpha} \sum_{i=1}^{d} m_i + \sum_{i=1}^{d} \sum_{j=1}^{m_i} \log(1 - \mathrm{e}^{-\eta(t_{ij})}) - \sum_{i=1}^{d} \sum_{j=1}^{m_i} \frac{(1 - \mathrm{e}^{-\eta(t_{ij})})^{\alpha} \log(1 - \mathrm{e}^{-\eta(t_{ij})})}{\left(1 - (1 - \mathrm{e}^{-\eta(t_{ij})})^{\alpha}\right)}.$$

These likelihood equations can be solved to obtain the required MLEs of the parameters a, b, and α. Since these equations are highly nonlinear and solving them analytically is difficult and unfeasible, some numerical techniques such as the Newton–Raphson method or a quasi-Newton method (e.g. [29]) have to be used for solving these equations.

4. Bayesian estimation

In this section, the Bayes estimates of model parameters a, b, and α are obtained under squared error loss function. It is assumed that the parameters a and b have non-informative priors and α has a gamma prior with hyper-parameters (p, q). That is, the prior distributions of a, b, and α are given as follows:

$$\pi_1(a) \propto \frac{1}{a}, \quad a > 0, \quad \pi_2(b) \propto \frac{1}{b}, \quad b > 0,$$

and

$$\pi_3(\alpha) \propto \alpha^{p-1} \,\mathrm{e}^{-\alpha/q}, \quad \alpha > 0, \; p, q > 0.$$

We further assume that a, b, and α are independent. Then, the joint prior PDF of a, b and α is given by

$$\pi(a, b, \alpha) \propto \frac{\alpha^{p-1}}{ab} \,\mathrm{e}^{-\alpha/q}, \quad a > 0, b > 0, \alpha > 0.$$

Thus, the joint posterior density of a, b, and α given $\boldsymbol{t} = (t_{11}, \ldots, t_{1m_1}, \ldots, t_{d1}, \ldots, t_{dm_d})$ can be obtained as

$$\pi(a, b, \alpha \mid \boldsymbol{t}) \propto L(a, b, \alpha) \pi(a, b, \alpha)$$

$$\propto \alpha^{p-1+\sum_{i=1}^{d} m_i} a^{-1+2\sum_{i=1}^{d} m_i} e^{-\alpha/q} \frac{(b+1)^{-\sum_{i=1}^{d} m_i}}{b} \times \prod_{i=1}^{d}\prod_{j=1}^{m_i} (v_i^b t_{ij}^{b+1})^2 e^{-\eta(t_{ij})} (1-e^{-\eta(t_{ij})})^{\alpha-1} (1-(1-e^{-\eta(t_{ij})})^{\alpha})^{R_{ij}}.$$

Under squared error loss function, the Bayes estimate of a parameter is equal to its posterior mean. Since it is difficult to find the analytical Bayes estimates for these parameters a, b, and α, we will use the Markov chain Monte Carlo (MCMC) technique to obtain the desired estimates.

4.1. MCMC method

To obtain the Bayes estimates of the parameters a, b, and α under the progressive-stress ALT, samples are generated from the posterior distribution. The conditional posterior distributions for the model parameters a, b, and α are, respectively, given by

$$\pi(a \mid b, \alpha, \boldsymbol{t}) = a^{-1+2\sum_{i=1}^{d} m_i} \prod_{i=1}^{d}\prod_{j=1}^{m_i} e^{-\eta(t_{ij})} (1-e^{-\eta(t_{ij})})^{\alpha-1} (1-(1-e^{-\eta(t_{ij})})^{\alpha})^{R_{ij}},$$

$$\pi(b \mid a, \alpha, \boldsymbol{t}) = \frac{(b+1)^{-\sum_{i=1}^{d} m_i}}{b} \prod_{i=1}^{d}\prod_{j=1}^{m_i} v_i^{2b} t_{ij}^{2b+1} e^{-\eta(t_{ij})} (1-e^{-\eta(t_{ij})})^{\alpha-1} \times (1-(1-e^{-\eta(t_{ij})})^{\alpha})^{R_{ij}},$$

and

$$\pi(\alpha \mid a, b, \boldsymbol{t}) = \alpha^{p-1+\sum_{i=1}^{d} m_i} e^{-\alpha/q} \prod_{i=1}^{d}\prod_{j=1}^{m_i} (1-e^{-\eta(t_{ij})})^{\alpha-1} (1-(1-e^{-\eta(t_{ij})})^{\alpha})^{R_{ij}}.$$

Since the conditional posterior distributions of the parameters a, b, and α are not reducible into the forms of some well-known distributions, we use Metropolis–Hastings algorithm proposed by Metropolis et al. [30] and Hastings [31] to generate posterior samples. If the conditional posterior distributions of the parameters are unimodal and are roughly symmetric, then these can be approximated by normal distribution. Figures 1–3 show that the conditional distributions of a, b, and α are unimodal and very much symmetric by visual inspection. Therefore, to generate random samples from these conditional distributions, we use the following steps from Metropolis–Hastings algorithm:

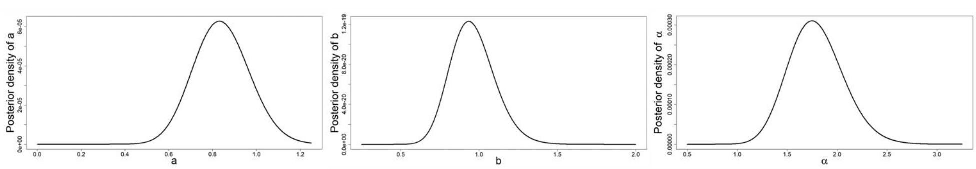

Figure 1. The posterior density of a (left), b (middle), and α (right) for $d = 2$.

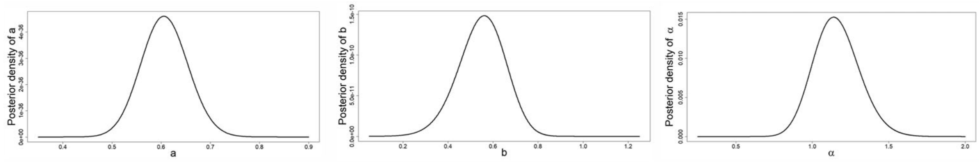

Figure 2. The posterior density of a (left), b (middle), and α (right) for $d = 3$.

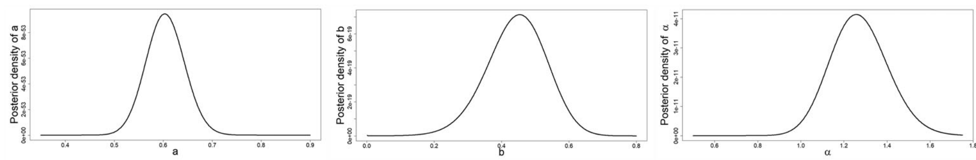

Figure 3. The posterior density of a (left), b (middle) and α (right) for $d = 4$.

Step 1: Set initial guesses for the parameters (a, b, α) as (a_0, b_0, α_0).
Step 2: Set $i = 1$.
Step 3: Generate $a \sim N(a_{i-1}, \sigma_{11})$, $b \sim N(b_{i-1}, \sigma_{22})$, and $\alpha \sim N(\alpha_{i-1}, \sigma_{33})$, where σ_{ii} denotes the (i, i)th entry of the variance–covariance matrix $\mathbf{\Sigma}$.
Step 4: Compute $P = \pi(a_i, b_i, \alpha_i \mid \boldsymbol{t})/\pi(a_{i-1}, b_{i-1}, \alpha_{i-1} \mid \boldsymbol{t})$.
Step 5: Accept (a_i, b_i, α_i) with probability $\min\{1, P\}$.
Step 6: Repeat Steps Step 3–5 B times to obtain B number of samples for the parameters (a, b, α).

Finally, remove the first B_0 samples to discard the possible dependency on the initial guesses to obtain the required approximate estimates under the squared error loss function as follows:

$$\hat{a}^* = \frac{1}{B'} \sum_{j=1}^{B'} a_j, \quad \hat{b}^* = \frac{1}{B'} \sum_{j=1}^{B'} b_j, \quad \text{and} \quad \hat{\alpha}^* = \frac{1}{B'} \sum_{j=1}^{B'} \alpha_j,$$

where $B' = B - B_0$ and B_0 is also known as the number of burn-in samples.

5. Interval estimations

In this section, we consider the construction of asymptotic confidence and the HPD intervals for the parameters a, b, and α.

5.1. Asymptotic confidence interval

Here, we obtain the asymptotic confidence interval by using the asymptotic normality property of MLEs of the parameters a, b, and α. It is well known that the MLEs $(\hat{a}, \hat{b}, \hat{\alpha})$ is asymptotically normal with mean vector (a, b, α) and variance–covariance matrix $\mathbf{\Sigma}$. Hence, the $100(1 - \gamma)\%$ asymptotic confidence intervals for parameters $(\theta_1, \theta_2, \theta_3) =$

(a, b, α) are given as

$$(\hat{\theta}_i - z_{1-\gamma/2}\sqrt{\sigma_{ii}}\,,\ \hat{\theta}_i + z_{1-\gamma/2}\sqrt{\sigma_{ii}}), \quad i = 1, 2, 3,$$

where σ_{ii} is the (i, i)th element of the variance–covariance matrix $\mathbf{\Sigma}$.

5.2. Highest posterior density interval

In this subsection, the construction of HPD interval (L, U) is considered for a random quantity θ which is defined as

$$p(L \leq \theta \leq U) = \int_L^U \pi^*(\theta \mid \boldsymbol{t})\,\mathrm{d}\theta = 1 - \gamma.$$

To obtain the HPD intervals for the parameters (a, b, α), we can use the posterior samples obtained in the Section 4 and follow the method discussed by Chen and Shao [32].

6. Illustrative example

In the section, we analyse a real data set reported in Stone [33] (also see, Lawless [34]) for illustration of the methods discussed in this paper. The report by Stone [33] gave the lifetimes of specimens of solid epoxy electrical-insulation studied under an accelerated voltage life test with three levels of voltage: 52.5, 55.0 and 57.5 kV. The obtained failure times with their voltage levels are tabulated in Table 1.

The failure times are given in minutes and for computational relevance the data is divided by 1440 to convert the unit of the data into days. To check the goodness-of-fit of the data to Burr X distribution, the Kolmogorov–Smirnov (K-S) distance as well as p-values are obtained separately for the data obtained under the three voltage levels and are tabulated in the Table 2. It is observed that the p-values for the data under the three ramp-stress levels are all greater than 0.05 implying that the data sets show good fit to Burr X distribution.

We obtain the MLEs and also the Bayes estimates under non-informative prior setup for the parameters a, b, and α and are obtained as (0.2726, 0.1689, 0.2759) and (0.351302, 0.1515, 0.3148), respectively. Further, constructions of 95% asymptotic and HPD intervals are considered. The asymptotic confidence intervals for the parameters a, b, and α are obtained as $(0.2106, 0.3345)$, $(0.1307, 0.2071)$, and $(0.1984, 0.3536)$, respectively.

Table 1. Failure times of epoxy electrical-insulation specimens at various voltage levels.

Voltage (kV)	Failure times (min)
52.5	4690, 740, 1010, 1190, 2450, 1390, 350, 6095, 3000, 1458, 6200*, 550, 1690, 745, 1225, 1480, 245, 600, 246, 1805
55.0	258, 114, 312, 772, 498, 162, 444, 1464, 132, 1740*, 1266, 300, 2440*, 520, 1240, 2600*, 222, 144, 745, 396
57.5	510, 1000*, 252, 408, 528, 690, 900*, 714, 348, 546, 174, 696, 294, 234, 288, 444, 390, 168, 558, 288

*Denote censoring times of the specimens.

The HPD intervals for a, b, and α are obtained as $(0.3322, 0.3742)$, $(0.1374, 0.1681)$, and $(0.2839, 0.3590)$, respectively. It is observed that the Bayes estimates are better than MLEs in terms of associated risk. The HPD intervals are better than asymptotic confidence intervals in terms of interval lengths.

Furthermore, the MLEs of the scale parameter σ under normal operating condition with the design stress (voltage) $S_0 = 50\,\text{kV}$ is obtained as $\hat{\gamma} = 1/(\hat{a}[S_0]^{\hat{b}}) = 1.8946$. Using the MLEs $\hat{\alpha}$ and $\hat{\gamma}$, the MLEs of failure rate, reliability function and mean time to failure (MTTF) can also be obtained under normal operating condition.

7. Simulation study

For investigating the performance of the methods discussed in this paper, a Monte Carlo simulation is conducted. The MLEs and the Bayes estimates are compared in terms of their MSEs. The asymptotic confidence interval (ACI) and HPD interval are compared in terms of average interval lengths and coverage probabilities. The simulation study is conducted by designing three different progressive-stress ALTs. The first is a simple ramp-stress ALT with only 2 stress levels ($d = 2$), the second a multiple ramp-stress ALT with 3 stress levels ($d = 3$), and the third a multiple ramp-stress ALT with 4 stress levels ($d = 4$). We consider three censoring schemes as follows:

$$\text{CS1:}\quad R_{ij} = \begin{cases} n_i - m_i, & j = 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\text{CS2:}\quad R_{ij} = \begin{cases} 1, & j = 1, \ldots, n_i - m_i, \\ 0, & \text{otherwise,} \end{cases}$$

$$\text{CS3:}\quad R_{ij} = \begin{cases} 2m_i - n_i + j, & j = 1, \ldots, n_i - m_i, \\ 0, & \text{otherwise.} \end{cases}$$

The true values of parameters a, b, and α are assumed to be 0.75, 0.5, and 0.95, respectively. For Bayesian estimation, we set hyper-parameters to be $p = 5$ and $q = 5.263158$. For the first ramp-stress ALT, we set ramp-stresses as $v_1 = 2$ and $v_2 = 4$; for the second ramp-stress ALT, we set ramp-stresses as $v_1 = 2$, $v_2 = 3$, and $v_3 = 4$; and for the third ramp-stress ALTs, the ramp-stresses are set as $v_1 = 2$, $v_2 = 2.5$, $v_3 = 3$, and $v_4 = 4$. The results obtained in the simulation are presented in the Tables 3–5. All the results are based on 1000 simulation samples.

It can be observed that, for all the three ramp-stress ALT designs, the MSEs for parameters using the Bayes method are smaller than those using maximum likelihood method except for a few cases. The interval lengths of HPD intervals are smaller than those of

Table 2. The K-S distance and the p-values.

Voltage level	52.5 kV	55.0 kV	57.5 kV
K-S distance	0.21733	0.20447	0.12199
p-value	0.26110	0.25450	0.96201

Table 3. Average estimates with MSEs (in parenthesis) and average interval lengths with coverage probabilities (in parenthesis) for simple ramp-stress ALT ($d = 2$).

n_i	m_i	CS	θ	Point estimate: MLE	Point estimate: Bayes	Interval length: ACI	Interval length: HPD
$n_i = \begin{cases} 15, & i=1 \\ 20, & i=2 \end{cases}$	$m_i = \begin{cases} 10, & i=1 \\ 15, & i=2 \end{cases}$	1	a	0.5686428 (0.04167)	0.728257 (0.00311)	0.128548 (0.967)	0.054039 (0.988)
			b	0.707544 (0.08451)	0.243851 (0.11417)	0.510219 (0.991)	0.341708 (0.998)
			α	1.184469 (0.16425)	0.962667 (0.00213)	0.123959 (0.958)	0.049937 (0.972)
		2	a	0.656560 (0.01368)	0.730760 (0.00367)	0.142887 (0.960)	0.069862 (0.975)
			b	0.660162 (0.03387)	0.261288 (0.11102)	0.581764 (0.986)	0.461087 (0.991)
			α	1.397855 (0.31472)	0.968965 (0.00370)	0.143346 (0.899)	0.054342 (0.953)
		3	a	0.785169 (0.01206)	0.769672 (0.00628)	0.167373 (0.966)	0.101260 (0.972)
			b	0.689683 (0.05471)	0.465112 (0.07019)	0.600910 (0.978)	0.581058 (0.983)
			α	1.240655 (0.16792)	0.959870 (0.00228)	0.123356 (0.961)	0.056087 (0.970)
$n_i = \begin{cases} 20, & i=1 \\ 30, & i=2 \end{cases}$	$m_i = \begin{cases} 16, & i=1 \\ 25, & i=2 \end{cases}$	1	a	0.555531 (0.04077)	0.742707 (0.00282)	0.123959 (0.948)	0.062169 (0.957)
			b	0.826988 (0.11488)	0.397588 (0.03997)	0.314775 (0.938)	0.304089 0.963
			α	1.045095 (0.04910)	0.962229 (0.00372)	0.123959 (0.964)	0.061185 (0.967)
		2	a	0.557517 (0.04207)	0.747931 (0.00280)	0.123615 (0.955)	0.063008 (0.977)
			b	0.815429 (0.10678)	0.370585 (0.04427)	0.311442 (0.979)	0.301191 (0.982)
			α	1.005728 (0.02425)	0.963509 (0.00424)	0.141579 (0.963)	0.078726 (0.971)
		3	a	0.698664 (0.00882)	0.773995 (0.00325)	0.141967 (0.956)	0.079108 (0.974)
			b	0.778876 (0.09369)	0.521607 (0.04175)	0.338361 (0.979)	0.340890 (0.981)
			α	0.995957 (0.01249)	0.945809 (0.00325)	0.139550 (0.957)	0.069287 (0.968)
$n_i = \begin{cases} 40, & i=1 \\ 50, & i=2 \end{cases}$	$m_i = \begin{cases} 30, & i=1 \\ 35, & i=2 \end{cases}$	1	a	0.563077 (0.03885)	0.721477 (0.00327)	0.125281 (0.973)	0.062860 (0.953)
			b	0.783415 (0.10257)	0.394168 (0.04746)	0.280779 (0.969)	0.222065 (0.974)
			α	1.143049 (0.13660)	0.983518 (0.00340)	0.123959 (0.963)	0.063125 (0.959)
		2	a	0.594139 (0.02721)	0.721040 (0.00500)	0.129540 (0.946)	0.068625 (0.955)
			b	0.747829 (0.06784)	0.391797 (0.03117)	0.290667 (0.978)	0.249161 (0.984)
			α	1.189172 (0.07672)	0.982197 (0.01301)	0.184072 (0.947)	0.091023 (0.939)
		3	a	0.752345 (0.00445)	0.769979 (0.00327)	0.131654 (0.961)	0.072651 (0.972)
			b	0.740425 (0.06833)	0.534372 (0.03025)	0.308323 (0.974)	0.295047 (0.986)
			α	1.116402 (0.04392)	0.950360 (0.00277)	0.123959 (0.953)	0.065471 (0.961)

Table 4. Average estimates with MSEs (in parenthesis) and average interval lengths with coverage probabilities (in parenthesis) for multiple ramp-stress ALT ($d = 3$).

n_i	m_i	CS	θ	Point estimate: MLE	Point estimate: Bayes	Interval length: ACI	Interval length: HPD
$n_i = \begin{cases} 15, & i=1 \\ 10, & i=2 \\ 10, & i=3 \end{cases}$	$m_i = \begin{cases} 12, & i=1 \\ 7, & i=2 \\ 6, & i=3 \end{cases}$	1	a	0.587664 (0.03328)	0.728013 (0.00349)	0.127143 (0.966)	0.055664 (0.971)
			b	0.730694 (0.07891)	0.233964 (0.12703)	0.484887 (0.972)	0.364939 (0.985)
			α	1.228186 (0.19051)	0.970446 (0.00510)	0.126917 (0.938)	0.054177 (0.969)
		2	a	0.627356 (0.02076)	0.739907 (0.00302)	0.128707 (0.938)	0.056902 (0.953)
			b	0.719056 (0.06062)	0.240543 (0.11649)	0.505154 (0.974)	0.431871 (0.983)
			α	1.233161 (0.13607)	0.953197 (0.00194)	0.123959 (0.962)	0.049405 (0.981)
		3	a	0.771628 (0.01396)	0.778156 (0.00801)	0.168108 (0.955)	0.100417 (0.948)
			b	0.713134 (0.06882)	0.376004 (0.08762)	0.518257 (0.982)	0.605123 (0.988)
			α	1.063044 (0.03881)	0.957370 (0.00222)	0.123132 (0.963)	0.055461 (0.974)
$n_i = \begin{cases} 20, & i=1 \\ 15, & i=2 \\ 15, & i=3 \end{cases}$	$m_i = \begin{cases} 15, & i=1 \\ 13, & i=2 \\ 13, & i=3 \end{cases}$	1	a	0.562749 (0.04005)	0.732683 (0.00315)	0.129803 (0.961)	0.065229 (0.970)
			b	0.820774 (0.11923)	0.413726 (0.04423)	0.333031 (0.991)	0.308172 (0.994)
			α	1.015684 (0.04782)	0.966450 (0.00467)	0.148844 (0.958)	0.073812 (0.972)
		2	a	0.565982 (0.03794)	0.741043 (0.00213)	0.123772 (0.971)	0.062982 (0.977)
			b	0.818243 (0.10717)	0.374838 (0.05009)	0.327227 (0.980)	0.307380 (0.992)
			α	1.022265 (0.02400)	0.969040 (0.00319)	0.125978 (0.973)	0.066176 (0.967)
		3	a	0.692774 (0.00754)	0.757686 (0.00220)	0.124121 (0.966)	0.063525 (0.972)
			b	0.812255 (0.10417)	0.547536 (0.03622)	0.343732 (0.997)	0.352176 (0.993)
			α	0.978731 (0.01277)	0.943573 (0.00287)	0.123959 (0.965)	0.060432 (0.966)
$n_i = \begin{cases} 35, & i=1 \\ 30, & i=2 \\ 25, & i=3 \end{cases}$	$m_i = \begin{cases} 30, & i=1 \\ 20, & i=2 \\ 15, & i=3 \end{cases}$	1	a	0.567882 (0.03865)	0.737851 (0.00260)	0.122617 (0.957)	0.055957 (0.971)
			b	0.759067 (0.09098)	0.296863 (0.07547)	0.290312 (0.984)	0.217999 (0.979)
			α	1.170235 (0.18246)	0.984202 (0.00801)	0.144478 (0.949)	0.074242 (0.952)
		2	a	0.612112 (0.02113)	0.737359 (0.00290)	0.130072 (0.945)	0.062699 (0.966)
			b	0.737248 (0.06012)	0.347622 (0.04897)	0.305776 0.976	0.269518 (0.992)
			α	1.187781 (0.07189)	0.972113 (0.00532)	0.140077 (0.967)	0.077979 (0.973)
		3	a	0.751516 (0.00429)	0.773339 (0.00300)	0.126231 (0.956)	0.066446 (0.973)
			b	0.744431 (0.07209)	0.500450 (0.02636)	0.314939 (0.991)	0.307502 (0.988)
			α	1.094196 (0.02903)	0.949368 (0.00301)	0.123959 (0.974)	0.062630 (0.982)

Table 5. Average estimates with MSEs (in parenthesis) and average interval lengths with coverage probabilities (in parenthesis) for multiple ramp-stress ALT ($d = 4$).

				Point estimate		interval length	
n_i	m_i	CS	θ	MLE	Bayes	ACI	HPD
$n_i = \begin{cases} 10, & i=1 \\ 10, & i=2 \\ 10, & i=3 \\ 5, & i=4 \end{cases}$	$m_i = \begin{cases} 8, & i=1 \\ 7, & i=2 \\ 7, & i=3 \\ 3, & i=4 \end{cases}$	1	a	0.610431	0.740282	0.130383	0.056286
				(0.02675)	(0.00265)	(0.951)	(0.966)
			b	0.725977	0.230053	0.500096	0.423988
				(0.06840)	(0.12608)	(0.969)	(0.987)
			α	1.240787	0.967178	0.122972	0.050850
				(0.20276)	(0.00280)	(0.954)	(0.971)
		2	a	0.636283	0.735512	0.135427	0.061978
				(0.02447)	(0.00485)	(0.966)	(0.982)
			b	0.743451	0.309270	0.503084	0.471396
				(0.06746)	(0.12842)	(0.963)	(0.957)
			α	1.222220	0.957003	0.123959	0.054703
				(0.11374)	(0.00292)	0.933	0.950
		3	a	0.792861	0.763362	0.166689	0.101501
				(0.02155)	(0.00533)	(0.942)	(0.937)
			b	0.720829	0.380858	0.532976	0.565468
				(0.07104)	(0.10967)	(0.964)	(0.955)
			α	1.068871	0.945560	0.122728	0.049741
				(0.04151)	(0.00181)	(0.931)	(0.943)
$n_i = \begin{cases} 15, & i=1 \\ 13, & i=2 \\ 12, & i=3 \\ 10, & i=4 \end{cases}$	$m_i = \begin{cases} 13, & i=1 \\ 10, & i=2 \\ 10, & i=3 \\ 8, & i=4 \end{cases}$	1	a	0.587099	0.734379	0.125612	0.059830
				(0.03238)	(0.00264)	(0.953)	(0.955)
			b	0.822748	0.409939	0.335165	0.333577
				(0.11406)	(0.05495)	(0.960)	(0.973)
			α	1.038893	0.974319	0.127987	0.071568
				(0.04178)	(0.00551)	(0.954)	(0.959)
		2	a	0.585140	0.746663	0.123675	0.055150
				(0.03162)	(0.00234)	(0.956)	(0.966)
			b	0.798213	0.327170	0.340084	0.313187
				(0.09345)	(0.06844)	(0.963)	(0.971)
			α	1.021276	0.970991	0.125814	0.064987
				(0.01795)	(0.00422)	(0.944)	(0.952)
		3	a	0.705557	0.768714	0.127049	0.065061
				(0.00730)	(0.00303)	(0.918)	(0.927)
			b	0.786791	0.497537	0.3619241	0.369320
				(0.08764)	(0.04862)	(0.943)	(0.947)
			α	1.018823	0.946112	0.123959	0.060926
				(0.01773)	(0.00295)	(0.926)	(0.939)
$n_i = \begin{cases} 15, & i=1 \\ 13, & i=2 \\ 12, & i=3 \\ 10, & i=4 \end{cases}$	$m_i = \begin{cases} 13, & i=1 \\ 10, & i=2 \\ 10, & i=3 \\ 8, & i=4 \end{cases}$	1	a	0.579791	0.718424	0.122967	0.056159
				(0.03264)	(0.00307)	(0.954)	(0.957)
			b	0.754913	0.339442	0.311662	0.245782
				(0.08514)	(0.06166)	(0.963)	(0.960)
			α	1.171164	0.987788	0.130850	0.063200
				(0.12418)	(0.00781)	(0.964)	(0.970)
		2	a	0.616809	0.724489	0.126834	0.059665
				(0.01911)	(0.00273)	(0.951)	(0.963)
			b	0.749321	0.371549	0.316667	0.291490
				(0.06479)	(0.04059)	(0.972)	(0.982)
			α	1.131062	0.960194	0.133706	0.066742
				(0.04310)	(0.00342)	(0.955)	(0.963)
		3	a	0.772081	0.776622	0.128714	0.067416
				(0.00979)	(0.00495)	(0.954)	(0.973)
			b	0.728740	0.474201	0.337036	0.339555
				(0.07209)	(0.04654)	(0.969)	(0.955)
			α	1.049155	0.949439	0.123959	0.060040
				(0.02136)	(0.00345)	(0.957)	(0.971)

ACIs. We also observe that, in most of the cases, the coverage probabilities of HPD intervals is higher than those of ACIs. Looking at the various results, it can be concluded that the Bayes estimates show better result than the counterpart. Moreover, on comparing the results for the simple ramp-stress and the two multiple ramp-stress ALT designs, it is difficult to choose a better one though simple ramp-stress ALT design performs better in many cases.

8. Conclusion

In this article, a progressive-stress ALT is studied for Burr X distribution under progressively type-II censored data. Classical and Bayes estimates for model parameters are obtained using the maximum likelihood method as well as MCMC method. Further, the asymptotic confidence intervals are constructed and HPD intervals are also constructed using the posterior samples obtained by Metropolis–Hastings algorithm. A numerical example is studied for illustrating the proposed methods. A simulation study is also performed. In the simulation study, the MLEs and Bayes estimates are compared by the average estimates and MSEs. The asymptotic confidence intervals and HPD intervals are compared in terms of their average interval lengths and the coverage probabilities. The simulation results show that the Bayes estimates and HPD intervals perform better than the classical point estimates and confidence intervals.

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No potential conflict of interest was reported by the authors.

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