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The Evaluation on the Process Capability Index C_L for Exponentiated Frech'et Lifetime Product under Progressive Type I Interval Censoring

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Abstract: We present the likelihood inferences on the lifetime performance index C_L to evaluate the performance of lifetimes of products following the skewed Exponentiated Frech'et distribution in many manufacturing industries. This research is related to the topic of skewed Probability Distributions and Applications across Disciplines. Exponentiated Frech'et distribution is a generalization of some lifetime distributions. The maximum likelihood estimator for C_L for lifetimes with exponentiated Frech'et distribution is derived to develop a computational testing procedure so that experimenters can implement it to test whether the lifetime performance reached the pre-assigned level of significance with a given lower specification limit under progressive type I interval censoring. At the end, two examples are provided to demonstrate the implementation on the algorithm for our proposed computational testing procedure.

Keywords: progressive type I interval censoring; lifetime performance index; exponentiated Frech'et distribution; maximum likelihood estimator; hypothesis testing procedure

1. Introduction

In this artificial intelligence era, the constantly changing of technology makes production techniques become sophisticated and complicated. The longer lifetime of products has an economic benefit related to an increase in the overall competitiveness of companies via the increase of the value added to products (see Montalvo et al. [1]). There are many process capability indices are developed (see Montgomery [2]) to control the quality of products with unilateral specifications or bilateral specifications. The lifetime performance index C_L we used is an index with unilateral specification and the larger lifetime will result in a larger index. In this research, we make use of this index to assess the lifetime performance for products with exponentiated Frech'et distribution lifetime. For other kinds of lifetime distributions, Tong et al. [3] proposed a testing algorithm for exponential distribution lifetime based on a complete sample.

For some reasons including the limitation of time, shortage of material resources, some restrictions on the cost or time, the experimenters will not be able to collect data of lifetimes of all products completely and thus only the censored data is collected. There are two types of censoring frequently considered by researchers. One is the type II censoring which has fa ixed number of samples observed and the other is type I censoring which terminates the experiment at fixed time T. In this research, we focus on the progressive type I interval sampling since it has the advantage of the convenience of collecting data. There are some notations for this type of censoring needed to be addressed as follows: Set up the experimental time as T and the number of intervals as m. Therefore, the quality engineer will collect the data of number of failure items $(X_1, \ldots, X_m)$ at the time points $(t_1, \ldots, t_m)$, where $T = t_m$. At each time point, $(R_1, \ldots, R_m)$ are removed sequentially with removing rates of $(p_1, \ldots, p_m)$, where $R_i \sim$ binomial$(n - \sum_{j=1}^{i} X_j - \sum_{j=1}^{i-1} R_j, p_i)$, $0 \le p_i \le 1$ and $p_m = 1$.

For this type of interval censoring, Utilize the maximum likelihood estimator (MLE) for C_L to propose a hypothesis testing procedure for various kinds of lifetime distributions can be referred to Wu and Lin [4] for one-parameter exponential distribution; Wu and Lin [5] for Weibull distribution; Wu and Lu [6] for Pareto products; Wu [7] for Chen lifetime distribution; Wu et al. [8] for Burr XII distribution; Wu et al. [9] for Rayleigh distribution; Wu and Hsieh [10] for Gompertz distribution. The Frech'et distribution is first introduced in Frech'et [11] and it has good applications on flood flows. To generalize the Frech'et distribution, Nadarajah and Kotz [12] proposed a new lifetime distribution called the exponentiated Frech'et distribution (EF distribution) through exponentiation of Frech'et distribution. It also has good application on lifetimes of products (see examples in Rao et al. [13]). In this paper, we derived the maximum likelihood estimators of the index of C_L for lifetimes following EF distribution and utilized this estimator to construct a testing procedure to reach a specified level of significance under a lower specification limit L.

The organization of the whole manuscript is summarized as follows: We introduced the lifetime performance index for products following EF lifetimes and the monotonic increasing relationship between this index and the conforming rate is further addressed in Section 2. In Section 3.1, we derived the maximum likelihood estimator of this index and its asymptotic distribution so that we can construct a hypothesis testing procedure based on the asymptotic sampling distribution of this estimator. We plotted power curves and constructed some tables to analyze the power performance of our proposed testing procedure and the impacts of different setup are also analyzed. In order to guide users to implement our proposed procedures, we give two practical examples to demonstrate the steps to execute the proposed testing procedure and make a decision to see if the process meets the pre-assigned lower specification limit in Section 3.2. In the end, we summarized the findings and conclusions for this paper in the Conclusions.

2. The Monotonic Relationship between the Lifetime Performance Index and the Conforming Rate

Let the random variable U represent the lifetime of the product following an EF distribution. The probability density function (pdf) $f_U(u)$, cumulative distribution function (cdf) $F_U(u)$ and hazard function $h_U(u)$ are defined in Equations (1)–(3):

$$f_U(u) = \theta \delta u^{-\delta-1} e^{-u^{-\delta}} \left(1 - e^{-u^{-\delta}}\right)^{\theta-1}, \ u > 0, \delta > 0, \theta > 0, \tag{1}$$

$$F_U(u) = 1 - \left(1 - e^{-u^{-\delta}}\right)^{\theta}, \ u > 0, \delta > 0, \theta > 0. \tag{2}$$

$$h_U(u) = \frac{f_U(u)}{1 - F_U(u)} = \theta \delta\, u^{-\delta-1} e^{-u^{-\delta}} \left(1 - e^{-u^{-\delta}}\right)^{-1}, \ u > 0, \delta > 0, \theta > 0. \tag{3}$$

The pdf for δ = 1, 3, 5 under θ = 2 is displayed in Figure 1a and for θ = 2, 4, 6 under δ = 1 is displayed in Figure 1b. It is shown that it's a right-skewed distribution. The hazard function for δ = 1, 3, 5 under θ = 2 is displayed in Figure 2a and for θ = 2, 4, 6 under δ = 1 is displayed in Figure 2b. It is shown that the hazard function has mono peak shape and the hazard rate is an increasing function of parameter δ when θ is fixed and is also an increasing function of parameter θ when δ is fixed.

Transform the lifetime U to the new lifetime variable Y through $Y = -\ln\left(1 - e^{-U^{-\delta}}\right)$. Then, this new random variable Y has one-parameter exponential distribution and its pdf, cdf and hazard function are given in Equations (4)–(6):

$$f_Y(y) = \theta e^{-\theta y}, \ y > 0, \theta > 0 \tag{4}$$

$$F_Y(y) = 1 - e^{-\theta y}, \ y > 0, \theta > 0 \tag{5}$$
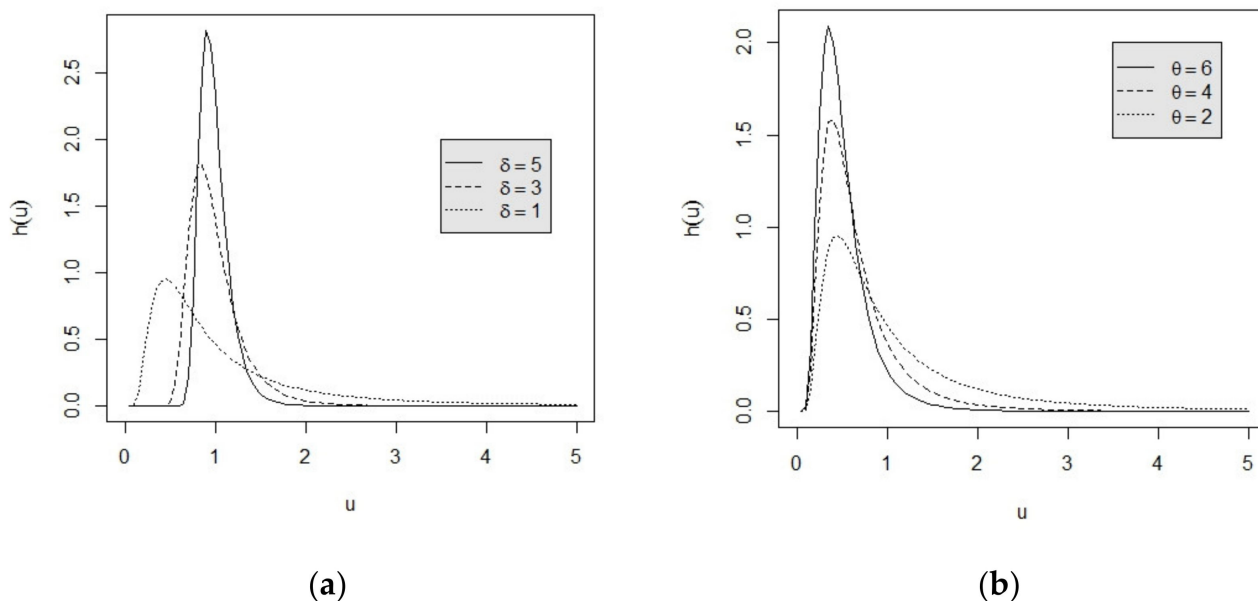
$$h_Y(y) = \frac{f_y(y)}{1 - F_y(y)} = \theta \tag{6}$$

(a)   **(b)**

Figure 1. (**a**) pdf curve for δ = 1, 3, 5 under θ = 2; (**b**) pdf curve for θ = 2, 4, 6 under δ = 1.

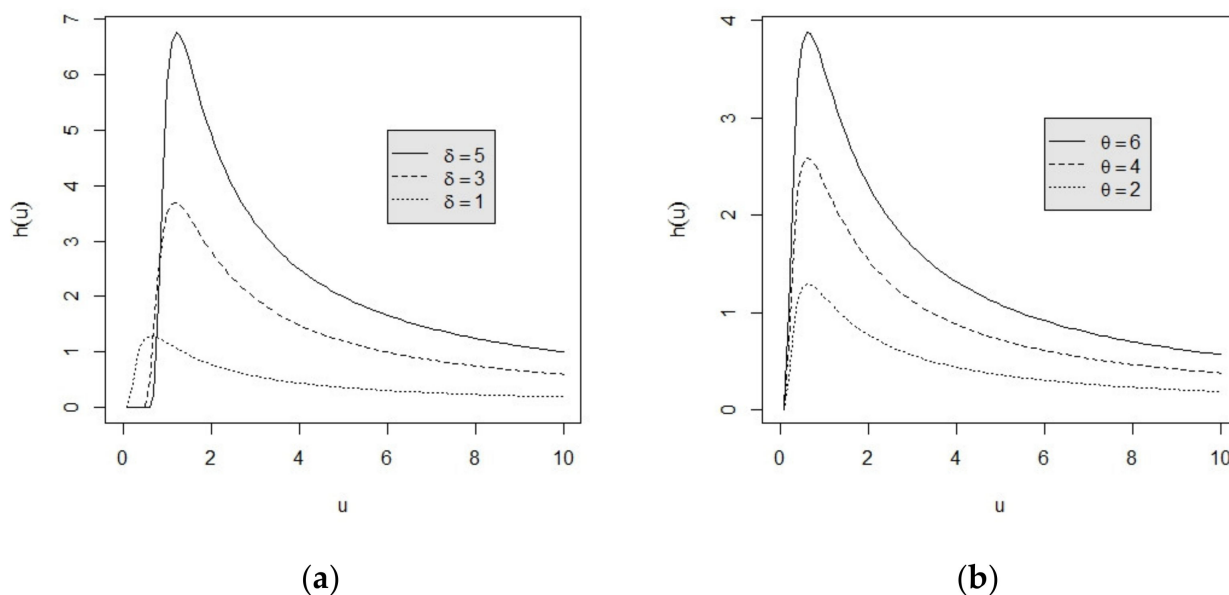**(a)**   **(b)**

Figure 2. (**a**) hazard function for δ = 1, 3, 5 under θ = 2; (**b**) hazard function for θ = 2, 4, 6 under δ = 1.

The parameter δ is assumed to be known and this value can be determined by the Gini test with maximum *p*-value, where the Gini test (see Gill and Gastwirth [14]) is a scale-free goodness of fit for exponential distribution. The steps to implement this test is depicted as follows: At first, we set up the null hypothesis as $H_0 : U_i \sim F_U(u) = 1 - \left(1 - e^{-u^{-\delta}}\right)^{\theta}$, $u > 0$, $\delta > 0, \theta > 0$. Secondly, calculate the Gini test statistic $G_n = \dfrac{\sum\limits_{i=1}^{n-1} i(n-i)(Y_{(i+1)} - Y_{(i)})}{(n-1)\sum\limits_{i=1}^{n} (n-i+1)(Y_{(i)} - Y_{(i-1)})}$, where $Y_{(i)} = -\ln(1 - e^{-U_{(i)}^{-\delta}})$ and $U_{(1)} < U_{(2)} < \ldots < U_{(n)}$. For large sample, the asymmetric distribution of $Z = \sqrt{12(n-1)}(G_n - 0.5)$ is a standard normal distribution. Then, we calculate the observed value of Z as z. Hence, the *p*-value to test if the data is following the EF distribution can be calculated as $2P(Z > |z|)$. The *p*-value is a function of δ. The value of δ is determined with the maximum *p*-value. The lifetime of products is a unilateral

specification quality characteristic and the larger the lifetime would be more attractive to picky consumers and to increase the sales in these competitive emerging markets. Let L be the pre-assigned lower specification limit in the quality control and an item is considered to be conformative if its lifetime exceeds this lower specification. Montgomery [2] proposed the process capability index C_L in Equation (7), where μ represents the mean of the process and σ represents the standard deviation of the process.

$$C_L = \frac{\mu - L}{\sigma} \tag{7}$$

This index is called the lifetime performance index this index and it is frequently used to assess the performance of lifetime. Using the pdf defined in Equation (4), the mean and standard deviation for the new lifetime variable Y can be found as $\mu = E(Y) = \frac{1}{\theta}$ and $\sigma = \sqrt{Var(Y)} = \frac{1}{\theta}$. If L_U is the given lower specification limit for lifetime variable U, then $L = -\ln\left(1 - e^{-L_U^{-\delta}}\right)$ is the new lower specification limit for new lifetime variable Y. Replacing μ and σ by $\frac{1}{\theta}$ in Equation (7), this index C_L can be rewritten as

$$C_L = \frac{\mu - L}{\sigma} = \frac{\frac{1}{\theta} - L}{\frac{1}{\theta}} = 1 - \theta L \tag{8}$$

Observe that this index is a non-increasing function of scale parameter and so is the hazard function. It means that the larger this index, the smaller the hazard rate.

The conforming rate is defined as the proportion of products with lifetime exceeding the given lower specification limit and it is calculated in Equation (9):

$$P_r = P(U \geq L_U) = P(Y \geq L) = \exp(-\theta L) = \exp(C_L - 1), -\infty < C_L < 1. \tag{9}$$

It is observed that there is a monotonic increasing relationship between the conforming rate P_r and the lifetime performance index C_L. For example, if the experimenter hoped P_r to be greater than 0.8187308, then C_L must be greater than 0.80 to attain the desired conforming rate.

3. Results

3.1. The Maximum Likelihood Estimator for the Lifetime Performance Index and the Testing Procedure

Let $X_1, \ldots, X_m$ be the progressive type I interval censored sample observed at time points $t_1, \ldots, t_m$ under random progressive censoring scheme $R_1, \ldots, R_m$ with removal percentages $p_1, \ldots, p_m$, where $0 \leq p_i \leq 1$ and $p_m = 1$. The random variable X_i is the number of failure items among $n - \sum_{i=1}^{i-1} X_i - \sum_{i=1}^{i-1} R_i$ items on the life test in the i*th* time interval $[t_{i-1}, t_i]$. The failure rate q_i in the ith time interval is

$$q_i = \frac{F_U(t_i) - F_U(t_{i-1})}{1 - F_U(t_{i-1})} = \frac{\left(1 - e^{-t_{i-1}^{-\delta}}\right)^{\theta} - \left(1 - e^{-t_i^{-\delta}}\right)^{\theta}}{\left(1 - e^{-t_{i-1}^{-\delta}}\right)^{\theta}} = 1 - \left(\frac{1 - e^{-t_i^{-\delta}}}{1 - e^{-t_{i-1}^{-\delta}}}\right)^{\theta}.$$

The distribution of X_i is denoted as

$$X_i | X_{i-1}, \ldots, X_1, R_{i-1}, \ldots, R_1 \sim \text{Binomial}\left(n - \sum_{i=1}^{i-1} X_i - \sum_{i=1}^{i-1} R_i, q_i\right), i = 1, \ldots, m. \tag{10}$$

The random variable R_i is the number of removing items at time point t_i from the remaining $n - \sum_{i=1}^{i} X_i - \sum_{i=1}^{i-1} R_i$ items on the life test.

The distribution of R_i is denoted as

$$R_i | X_i, \ldots, X_1, R_{i-1}, \ldots, R_1 \sim \text{Binomial}\left(n - \sum_{i=1}^{i} X_i - \sum_{i=1}^{i-1} R_i, p_i\right), i = 1, \ldots, m. \quad (11)$$

From the distributions of $X_1, \ldots, X_m$ and $R_1, \ldots, R_m$ given in Equations (10) and (11), we can obtain the likelihood function based on progressive type I interval censored sample $X_1, \ldots, X_m$ as

$$L(\theta) \propto \prod_{i=1}^{m} (F_U(t_i) - F_U(t_{i-1}))^{X_i} (1 - F_U(t_i))^{R_i}$$

$$\propto \prod_{i=1}^{m} \left(\left(1 - e^{-t_{i-1}^{-\delta}}\right)^{\theta} - \left(1 - e^{-t_i^{-\delta}}\right)^{\theta} \right)^{X_i} \left(\left(1 - e^{-t_i^{-\delta}}\right)^{\theta} \right)^{R_i}$$

$$\propto \prod_{i=1}^{m} \left(1 - \left(\frac{1 - e^{-t_i^{-\delta}}}{1 - e^{-t_{i-1}^{-\delta}}}\right)^{\theta} \right)^{X_i} \left(\left(1 - e^{-t_{i-1}^{-\delta}}\right)^{\theta} \right)^{X_i} \left(\left(1 - e^{-t_i^{-\delta}}\right)^{\theta} \right)^{R_i} \quad (12)$$

We can obtain the log-likelihood function

$$\ln L(\theta) = \sum_{i=1}^{m} X_i \ln\left(1 - \left(\frac{1 - e^{-t_i^{-\delta}}}{1 - e^{-t_{i-1}^{-\delta}}}\right)^{\theta} \right) - \theta \sum_{i=1}^{m} \left(R_i \ln\left(1 - e^{-t_i^{-\delta}}\right) + X_i \ln\left(1 - e^{-t_{i-1}^{-\delta}}\right) \right). \quad (13)$$

Then, the log-likelihood equation can be obtained as

$$\frac{d}{d\theta} \ln L(\theta) = \sum_{i=1}^{m} X_i \frac{\left(\ln\left(1 - e^{-t_{i-1}^{-\delta}}\right) - \ln\left(1 - e^{-t_i^{-\delta}}\right) \right) \left(1 - \left(\frac{1-e^{-t_i^{-\delta}}}{1-e^{-t_{i-1}^{-\delta}}}\right)^{\theta} \right)}{1 - \left(\frac{1-e^{-t_i^{-\delta}}}{1-e^{-t_{i-1}^{-\delta}}}\right)^{\theta}}$$

$$- \sum_{i=1}^{m} \left(R_i \ln\left(1 - e^{-t_i^{-\delta}}\right) + X_i \ln\left(1 - e^{-t_{i-1}^{-\delta}}\right) \right) = 0 \quad (14)$$

The solution of the above log-likelihood equation is the MLE of θ and is denoted by $\hat{\theta}$. There is no close-form for this solution and it can only be solved numerically. From chapter 10 of Casella and Berger [15], the limiting distribution of MLE is a normal distribution with mean of θ and variance of the reciprocal of the Fisher's information $I(\lambda)$, where

$$I(\lambda) = -E\left[\frac{d^2 \ln L(\theta)}{d\theta^2}\right].$$

From Equation (14), we can obtain

$$\frac{d^2}{d\theta^2} \ln L(\lambda) = -\sum_{i=1}^{m} X_i \frac{\left(\left(\ln\left(1 - e^{-t_{i-1}^{-\delta}}\right) - \ln\left(1 - e^{-t_i^{-\delta}}\right) \right) \right)^2 \left(\frac{1-e^{-t_i^{-\delta}}}{1-e^{-t_{i-1}^{-\delta}}}\right)^{\theta}}{\left(1 - \left(\frac{1-e^{-t_i^{-\delta}}}{1-e^{-t_{i-1}^{-\delta}}}\right)^{\theta} \right)^2} \quad (15)$$

Refer to Equation (15) of Wu [7], we have

$$E(X_i) = EE(X_i | X_{i-1}, \ldots, X_1, R_{i-1}, \ldots, R_1)$$

$$= n q_i \prod_{j=1}^{i-1} (1 - p_j)(1 - q_j), i = 1, \ldots, m, \quad (16)$$

where

$$q_i = 1 - \left(\frac{1 - e^{-t_i^{-\delta}}}{1 - e^{-t_{i-1}^{-\delta}}}\right)^\theta, i = 1, \ldots, m.$$

Using Equation (16), we can obtain the Fisher's information number as

$$I(\theta) = -E\left[\frac{d^2 \ln L(\theta)}{d\theta^2}\right]$$

$$= -\sum_{i=1}^m nq_i \frac{\left(\left(\ln\left(1 - e^{-t_{i-1}^{-\delta}}\right) - \ln\left(1 - e^{-t_i^{-\delta}}\right)\right)\right)^2 \left(\frac{1 - e^{-t_i^{-\delta}}}{1 - e^{-t_{i-1}^{-\delta}}}\right)^\theta}{\left(1 - \left(\frac{1 - e^{-t_i^{-\delta}}}{1 - e^{-t_{i-1}^{-\delta}}}\right)^\theta\right)^2} \prod_{j=1}^{i-1}(1 - p_j)(1 - q_j) \tag{17}$$

Then, the limiting distribution of the MLE of θ is determined as $\hat{\theta} \to N(\theta, I^{-1}(\theta))$.
By the property of the invariance of MLE, the MLE of C_L can be obtained as

$$\hat{C}_L = 1 - \hat{\theta}L \tag{18}$$

with limiting distribution as

$$\hat{C}_L = 1 - \hat{\theta}L \xrightarrow[m \to \infty]{d} N\left(C_L, L^2 V(\hat{\theta})\right) \tag{19}$$

It's always more convenient for experimenter to collect sample by considering equal interval lengths t for all m intervals such that $t_i - t_{i-1} = t, i = 1, \ldots, m$. Therefore, we have $t_i = it, i = 1, \ldots, m$. For this case, the log-likelihood equation in Equation (14) became

$$\frac{d}{d\theta} \ln L(\theta) = \sum_{i=1}^m X_i \frac{\left(\ln\left(1 - e^{-((i-1)t)^{-\delta}}\right) - \ln\left(1 - e^{-(it)^{-\delta}}\right)\right)\left(1 - \left(\frac{1 - e^{-(it)^{-\delta}}}{1 - e^{-((i-1)t)^{-\delta}}}\right)^\theta\right)}{1 - \left(\frac{1 - e^{-(it)^{-\delta}}}{1 - e^{-((i-1)t)^{-\delta}}}\right)^\theta}$$

$$- \sum_{i=1}^m \left(R_i \ln\left(1 - e^{-(it)^{-\delta}}\right) + X_i \ln\left(1 - e^{-((i-1)t)^{-\delta}}\right)\right) = 0 \tag{20}$$

The information number became

$$I(\theta) = \sum_{i=1}^m nq_i \frac{\left(\ln\left(1 - e^{-((i-1)t)^{-\delta}}\right) - \ln\left(1 - e^{-(it)^{-\delta}}\right)\right)^2 \left(\frac{1 - e^{-(it)^{-\delta}}}{1 - e^{-((i-1)t)^{-\delta}}}\right)^\theta}{\left(1 - \left(\frac{1 - e^{-(it)^{-\delta}}}{1 - e^{-((i-1)t)^{-\delta}}}\right)^\theta\right)^2} \prod_{j=1}^{i-1}(1 - p_j)(1 - q_j) \tag{21}$$

Making use of the maximum likelihood estimator of C_L as the testing statistic, a testing procedure is developed as follows:

Let c_0 be the desired level for a quality engineer so that the manufacturing process is capable if the lifetime performance index C_L exceeds c_0. The null hypothesis and alternative hypothesis are set up as follows:

$H_0 : C_L \le c_0$ (the process is not capable) vs. $H_a : C_L > c_0$ (the process is capable). Let C_L^0 be the critical value and C_L^0 is determined as follows:

$$\sup P\left(\hat{C}_L > C_L^0 | C_L \le c_0\right) = \sup P\left(1 - \hat{\theta}L > C_L^0 \middle| \theta \ge \frac{1 - c_0}{L}\right) = \sup P\left(\hat{\theta} < \frac{1 - C_L^0}{L} \middle| \theta \ge \frac{1 - c_0}{L}\right)$$

$$= \sup P\left(Z < \left(\frac{1 - C_L^0}{L} - \theta\right) \middle/ \sqrt{w(\theta)} \middle| \theta \ge \frac{1 - c_0}{L}\right) = \alpha,$$

where

$$w(\theta) = V(\hat{\theta}), \ Z = \frac{\hat{\theta} - \theta}{\sqrt{w(\theta)}} \xrightarrow[m \to \infty]{d} N(0, 1).$$

When $\theta = \theta_0 = \frac{1-c_0}{L}$, the sup can be attained.
Therefore,

$$P\left(Z < \left(\frac{1 - C_L^0}{L} - \theta_0\right)\Big/\sqrt{w(\theta_0)}\,\Big|\,\theta_0 = \frac{1-c_0}{L}\right) = \alpha \Rightarrow$$

We yield the critical value as

$$C_L^0 = 1 - L\left(\theta_0 + Z_\alpha\sqrt{w(\theta_0)}\right), \tag{22}$$

where Z_α denotes the left-tailed αth percentile of a standard normal distribution.
Thus, the critical values can be determined as

$$C_L^0 = 1 - L\left(\theta_0 + Z_\alpha\sqrt{w(\theta_0)}\right), \text{ where } \theta_0 = \frac{1-c_0}{L} \tag{23}$$

This manufacturing process is concluded to be capable if $\hat{C}_L > C_L^0$.

The steps to implement the hypothesis are summarized as follows:

Step 1: Specify the level of significance α and the lower specification L_U for lifetime U. Thus we can obtain the lower specification $L = -\ln\left(1 - e^{-L_U^{-\delta}}\right)$ for the new lifetime Y. The progressive type I interval censored sample $X_1, \ldots, X_m$ is collected at the pre-set times $t_1, \ldots, t_m$ with censoring schemes of $R_1, \ldots, R_m$ with removal probability $p_1, \ldots, p_m$ from the EF distribution.

Step 2: For a given conforming rate P_r, one can determine the required level $c_0 = 1 + \ln(P_r)$. Then, the null and alternative hypothesis $H_0 : C_L \leq c_0$ and $H_a : C_L > c_0$ are determined.

Step 3: Find the maximum likelihood estimator of C_L as $\hat{C}_L = 1 - \hat{\theta}L$, where $\hat{\theta}$ is the maximum likelihood estimator of θ by solving the log-likelihood equation defined in Equation (14).

Step 4: Compute the critical value $C_L^0 = 1 - L\left(\theta_0 + Z_\alpha\sqrt{w(\theta_0)}\right)$, where $\theta_0 = \frac{1-c_0}{L}$.

Step 5: If $\hat{C}_L \in (C_L^0, \infty)$, we can conclude that this manufacturing process is capable. Otherwise, this process is not capable.

In order to conduct the power analysis, the power of the proposed procedure is needed to be computed.

At the point of $C_L = c_1 > c_0$ in alternative hypothesis, let $\delta = c_1 - c_0$ be the deviation of true parameter c_1 from the desired target c_0. The power $h(\delta)$ is

$$h(\delta) = P\left(\hat{C}_L > C_L^0 | c_1 = 1 - \theta_1 L\right)$$

$$= P\left(1 - \hat{\theta}L > 1 - L\left(\theta_0 + Z_\alpha\sqrt{w(\theta_0)}\right)\Big|\,\theta_1 = \frac{1-c_1}{L}\right)$$

$$= P\left(\hat{\theta} < \theta_0 + Z_\alpha\sqrt{w(\theta_0)}\,\Big|\,\theta_1 = \frac{1-c_1}{L}\right)$$

$$= P\left(\frac{\hat{\theta} - \theta_1}{\sqrt{w(\theta_1)}} < \frac{\left(\theta_0 + Z_\alpha\sqrt{w(\theta_0)}\right) - \theta_1}{\sqrt{w(\theta_1)}}\,\Big|\,\theta_1 = \frac{1-c_1}{L}\right)$$

$$= \Phi\left(\frac{\delta/L + Z_\alpha\sqrt{w(\theta_0)}}{\sqrt{w(\theta_1)}}\right) \tag{24}$$

where $\Phi(\cdot)$ is the cdf for Z and $Z \sim N(0,1)$, $\theta_0 = \frac{1-c_0}{L}$ and $\theta_1 = \frac{1-c_1}{L}$. Apparently, the power is an increasing function of δ.

Given the lower specification limit $L_U = 0.06306$, then the lower specification limit for new lifetime variable Y is $L = 0.05$. Let the total experimental time be $T = 0.5$. Suppose

the quality engineer wants the conforming rate P_r to be greater than 0.8187308, then the target level $c_0 = 0.8$, i.e., the null hypothesis is set up to be $H_0 : C_L \leq 0.8$. Testing this hypothesis, the power $h(\delta)$ in Equation (24) are computed and listed in Tables 1 and A1 and Table A2 in Appendix A at $\alpha = 0.01, 0.05, 0.1$ respectively for $\delta = 0(0.025)0.125$, $m = 3(1)6$, $n = 60(20)100$ and $p = 0, 0.05, 0.075$. For some typical cases, the power curves are obtained in Figures 3–6.

Table 1. The power $h(\delta)$ at $\alpha = 0.01$.

					δ			
m	*n*	*p*	0.000	0.025	0.050	0.075	0.100	0.125
3	60	0.000	0.0100	0.0344	0.1114	0.3130	0.6696	0.9540
		0.050	0.0100	0.0333	0.1054	0.2939	0.6382	0.9403
		0.075	0.0100	0.0328	0.1025	0.2845	0.6223	0.9324
	80	0.000	0.0100	0.0436	0.1643	0.4672	0.8540	0.9952
		0.050	0.0100	0.0421	0.1552	0.4418	0.8301	0.9927
		0.075	0.0100	0.0414	0.1508	0.4293	0.8174	0.9910
	100	0.000	0.0100	0.0534	0.2225	0.6082	0.9447	0.9996
		0.050	0.0100	0.0514	0.2101	0.5802	0.9311	0.9994
		0.075	0.0100	0.0504	0.2041	0.5662	0.9234	0.9991
4	60	0.000	0.0100	0.0386	0.1365	0.3933	0.7839	0.9868
		0.050	0.0100	0.0369	0.1263	0.3621	0.7443	0.9787
		0.075	0.0100	0.0360	0.1215	0.3471	0.7238	0.9733
	80	0.000	0.0100	0.0497	0.2027	0.5689	0.9282	0.9994
		0.050	0.0100	0.0473	0.1873	0.5309	0.9054	0.9986
		0.075	0.0100	0.0461	0.1801	0.5122	0.8924	0.9980
	100	0.000	0.0100	0.0616	0.2746	0.7140	0.9803	1.0000
		0.050	0.0100	0.0582	0.2541	0.6761	0.9707	0.9999
		0.075	0.0100	0.0567	0.2443	0.6569	0.9647	0.9999
5	60	0.000	0.0100	0.0418	0.1562	0.4539	0.8502	0.9956
		0.050	0.0100	0.0394	0.1418	0.4114	0.8071	0.9908
		0.075	0.0100	0.0383	0.1352	0.3913	0.7839	0.9873
	80	0.000	0.0100	0.0544	0.2329	0.6396	0.9609	0.9999
		0.050	0.0100	0.0509	0.2112	0.5914	0.9409	0.9996
		0.075	0.0100	0.0493	0.2012	0.5676	0.9288	0.9994
	100	0.000	0.0100	0.0678	0.3149	0.7806	0.9918	1.0000
		0.050	0.0100	0.0632	0.2863	0.7363	0.9852	1.0000
		0.075	0.0100	0.0611	0.2730	0.7134	0.9807	1.0000

From Table 1, Table A1, Table A2 and Figures 3–6, we found that the power $h(\delta)$ is an increasing function of δ for any combinations of n, m, p and α. That is if the true index c_1 is further deviated from the target level c_0, we obtained the higher test power as expected; The power $h(\delta)$ is an increasing function of n for fixed m, p and α as expected; Not surprisingly, the power $h(\delta)$ is an increasing function of m for fixed n, p and α; The power $h(\delta)$ is an increasing function of α for any combinations of n, m and p as expected; The power $h(\delta)$ is a decreasing function of p for fixed n, m and α as expected.
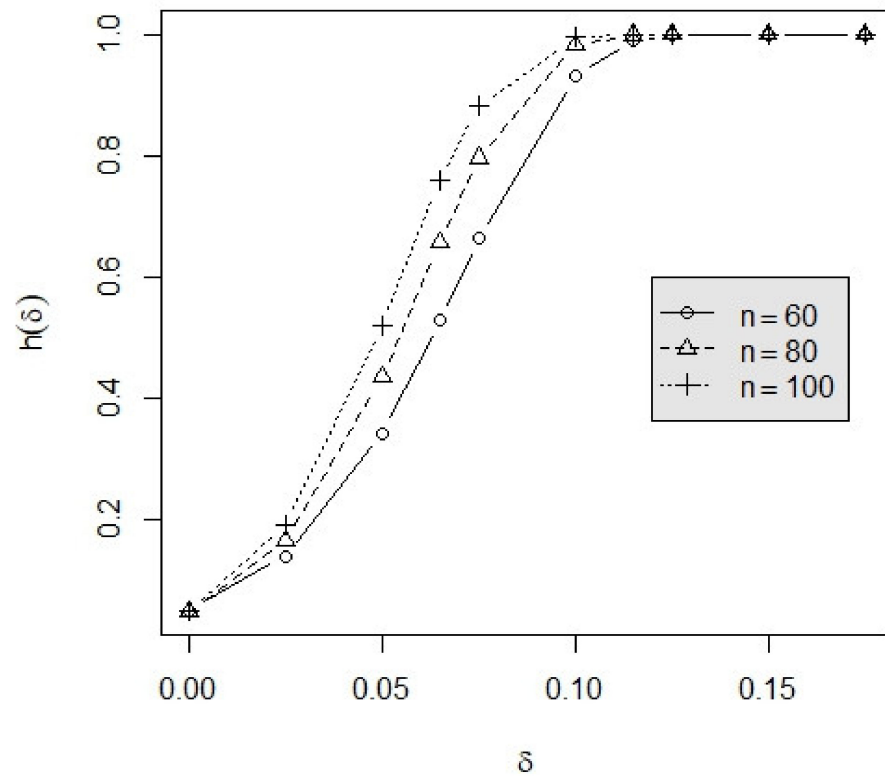
Figure 3. Power function for the test at $\alpha = 0.05$ under $m = 3$ and $p = 0.05$.
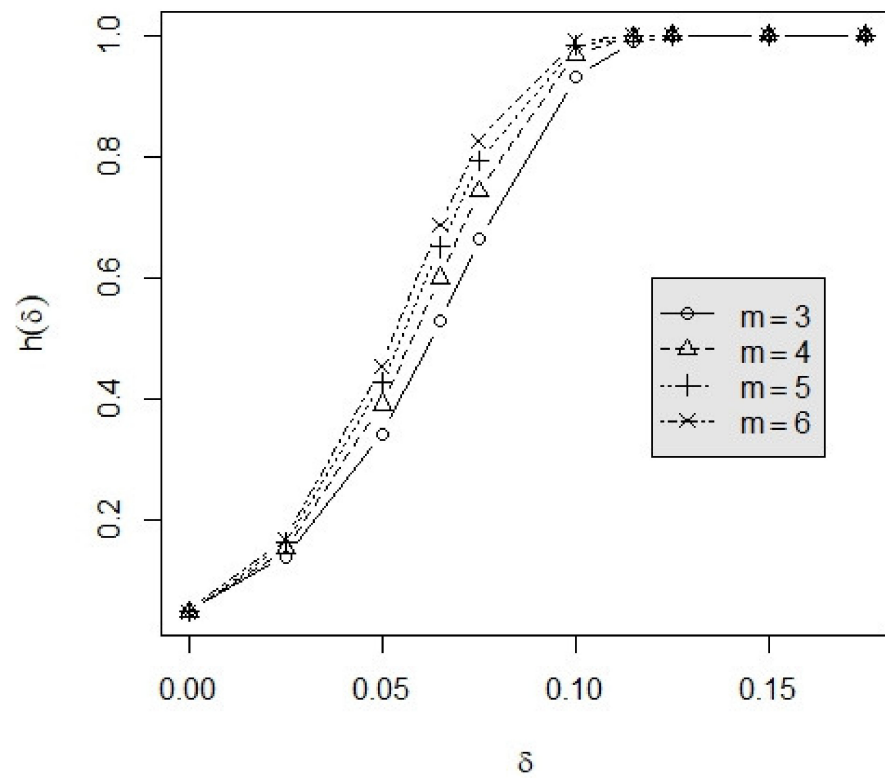

Figure 4. Power function for the test at $\alpha = 0.05$ under $n = 60$ and $p = 0.05$.
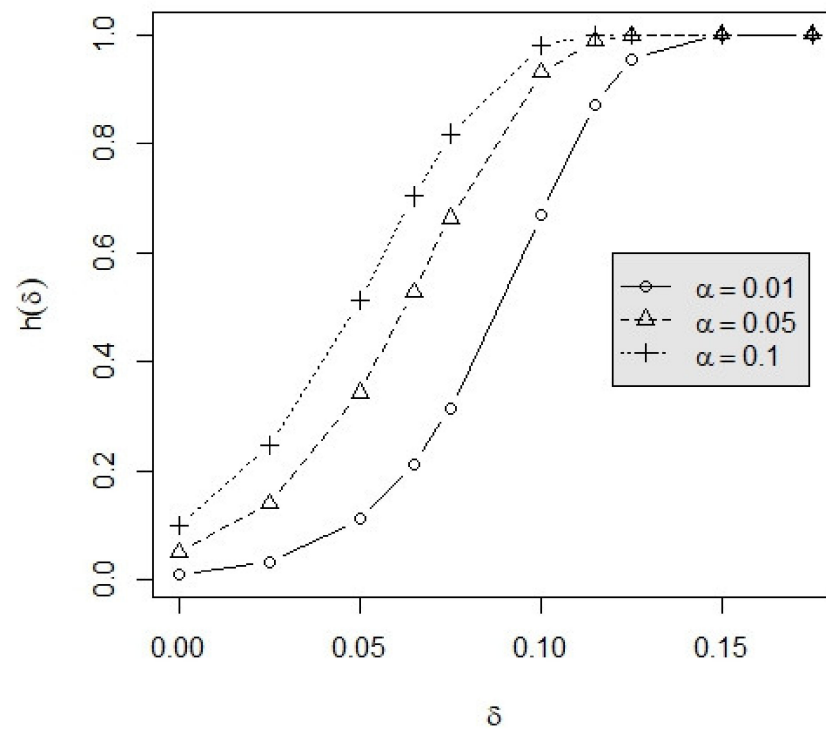
Figure 5. Power function for the test under $n = 60$, $m = 3$ and $p = 0.05$.
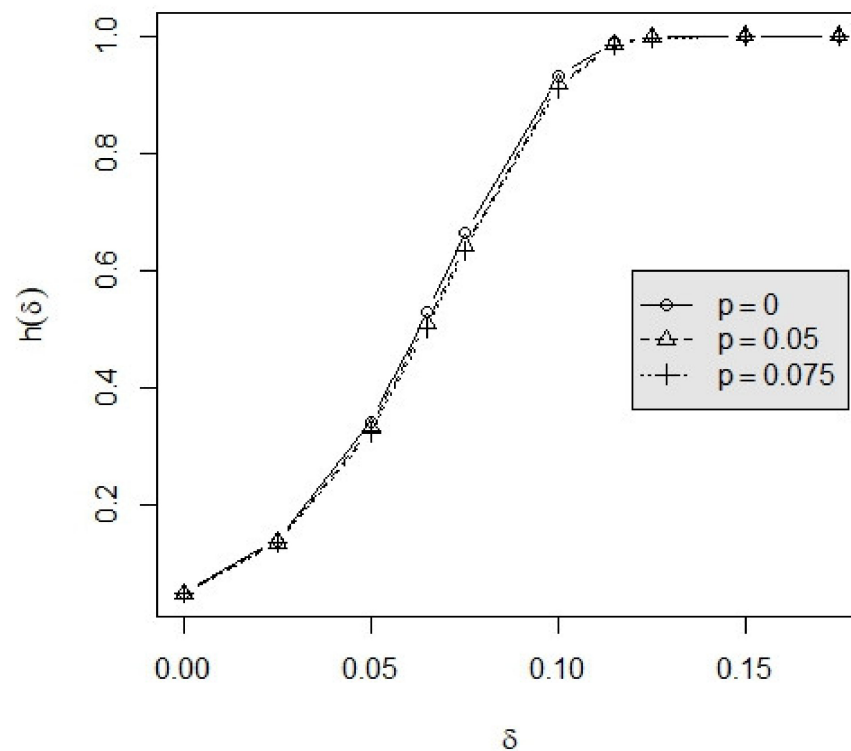

Figure 6. Power function for the test at $\alpha = 0.05$ under $n = 60$ and $m = 3$.

The simulated type I error under $\delta = 0.000$ and the simulated test power under $\delta = 0.025, 0.050, 0.100$ for the case of $\alpha = 0.01, 0.05$, $p = 0.05$ and $n = 10, 20, 40, 60, 80, 100, 200, 500$ using 100,000 repetitions by Monte-Carlo method are listed in Table 2. Referring to $\alpha = 0.05$ and smaller sample sizes $n = 10, 20, 40$, the simulated type I error are 0.01448,

0.02244, 0.02930. They are not equal to the nominal type I error α = 0.05 and all of them are much smaller than the nominal one. For larger sample sizes n = 60, 80, 100, 200, 500, the simulated type I error are 0.03189, 0.03411, 0.03565, 0.03987, 0.04267. They are also not equal to the nominal type I error and smaller than the nominal one. However, the simulated type I error for larger sample sizes are closer to the nominal one. Apparently, the simulated type I errors are approaching to the nominal one when the sample size is increasing and no one can really reach the nominal type I error. Under δ = 0.100, the simulated test power for sample sizes n = 10, 20, 40, 60, 80, 100, 200, 500 are 0.28509, 0.61186, 0.91324, 0.98347, 0.99741, 0.99955, 1.00000, 1.00000. As you can see, the simulated test power is increasing when the sample size is increasing and they are reaching one for sample size being at least 60. To sum up, we will suggest users to consider sample size n $\geq$ 60 to get type I error closer to the nominal one and to yield much higher test power.

Table 2. The simulated type I error and test power.

			δ			
			α = 0.01			
m	*p*	*n*	0.000	0.025	0.050	0.100
5	0.05	10	0.00016	0.00064	0.00205	0.02280
		20	0.00094	0.00407	0.01661	0.19383
		40	0.00202	0.01503	0.07845	0.65924
		60	0.00340	0.02959	0.16399	0.89157
		80	0.00352	0.04384	0.25984	0.97334
		100	0.00416	0.06086	0.36242	0.99392
		200	0.00571	0.15585	0.74854	1.00000
		500	0.00667	0.47964	0.99508	1.00000
			α = 0.05			
m	*p*	*n*	0.000	0.025	0.050	0.100
5	0.05	10	0.01448	0.03333	0.07094	0.28509
		20	0.02244	0.06323	0.16004	0.61186
		40	0.02930	0.11266	0.32399	0.91324
		60	0.03189	0.15746	0.46820	0.98347
		80	0.03411	0.20030	0.58476	0.99741
		100	0.03565	0.23446	0.68200	0.99955
		200	0.03987	0.41045	0.92596	1.00000
		500	0.04267	0.75502	0.99968	1.00000

3.2. Example

In order to guide users to fulfill our proposed hypothesis testing procedure to practical problems, one numerical example given below is taken into account.

We consider the data of relief times (in hours) for 50 arthritic patients in Wingo [16]. The parameter δ in [1,3] is determined by the Gini test with the maximum *p*-value in Figure 7. It shows that the optimal value of δ is identified as δ = 1.755 with the maximum p-value of 0.992592 and it indicates this data fits EF distribution the most at δ = 1.755.
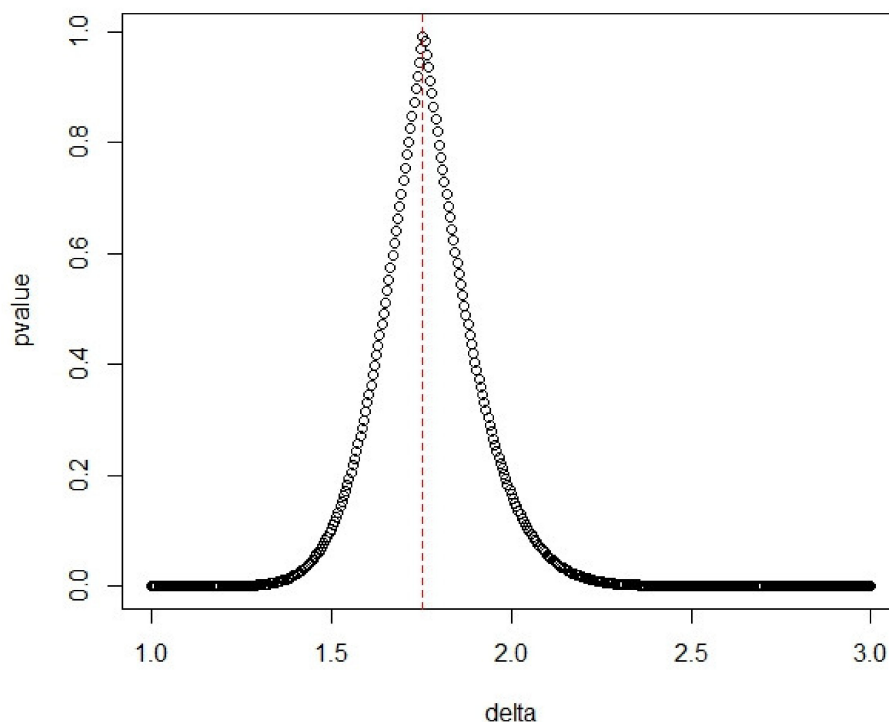
Figure 7. The p-values versus the parameter δ.

In this example, we consider the setup of $m = 5$, $t = 0.2$ (h) and $(p_1, p_2, p_3, p_4, p_5) = (0.15, 0.15, 0.15, 0.15, 1)$. Now the steps to do the proposed testing procedure about C_L are enumerated as follows:

Step 1: Specify the level of significance α as 0.05 and the lower specification $L_U = 0.3611833$ for lifetime U. Thus we can obtain the lower specification $L = -\ln\left(1 - e^{-0.3611833^{-1.755}}\right) = 0.00255$ for the new lifetime Y. The progressive type I interval censored sample are created as $(X_1, X_2, X_3, X_4, X_5, X_6) = (19,3,2,0,0,1)$ at the time points of $(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) = (5,10,15,20,25,30)$ with progressive censoring schemes of $(R_1, R_2, R_3, R_4, R_5, R_6) = (2,0,1,0,1,1)$.

Step 2: If the user wishes the conforming rate to be greater than $P_r = 0.860708$, then the corresponding lifetime performance index target value should be $c_0 = 1 + \ln(P_r) = 0.85$. Then, the null and alternative hypothesis $H_0 : C_L \leq 0.85$ and $H_a : C_L > 0.85$ are determined.

Step 3: Solving Equation (14) numerically to find the MLE of θ as $\hat{\theta} = 9.252991$. Then, the maximum likelihood estimator of C_L can be found as $\hat{C}_L = 1 - \hat{\theta}L = 1 - 9.252991$ $(0.00255) = 0.9764049$.

Step 4: The critical value can be computed as $C_L^0 = 1 - L\left(\theta_0 + Z_{1-\alpha}\sqrt{w(\theta_0)}\right) = 0.9039645$.

Step 5: Since $\hat{C}_L = 0.9764049 \in \left(C_L^0, \infty\right)$, we concluded that there is enough evidence to support $H_a : C_L > 0.85$ so that this manufacturing process is capable.

4. Conclusions

It's very crucial to upgrade the quality of products in the competitive emerging markets and the lifetime performance index is an effective measurement on the quality of products in terms of lifetime. The attractive property of progressive type I interval censoring is the convenient collection of data for a quality engineer. We presented an algorithm for users to fulfill the testing procedure about the lifetime performance index for EF distribution based on the asymptotic distribution of the maximum likelihood estimator under this censoring. Since we use normal approximation to the MLE distribution to develop the testing procedure and compute the test power, this approximation would be good if we have a large enough sample size. The power analysis shows that the proposed procedure reached the pre-assigned level of significance. The impact of the different setup of values of

n, m, p and α on the test power is also analyzed and concluded. For the guideline of users, we give two numerical examples to demonstrate the enumerated algorithm to conduct the testing procedure. In the future, we can investigate the inferences on the lifetime performance index for other lifetime distribution like exponentiated Weibull distribution.

Appendix A

Table A1. The power $h(\delta)$ at $\alpha = 0.05$.

			δ					
m	*n*	*p*	0.000	0.025	0.050	0.075	0.100	0.125
3	60	0.000	0.0500	0.1404	0.3432	0.6636	0.9298	0.9982
		0.050	0.0500	0.1372	0.3312	0.6429	0.9174	0.9972
		0.075	0.0500	0.1356	0.3253	0.6325	0.9106	0.9966
	80	0.000	0.0500	0.1664	0.4358	0.7960	0.9817	0.9999
		0.050	0.0500	0.1622	0.4208	0.7770	0.9766	0.9999
		0.075	0.0500	0.1601	0.4134	0.7673	0.9737	0.9998
	100	0.000	0.0500	0.1918	0.5206	0.8818	0.9957	1.0000
		0.050	0.0500	0.1866	0.5035	0.8667	0.9941	1.0000
		0.075	0.0500	0.1840	0.4950	0.8587	0.9931	1.0000
4	60	0.000	0.0500	0.1532	0.3919	0.7422	0.9670	0.9997
		0.050	0.0500	0.1482	0.3732	0.7142	0.9561	0.9995
		0.075	0.0500	0.1457	0.3642	0.7001	0.9497	0.9992
	80	0.000	0.0500	0.1833	0.4960	0.8631	0.9941	1.0000
		0.050	0.0500	0.1767	0.4733	0.8404	0.9909	1.0000
		0.075	0.0500	0.1735	0.4622	0.8285	0.9889	1.0000
	100	0.000	0.0500	0.2126	0.5882	0.9313	0.9991	1.0000
		0.050	0.0500	0.2045	0.5632	0.9155	0.9984	1.0000
		0.075	0.0500	0.2006	0.5508	0.9069	0.9979	1.0000

Table A1. *Cont.*

5	60	0.000	0.0500	0.1626	0.4275	0.7922	0.9823	0.9999
		0.050	0.0500	0.1558	0.4026	0.7590	0.9732	0.9999
		0.075	0.0500	0.1526	0.3907	0.7421	0.9676	0.9998
	80	0.000	0.0500	0.1957	0.5390	0.9010	0.9977	1.0000
		0.050	0.0500	0.1868	0.5093	0.8766	0.9957	1.0000
		0.075	0.0500	0.1826	0.4950	0.8634	0.9943	1.0000
	100	0.000	0.0500	0.2280	0.6350	0.9558	0.9997	1.0000
		0.050	0.0500	0.2170	0.6031	0.9405	0.9994	1.0000
		0.075	0.0500	0.2118	0.5874	0.9317	0.9991	1.0000

Table A2. The power $h(\delta)$ at $\alpha = 0.1$.

			δ					
m	*n*	*p*	0.000	0.025	0.050	0.075	0.100	0.125
3	60	0.000	0.1000	0.2473	0.5124	0.8179	0.9786	0.9998
		0.050	0.1000	0.2426	0.4991	0.8025	0.9737	0.9997
		0.075	0.1000	0.2403	0.4926	0.7946	0.9710	0.9996
	80	0.000	0.1000	0.2833	0.6076	0.9053	0.9959	1.0000
		0.050	0.1000	0.2775	0.5927	0.8937	0.9945	1.0000
		0.075	0.1000	0.2746	0.5853	0.8876	0.9936	1.0000
	100	0.000	0.1000	0.3170	0.6866	0.9524	0.9993	1.0000
		0.050	0.1000	0.3102	0.6712	0.9447	0.9989	1.0000
		0.075	0.1000	0.3068	0.6633	0.9405	0.9987	1.0000
4	60	0.000	0.1000	0.2657	0.5650	0.8730	0.9918	1.0000
		0.050	0.1000	0.2586	0.5454	0.8544	0.9883	1.0000
		0.075	0.1000	0.2551	0.5358	0.8447	0.9861	0.9999
	80	0.000	0.1000	0.3064	0.6658	0.9435	0.9990	1.0000
		0.050	0.1000	0.2975	0.6445	0.9314	0.9983	1.0000
		0.075	0.1000	0.2932	0.6340	0.9247	0.9978	1.0000
	100	0.000	0.1000	0.3444	0.7458	0.9760	0.9999	1.0000
		0.050	0.1000	0.3339	0.7246	0.9689	0.9998	1.0000
		0.075	0.1000	0.3288	0.7139	0.9649	0.9997	1.0000
5	60	0.000	0.1000	0.2791	0.6019	0.9049	0.9963	1.0000
		0.050	0.1000	0.2696	0.5767	0.8844	0.9937	1.0000
		0.075	0.1000	0.2651	0.5644	0.8735	0.9921	1.0000
	80	0.000	0.1000	0.3231	0.7051	0.9627	0.9997	1.0000
		0.050	0.1000	0.3113	0.6785	0.9508	0.9993	1.0000
		0.075	0.1000	0.3057	0.6654	0.9440	0.9990	1.0000
	100	0.000	0.1000	0.3642	0.7841	0.9861	1.0000	1.0000
		0.050	0.1000	0.3503	0.7585	0.9800	0.9999	1.0000
		0.075	0.1000	0.3437	0.7456	0.9763	0.9999	1.0000

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