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ON THE "FAIR" GAMES PROBLEM FOR THE WEIGHTED GENERALIZED PETERSBURG GAMES

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ABSTRACT. Let $S_n = \sum_{j=1}^{n} a_j Y_j$, $n \geq 1$, where $\{Y_n, n \geq 1\}$ are i.i.d. r.v.'s and $\{a_n, n \geq 1\}$ are real numbers. Interpreting $a_n Y_n$ as a player's winnings from the n–th game, a natural question is whether there is an entrance fee m_n to the n–th game such that $S_n/M_n \to 1$ in pr., where $M_n = \sum_{j=1}^{n} m_j$. Suppose that $\{Y_n\}$ represent the winnings from a sequence of generalized Petersburg games, that is, $\{Y_n, n \geq 1\}$ are i.i.d. random variables with $P\{Y_1 = q^{-k}\} = pq^{k-1}$, $0 < p = 1 - q < 1$, $k \geq 1$. It is shown that when $a_n > 0$, $\forall n = 1, 2, 3, \cdots$ and $\lim_{n\to\infty} \left[\left(\sum_{j=1}^{n} a_j \right) \Big/ \left(\max_{1 \leq j \leq n} a_j \right) \right] = \infty$, then there exist $\{M_n, n \geq 1\}$ such that $S_n/M_n \to 1$ in pr..

1. INTRODUCTION

Consider a sequence of games and a sequence of independent random variables $\{X_n, n \geq 1\}$ where for each $n \geq 1$, X_n represents a player's winnings from participating in game n. Suppose that the player pays the (nonrandom) entrance fee

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m_n for the opportunity to play the n–th game, $n \geq 1$. For the first n games, $S_n = \sum_{j=1}^{n} X_j$ represents the total winnings and $M_n = \sum_{j=1}^{n} m_j$ represents the total or accumulated entrance fees, $n \geq 1$. The sequence of entrance fees $\{m_n, n \geq 1\}$ is said to be "a fair solution in the weak (resp., strong) sense to the games" if $S_n/M_n \to 1$ in pr. (resp. $S_n/M_n \to 1$ almost surely (a.s.)).

In the current work, attention will primarily be focused on the weighted i.i.d. case consisting of X_n's of the form $a_n Y_n$ where $\{a_n, n \geq 1\}$ are real numbers and $\{Y_n, n \geq 1\}$ are i.i.d. random variables. Adler and Rosalsky [1, Theorem 3] provided a generalization of the Chow–Robbins Theorem [2] to the weighted i.i.d. case. They showed for i.i.d. random variables $\{Y_n, n \geq 1\}$ with

$$(*) \qquad E|Y_1| = \infty, \qquad n|a_n| \uparrow \quad \text{and} \quad \sum_{j=1}^{n} a_j = O(n|a_n|)\,,$$

then for each sequence of real numbers $\{M_n, n \geq 1\}$ either

$$\liminf_{n\to\infty} \left|\frac{S_n}{M_n}\right| = 0 \quad \text{a.s.} \qquad \text{or} \qquad \limsup_{n\to\infty} \left|\frac{S_n}{M_n}\right| = \infty \quad \text{a.s.}$$

and, consequently, $P\Big\{\lim_{n\to\infty} \frac{S_n}{M_n} = 1\Big\} = 0$.

The classical Petersburg game may be described as follows: A fair coin is repeatedly tossed. If "heads" occur for the first time on the k–th toss, the player wins 2^k dollars. Thus, the player wins X dollars where $P\{X = 2^k\} = 2^{-k}$, $k \geq 1$. In [1], Adler and Rosalsky consider the case where the underlying coin need not be fair, that is, suppose "heads" occur with probability p where $0 < p < 1$. Let α be a fixed constant. For the n–th game, if "heads" occur for the first time on the k–th toss, the player wins $n^\alpha q^{-k}$ dollars where $q = 1 - p$. In other words, the winnings X_n from the n–th game are of the form $X_n = n^\alpha Y_n$, where $\{Y_n, n \geq 1\}$ are i.i.d. random variables with

$$(1) \qquad P\{Y_1 = q^{-k}\} = pq^{k-1}\,, \qquad k \geq 1\,.$$

In this paper, we consider the weighted generalized Petersburg games. While no fair solution exists in the strong sense when the hypotheses of $(*)$ are satisfied, we try to find a fair solution in the weak sense. That is, for a sequence of real numbers $\{a_n, n \geq 1\}$ and i.i.d. random variables $\{Y_n, n \geq 1\}$ with $P\{Y_1 = q^{-k}\} = pq^{k-1}$, $k \geq 1$, $0 < p = 1-q < 1$, find conditions on $\{a_n, n \geq 1\}$ which ensure the existence of constants $\{M_n, n \geq 1\}$ for which

$$\frac{\sum_{j=1}^{n} a_j Y_j}{M_n} \longrightarrow 1 \qquad \text{in pr.}$$

obtains. Under such conditions, $\{M_n, n \geq 1\}$ can be found explicitly (Adler and Rosalsky proved in the special case where $a_n = n^{\alpha}$, $n \geq 1$, $\alpha > -1$).

2. RESULTS

Let Y be distributed as in (1). We introduce some properties given by Adler and Rosalsky [1].

Lemma 1. ([1]): *Let Y be a random variable with $P\{Y = q^{-k}\} = pq^{k-1}$, $k \geq 1$, $0 < q = 1 - p < 1$.*

(i) For all $a > 0$, $P\{Y > a\} < (qa)^{-1}$.

(ii) For all $a \geq 1$, $P\{Y > a\} \geq a^{-1}$.

(iii) For all $a \geq q^{-1}$, $EY^2 I(Y < a) = q^{-k-1} - q^{-1} < q^{-1}a$, where k is the largest integer such that $q^{-k} \leq a$.

(iv) For all $a \geq q^{-1}$, $EYI(Y \leq a) = kpq^{-1}$, where k is the largest integer such that $q^{-k} \leq a$.

For a given sequence of positive weights $\{a_n, n \geq 1\}$, define, for each $x > 0$, $u_n(x)$ as the expectation of the weighted random variable $a_n Y_n$ which is truncated at x, that is, $u_n(x) = E[a_n Y_n I(a_n Y_n \leq x)]$. In the following theorems, let

$$A_n = \left\{ x : \sum_{j=1}^{n} u_j(x) \geq x \right\},$$

and

$$M_n = \sup\left\{x : \sum_{j=1}^{n} u_j(x) \geq x\right\}.$$

Theorem 1. *Let $\{Y_n, n \geq 1\}$ be i.i.d. random variables with $P\{Y_1 = q^{-k}\} = pq^{k-1}$, $k \geq 1$, $0 < p = 1 - q < 1$. If*

$$(2) \qquad \lim_{n\to\infty} \frac{\sum_{j=1}^{n} a_j}{\max_{1\leq j\leq n} a_j} = \infty \quad \text{and} \quad a_n > 0, \qquad \forall n = 1, 2, 3, \cdots,$$

then

(i) *there exists an integer N_0, such that $A_n \neq \phi$, $\forall n \geq N_0$.*

(ii) $A_n \subset A_m$, $\forall m \geq n \geq N_0$.

(iii) *M_n is finite for each fixed $n \geq N_0$, and $M_n \uparrow \infty$.*

(Throughout, the symbol log denoted the logarithm to the base q^{-1}.)

Proof. (i) For all $x \geq q^{-1}$, Lemma 1(iv) ensures that

$$pq^{-1}(\log x - 1) < EY_1 I(Y_1 \leq x) \leq pq^{-1} \log x\,.$$

Therefore, for each $j = 1, 2, 3, \cdots, n$,

$$a_j pq^{-1}(\log a_j^{-1} x - 1) < u_j(x) = E a_j Y_j I(a_j Y_j \leq x) \leq a_j pq^{-1} \log a_j^{-1} x,$$

and hence,

$$(3) \qquad \begin{aligned} \sum_{j=1}^{n}[a_j pq^{-1}(\log a_j^{-1} x - 1)] - x \;&<\; \sum_{j=1}^{n} u_j(x) - x \\ &\leq\; \sum_{j=1}^{n}[a_j pq^{-1} \log a_j^{-1} x] - x\,. \end{aligned}$$

Let

$$h_n(x) = \sum_{j=1}^{n}[a_j pq^{-1}(\log a_j^{-1} x - 1)] - x\,,$$

and

$$g_n(x) = \sum_{j=1}^{n}[a_j pq^{-1} \log a_j^{-1} x] - x\,.$$

Then we can rewrite (3) as

$$(3)' \qquad h_n(x) < \sum_{j=1}^{n} u_j(x) - x \le g_n(x)\,.$$

Also, we note that $h_n(x)$ has a maximun value at $x_0^{(n)} = pq^{-1}\sum_{j=1}^{n} a_j \log e$, and

$$h_n(x_0^{(n)}) = pq^{-1}\left[\sum_{j=1}^{n} a_j \log\left(\frac{\sum_{j=1}^{n} a_j}{a_j} \frac{p\log e}{e}\right)\right].$$

Now, condition (2) implies that, for a given fixed $0 < p < 1$, there exists an integer N_0 such that

$$\frac{\sum_{j=1}^{n} a_j}{\max_{1\le j\le n} a_j} \ge \frac{e}{p\log e}\,, \qquad \forall n \ge N_0\,.$$

Hence for each $n \ge N_0$

$$h_n(x_0^{(n)}) = pq^{-1}\left[\sum_{j=1}^{n} a_j \log\left(\frac{\sum_{j=1}^{n} a_j}{a_j} \frac{p\log e}{e}\right)\right] \ge 0\,.$$

Therefore, by $(3)'$, we get

$$A_n = \left\{x : \sum_{j=1}^{n} u_j(x) \ge x\right\} \ne \phi\,, \qquad \forall n \ge N_0\,.$$

(ii) Since $u_j(x) \ge 0$, $\forall j$, we have $\sum_{j=1}^{m} u_j(x) \ge \sum_{j=1}^{n} u_j(x) \ge x$, $\forall m \ge n \ge N_0$.

Therefore, $A_n \subset A_m$, $\forall m \ge n \ge N_0$.

(iii) For fixed $n \ge N_0$,

$$g_n(x) = \sum_{j=1}^{n}[a_j pq^{-1} \log a_j^{-1} x] - x \to -\infty\,, \qquad \text{as} \qquad x \to \infty\,.$$

Thus, by (3)′,

$$M_n = \sup\Big\{x : \sum_{j=1}^{n} u_j(x) \geq x\Big\} \text{ is bounded away from infinity.}$$

To prove $M_n \uparrow \infty$ as $n \to \infty$, note that (by (ii))

$$A_n \subset A_m\,, \qquad \forall m \geq n \geq N_0$$

and thus $\qquad M_n \leq M_m\,, \qquad \forall m \geq n \geq N_0.$

Finally, under the definition of M_n and $x_0^{(n)}$, we have

$$M_n = \sup A_n \geq x_0^{(n)} = pq^{-1}\Big(\sum_{j=1}^{n} a_j\Big)\log e\,,$$

and this ensures that $M_n \to \infty$, as $n \to \infty$; since

$$\frac{\sum_{j=1}^{n} a_j}{\max\limits_{1\leq j\leq n} a_j} \to \infty \quad \text{if and only if} \quad \sum_{j=1}^{n} a_j \to \infty \quad \text{and} \quad \frac{a_n}{\sum_{j=1}^{n} a_j} \to 0$$

for positive numbers $\{a_n\}$.

Theorem 2. *Let $\{Y_n, n \geq 1\}$ be i.i.d. random variables with $P\{Y_1 = q^{-k}\} = pq^{k-1}$, $k \geq 1$, $0 < p = 1 - q < 1$. For a sequence $\{a_n, n \geq 1\}$ of positive real numbers, if $\lim\limits_{n\to\infty} \frac{\sum_{j=1}^{n} a_j}{\max\limits_{1\leq j\leq n} a_j} = \infty$, then*

(i) $\dfrac{M_n}{\max\limits_{1\leq j\leq n} a_j} \to \infty$, *as* $n \to \infty$.

(ii) $\dfrac{\sum_{j=1}^{n} u_j(M_n)}{M_n} \to 1$, *as* $n \to \infty$.

(iii) $\sum_{j=1}^{n} a_j = o(M_n)$.

Proof. (i) We recall that

$$M_n = \sup A_n \geq x_0^{(n)} = pq^{-1}\Big(\sum_{j=1}^{n} a_j\Big) \log e \,,$$

that is, $$\frac{M_n}{\max\limits_{1\leq j\leq n} a_j} \geq \frac{pq^{-1}\Big(\sum\limits_{j=1}^{n} a_j\Big)\log e}{\max\limits_{1\leq j\leq n} a_j}.$$

Now, letting $n \to \infty$ and using $\lim\limits_{n\to\infty} \dfrac{\sum\limits_{j=1}^{n} a_j}{\max\limits_{1\leq j\leq n} a_j} = \infty$

we conclude that (i) is true.

(ii) Since

$$M_n = \sup\Big\{x : \sum_{j=1}^{n} u_j(x) \geq x\Big\}\,,$$

then either

$$\frac{\sum\limits_{j=1}^{n} u_j(M_n)}{M_n} = 1 \quad \text{or} \quad \frac{\sum\limits_{j=1}^{n} u_j(M_n)}{M_n} > 1 \geq \frac{\sum\limits_{j=1}^{n} u_j(M_n^+)}{M_n}\,. \tag{4}$$

In the latter case, it follows

$$\begin{aligned} 0 \ \leq \ &\left| \frac{\sum\limits_{j=1}^{n} u_j(M_n)}{M_n} - \frac{\sum\limits_{j=1}^{n} u_j(M_n^+)}{M_n} \right| \leq \sum_{j=1}^{n} P\{Y_n \geq a_j^{-1} M_n\} \\ \leq \ & \sum_{j=1}^{n} (q a_j^{-1} M_n)^{-1} = q^{-1} \frac{\sum\limits_{j=1}^{n} a_j}{M_n}\,. \end{aligned} \tag{5}$$

Now, by (i) and the fact that $EY_n = \infty$, $\forall n$, we have

$$0 \le \frac{\frac{1}{M_n}\sum_{j=1}^{n} a_j}{\frac{1}{M_n}\sum_{j=1}^{n} u_j(M_n)} = \frac{\sum_{j=1}^{n} a_j}{\sum_{j=1}^{n} a_j u^*(a_j^{-1}M_n)}$$

$$\le \frac{1}{u^*\Big(M_n/\max_{1\le j\le n} a_j\Big)} \to 0$$

where

$$u^*(x) = EY_1 I(Y_1 \le x)\,.$$

Therefore, we get $\lim_{n\to\infty} \frac{\sum_{j=1}^{n} u_j(M_n)}{M_n} = 1.$

(iii) As in the proof of (ii), we have

$$0 \le \frac{\sum_{j=1}^{n} a_j}{\sum_{j=1}^{n} u_j(M_n)} \le \frac{1}{u^*\Big(M_n/\max_{1\le j\le n} a_j\Big)} \to 0\,, \qquad \text{as} \qquad n\to\infty\,.$$

Note that

$$\frac{\sum_{j=1}^{n} a_j}{M_n} = \frac{\sum_{j=1}^{n} a_j}{\sum_{j=1}^{n} u_j(M_n)} \frac{\sum_{j=1}^{n} u_j(M_n)}{M_n}\,.$$

It follows from (ii) that

$$\sum_{j=1}^{n} a_j = o(M_n)\,.$$

One preliminary lemma 2 will be established before stating the main results.

Lemma 2. *Let $\{X_n, n\ge 1\}$ be independent random variables and $\{b_n, n\ge 1\}$ be real numbers with $0 < b_n \uparrow \infty$. Then*

$$\frac{1}{b_n}\Big[\sum_{j=1}^{n} X_j - \sum_{j=1}^{n} EX_j I(|X_j| \le b_n)\Big] \longrightarrow 0 \qquad \text{in pr.}, \tag{6}$$

if

(7) (i) $$\sum_{j=1}^{n} P\{|X_j| > b_n\} = o(1),$$

(8) (ii) $$\frac{1}{b_n^2}\sum_{j=1}^{n} EX_j^2 I(|X_j| \le b_n) = o(1).$$

Proof. To prove (6), set

$$X_j' = X_j I(|X_j| \le b_n), \qquad Z_j' = X_j' - EX_j'.$$

Then (ii) entails

$$\frac{1}{b_n}\sum_{j=1}^{n} Z_j \longrightarrow 0 \quad \text{in pr.},$$

since $\forall \varepsilon > 0$,

$$P\Big\{\Big|\frac{1}{b_n}\sum_{j=1}^{n} Z_j\Big| > \varepsilon\Big\} \le \frac{\sum_{j=1}^{n} E(X_j')^2}{(\varepsilon b_n)^2} = \frac{\sum_{j=1}^{n} EX_j^2 I(|X_j| \le b_n)}{(\varepsilon b_n)^2} \longrightarrow 0,$$

as $n \to \infty$. That is,

(9) $$\frac{1}{b_n}\Big[\sum_{j=1}^{n} X_j' - \sum_{j=1}^{n} EX_j I(|X_j| \le b_n)\Big] \longrightarrow 0 \quad \text{in pr.}.$$

Next we will show that

$$\sum_{j=1}^{n} X_j - \sum_{j=1}^{n} X_j' \longrightarrow 0 \quad \text{in pr.}.$$

It follows from the first condition that

$$P\Big\{\sum_{j=1}^{n} X_j \ne \sum_{j=1}^{n} X_j'\Big\} \le \sum_{j=1}^{n} P\{X_j \ne X_j'\} = \sum_{j=1}^{n} P\{|X_j| > b_n\} = o(1).$$

Therefore, for all $\varepsilon > 0$,

$$P\Big\{\Big|\sum_{j=1}^{n} X_j - \sum_{j=1}^{n} X_j'\Big| > \varepsilon\Big\} \le P\Big\{\sum_{j=1}^{n} X_j \ne \sum_{j=1}^{n} X_j'\Big\} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

Hence

$$\text{(10)} \qquad \frac{1}{b_n}\Big[\sum_{j=1}^{n} X_j - \sum_{j=1}^{n} X_j'\Big] \longrightarrow 0 \quad \text{in pr.}.$$

Combining (9) with (10), we conclude that

$$\frac{1}{b_n}\Big[\sum_{j=1}^{n} X_j - \sum_{j=1}^{n} EX_jI(|X_j| \le b_n)\Big] \longrightarrow 0 \quad \text{in pr.}.$$

Under the condition (2), it will be shown that (6) is true if $\{X_n\}$ and $\{b_n\}$ are replaced by $\{a_nY_n\}$ and $\{M_n\}$, respectively. And with this, the main result can be established.

Theorem 3. *Let $\{Y_n, n \ge 1\}$ be i.i.d. random variables with $P\{Y_1 = q^{-k}\} = pq^{k-1}$, $k \ge 1$, $0 < p = 1 - q < 1$, and let $\{a_n, n \ge 1\}$ be a sequence of positive real numbers.*

$$\text{If} \quad \lim_{n\to\infty} \frac{\sum_{j=1}^{n} a_j}{\max_{1\le j\le n} a_j} = \infty \quad \text{then} \quad \frac{1}{M_n}\sum_{j=1}^{n} a_jY_j \longrightarrow 1 \quad \text{in pr.}.$$

Proof. By Lemma 1 (i) and (iii), we have

$$\sum_{j=1}^{n} P\{a_jY_j > M_n\} \le \sum_{j=1}^{n}(qa_j^{-1}M_n^{-1}) = \frac{\sum_{j=1}^{n} a_j}{qM_n},$$

and

$$\frac{1}{M_n^2}\sum_{j=1}^{n} E(a_jY_j)^2 I(a_jY_j \le M_n) \le \frac{\sum_{j=1}^{n} a_j}{qM_n}.$$

Now conditions (7) and (8) follow directly from Theorem 2 (iii), whence via Lemma 2

$$\text{(11)} \qquad \frac{1}{M_n}\Big[\sum_{j=1}^{n} a_jY_j - \sum_{j=1}^{n} Ea_jY_jI(a_jY_j \le M_n)\Big] \longrightarrow 0 \quad \text{in pr.}.$$

Finally, combining (11) with Theorem 2 (ii), we have

$$\frac{1}{M_n}\sum_{j=1}^{n} a_jY_j \longrightarrow 1 \quad \text{in pr.}.$$

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